Hadron dissociation in dense matter (I)

David.Blaschke@gmail.com   (Wroclaw University & JINR Dubna & MEPhI Moscow)

1. Path integral approach to the partition function

2. Ideal quantum gases
- neutral and charged scalar fields: Bose condensation
- fermion fields
- gauge fields and blackbody radiation

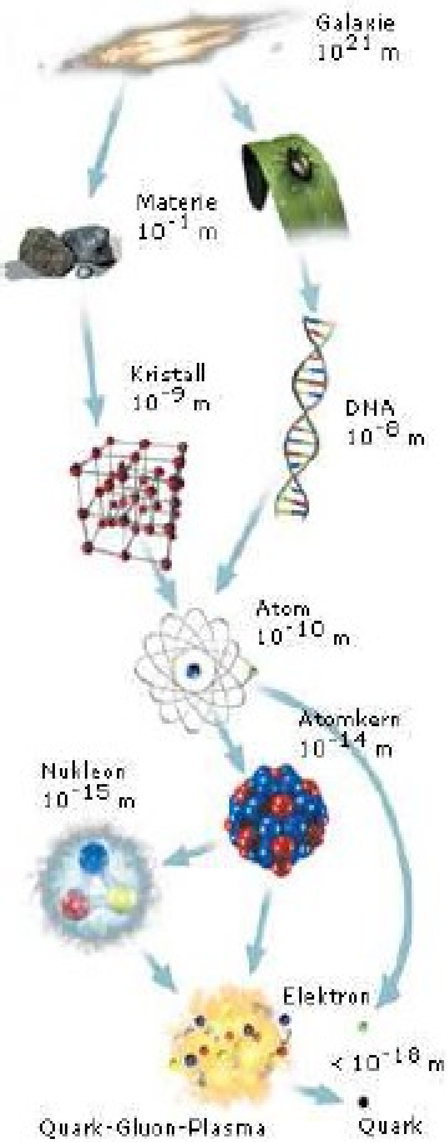
3. Strongly interacting quark matter
- Hubbard-Stratonovich trick: bosonization
- Nambu-Jona-Lasinio model: chiral symmetry breaking and color superconductivity
- Bound states and Mott effect
- Beth-Uhlenbeck equation of state

4. Lattice QCD & hadron resonance gas

<u>Literature:</u>

J.I. Kapusta: Finite Temperature Field Theory (Cambridge University Press, 1989)
K. Yagi, T. Hatsuda, Y. Miyake: Quark-Gluon Plasma (Cambridge University Press, 2005)

MANY PARTICLE SYSTEMS & QUANTUM FIELD THEORY



Elements	Bound states	System
humans, animals	couples, groups, parties	society
molecules, crystals	(bio)polymers	animals, plants
atoms	molecules, clusters, crystals	solids, liquids, ...
ions, electrons	atoms	plasmas
nucleons, mesons	nuclei	nuclear matter
quarks, anti-quarks	nucleons, mesons	quark matter

Highly Compressed Matter ⇔ Pauli Principle

Partition function: $Z = \mathrm{Tr}\left\{ \mathrm{e}^{-\beta(H-\mu_i Q_i)} \right\}$

Partition function for Quantum Chromodynamics (QCD)

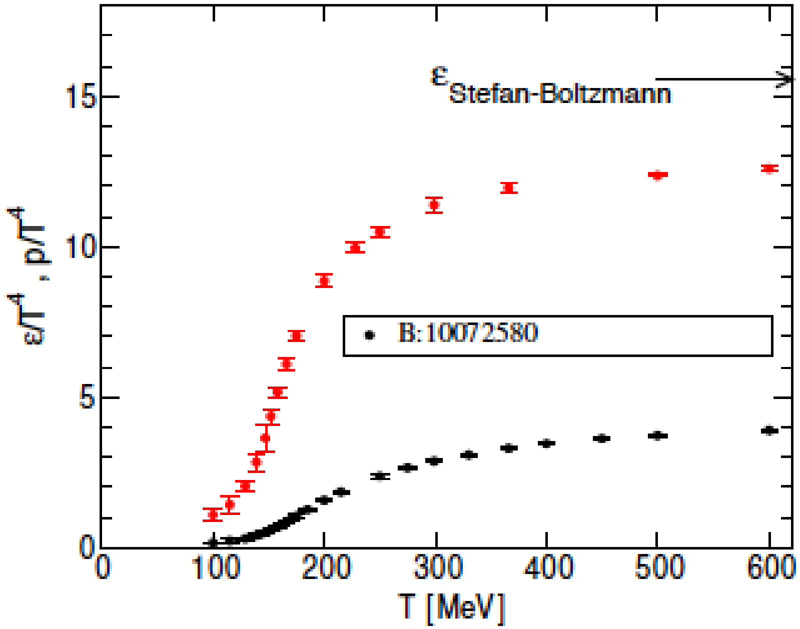
- Partition function as a Path Integral (imaginary time $\tau = i\,t,\ 0 \le \tau \le \beta = 1/T$)

$$Z[T,V,\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \exp\left\{-\int_0^\beta d\tau \int_V d^3x\ \mathcal{L}_{QCD}(\psi,\bar{\psi},A)\right\}$$

- QCD Lagrangian, non-Abelian gluon field strength: $F^a_{\mu\nu}(A) = \partial_\mu A^a \nu - \partial_\nu A^a_\mu + g\ f^{abc}[A^b_\mu, A^c_\nu]$

$$\mathcal{L}_{QCD}(\psi,\bar{\psi},A) = \bar{\psi}[i\gamma^\mu(\partial_\mu - igA_\mu) - m - \gamma^0\mu]\psi - \frac{1}{4}F^a_{\mu\nu}(A)F^{a,\mu\nu}(A)$$

- Numerical evaluation: Lattice gauge theory simulations (hotQCD, Wuppertal-Budapest)



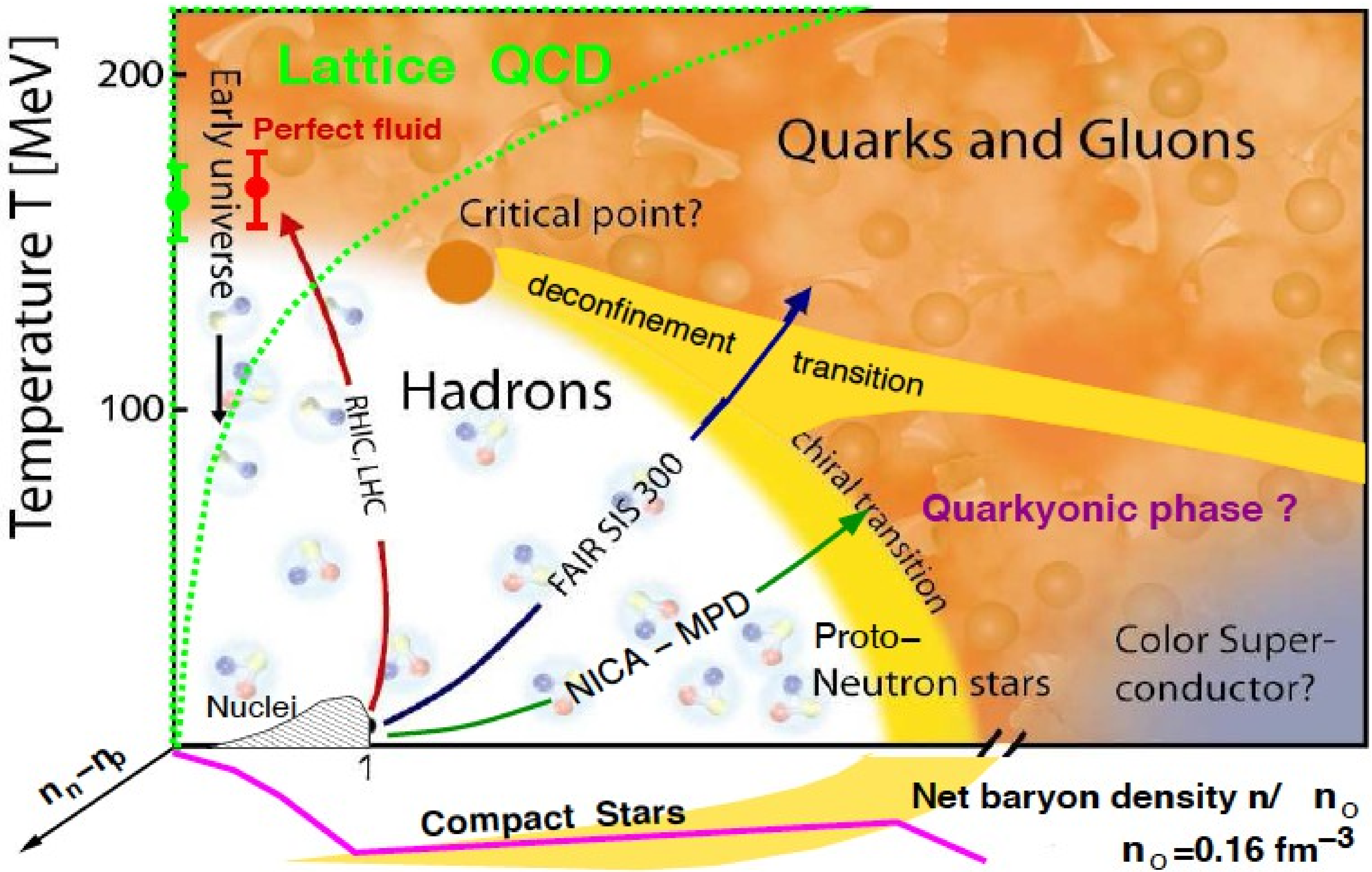
- Equation of state: $\varepsilon(T) = -\partial \ln Z[T,V,\mu]/\partial\beta$ (1/V)
- Phase transition at $T_c = 155$ MeV
- Problem: Interpretation ?

$\varepsilon/T^4 = \frac{\pi^2}{30}N_\pi \sim 1$ (ideal pion gas)

$\varepsilon/T^4 = \frac{\pi^2}{30}(N_G + \frac{7}{8}N_Q) \sim 15.6$ (quarks and gluons)

- Hadron resonance gas

Modern QCD Phase Diagram: Quarkyonic Matter



A. Andronic, D. Blaschke, et al., "Hadron production ...", Nucl. Phys. A 837 (2010) 65 - 86

Ensembles and Partition Function

- *microcanonical ensemble:* isolated system, fixed energy E, particle number N, volume V
- *canonical ensemble:* system with a heat reservoir at temperature T; fixed: T, N, and V
- *grand canonical ensemble:* system can exchange particles and energy with a reservoir. fixed variables: T, V, and the chemical potential μ

System: Hamiltonian $\hat{H}$ and conserved number operators $\hat{N}_i$ (hermitean, commute with $\hat{H}$)
In relativistic QED: $N_e = N_{e-} - N_{e+}$ is conserved, not N_{e-} or N_{e+} separately.

- statistical density matrix is: $\hat{\rho} = \exp\left[-\beta(\hat{H} - \mu_i \hat{N}_i)\right]$,
- ensemble average of an operator $\hat{A}$ is given by: $A = \mathrm{Tr}\hat{\rho}\hat{A}/\mathrm{Tr}\hat{\rho}$.
- grand canonical partition function is: $Z = \mathrm{Tr}\hat{\rho}$.

The partition function $Z = Z(T, V, \mu_1, \mu_2, \ldots)$ is the single most important function !
From it all other standard thermodynamic properties may be determined:
$P = T\partial \ln Z/\partial V$, (pressure) $N_i = T\partial \ln Z/\partial \mu_i$, (particle numbers)
$S = \partial(T \ln Z)/\partial T$, (entropy) $E = -PV + TS + \mu_i N_i$, (energy)

Note: Extension to the nonequilibrium situation possible; generalized Gibbs ensemble
Nonequilibrium characterized by further observables, e.g., currents or reaction variables.
relevant statistical operator $\Longrightarrow$ Zubarev formalism.

Path Integral approach to Partition Function

Partition function quantum statistics: sum over all (eigen-)states.

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(\hat{H}-\mu_i \hat{N}_i)} = \int d\phi_a \langle \phi_a | \mathrm{e}^{-\beta(\hat{H}-\mu_i \hat{N}_i)} | \phi_a \rangle \ ,$$

Similar to the transition amplitude (time evolution operator) in Quantum Field Theory,
Introduce *imaginary time* variable $\tau = \mathrm{i}\, t$ with integration limited to $0 < \tau < \beta$.
For system with conserved charges $\mathcal{N}(\pi, \phi)$, make the replacement

$$\mathcal{H}(\pi, \phi) \to \mathcal{K}(\pi, \phi) = \mathcal{H}(\pi, \phi) - \mu \mathcal{N}(\pi, \phi) \ ,$$

Representation of the partition function Z as a functional integral

$$Z = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp\left\{ \int_0^\beta \int d^3x \left(i\, \pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu \mathcal{N}(\pi, \phi) \right) \right\} \ .$$

“Periodic”: field integration constrained, so that $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$

Key lesson:
Quantization: Path integral over all admissible (constraints!) *classical field* configurations;
Statistical operator quantization on the other hand requires introduction of *field operators*.

Partition Function as a Path Integral - Equivalence

Be $\hat{\phi}(\vec{x},0)$ a field operator in the Schrödinger picture at time $t=0$ and $\hat{\pi}(\vec{x},0)$ the corresponding canonically conjugated field momentum operator. For eigenstates $\mid \phi\rangle$ of the field holds the eigenvalue equation

$$\hat{\phi}(\vec{x},0) \mid \phi\rangle = \phi(\vec{x}) \mid \phi\rangle \ ,$$

where $\phi(\vec{x})$ is the "eigenvalue" corresponding to the field operator. For the eigenstates of the fields completeness and orthonormality shall hold

$$\int d\phi(\vec{x}) \mid \phi\rangle\langle\phi \mid \ = \ 1 \ ; \quad \langle\phi_a \mid \phi_b\rangle = \delta\left[\phi_a(\vec{x}) - \phi_b(\vec{x})\right] \ .$$

For the field momentum operator and its eigenstates $\mid \pi\rangle$ holds analogously

$$\begin{aligned} \hat{\pi}(\vec{x},0) \mid \pi\rangle \ &= \ \pi(\vec{x}) \mid \pi\rangle \\ \int \frac{d\pi(\vec{x})}{2\pi} \mid \pi\rangle\langle\pi \mid \ &= \ 1 \ ; \quad \langle\pi_a \mid \pi_b\rangle = \delta\left[\pi_a(\vec{x}) - \pi_b(\vec{x})\right] \ . \end{aligned}$$

The transition amplitude between coordinates and momenta eigenstates $\langle x \mid p\rangle = \exp(ipx)$ is generalized to the quantum field theory case by

$$\langle\phi \mid \pi\rangle = \exp\left[i \int d^3x \, \pi(\vec{x})\phi(\vec{x})\right] \ .$$

Partition Function as a Path Integral - Equivalence

For a dynamical description of the system we require the Hamiltonian operator

$$\hat{H} = \int d^3x \mathcal{H}\left(\hat{\pi}, \hat{\phi}\right) .$$

Consider the state $\mid \phi_a\rangle$ at $t = 0$, which at a later time t_f has evolved to $e^{-i\hat{H}t_f} \mid \phi_a\rangle$. For the quantum statistical partition function, the system returns at $t = t_f$ to the initial state at $t = 0$. The time interval $(0, t_f)$ is decomposed into equidistant parts $\Delta t = t_f/N$. At each time step we introduce a complete set of field and field-momentum states

$$\begin{aligned}\langle \phi_a \mid e^{-iHt_f} \mid \phi_a\rangle \;=\; & \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \frac{d\pi_i d\phi_i}{2\pi} \right) \langle \phi_a \mid \pi_N\rangle\langle \pi_N \mid e^{-iH\Delta t} \mid \phi_N\rangle\langle \phi_N \mid \pi_{N-1}\rangle \\ & \times \langle \pi_{N-1} \mid e^{-iH\Delta t} \mid \phi_{N-1}\rangle \times \ldots \times \langle \phi_2 \mid \pi_1\rangle\langle \pi_1 \mid e^{-iH\Delta t} \mid \phi_1\rangle\langle \phi_1 \mid \phi_a\rangle\end{aligned}$$

We make use of the following expressions

$$\langle \phi_1 \mid \phi_a\rangle \;=\; \delta\left(\phi_1 - \phi_a\right) \; ; \; \langle \phi_{i+1} \mid \pi_i\rangle = \exp\left[i \int d^3x \, \pi_i(\vec{x})\phi_{i+1}(\vec{x}) \right] .$$

For $\Delta t \to 0$ the exponential function can be expanded with $H_i = \int d^3x \mathcal{H}\left(\pi_i(\vec{x}), \phi_i(\vec{x})\right)$

$$\langle \pi_i \mid e^{-i\hat{H}\Delta t} \mid \phi_i\rangle \;\simeq\; \langle \pi_i \mid (1 - \hat{H}\Delta t) \mid \phi_i\rangle = \langle \pi_i \mid \phi_i\rangle(1 - H_i\Delta t) = (1 - H_i\Delta t) \exp\left[i \int d^3x \, \pi_i(\vec{x})\phi_i(\vec{x}) \right]$$

Partition Function as a Path Integral - Equivalence

Taken all expressions together yields

$$\langle \phi_a \mid e^{-i\hat{H}t_f} \mid \phi_a \rangle = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \frac{d\pi_i d\phi_i}{2\pi} \right) \delta\left(\phi_1 - \phi_a\right) \exp\{-i\Delta t \sum_{j=1}^{N} \int d^3x [\mathcal{H}(\pi_j, \phi_j) - \pi_j \frac{\phi_{j+1} - \phi_j}{\Delta t}]\}$$

Here holds $\phi_{N+1} = \phi_a = \phi_1$. In the continuum limit we obtain

$$\langle \phi_a \mid e^{-i\hat{H}t_f} \mid \phi_a \rangle = \int \mathcal{D}\pi \int_{\phi(\vec{x},0)=\phi_a}^{\phi(\vec{x},t_f)=\pm\phi_a} \mathcal{D}\phi \exp\left[i \int_0^{t_f} dt \int d^3\vec{x} \underbrace{\left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H}(\phi, \pi) \right)}_{\mathcal{L}(\phi,\pi)} \right]$$

$\mathcal{D}\pi$ and $\mathcal{D}\phi$ stand for the Functional Integration over fields and their conjugate momenta. The result for the partition function reads

$$Z = \int [d\pi] \int_{\pm} [d\phi] \exp\left(\int_0^{\beta} d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu_i \mathcal{N}_i(\pi, \phi) \right) \right) \quad (1)$$

The index $\pm$ stands for the symmetry (antisymmetry) of the Bose (Fermi) fields at the borders of the imaginary time interval: $\phi(\vec{x}, 0) = \pm\phi(\vec{x}, \beta)$.

Example: Neutral Scalar Field

The most general renormalizable Lagrangian for a neutral scalar field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi)\ , \quad \text{with the potential } \ U(\phi) = g\phi^3 + \lambda\phi^4\ ,$$

and $\lambda \geq 0$ for stability of the vacuum. The momentum conjugate to the field is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t}\ ,$$

and the Hamiltonian is

$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\ .$$

There is no conserved charge.
The first step in evaluating the partition function is to return to the discretized version

$$\begin{aligned} Z \ &= \ \lim_{N\to\infty}\left(\Pi_{i=1}^N \int_{-\infty}^{\infty}\frac{d\pi_i}{2\pi}\int_{\text{periodic}} d\phi_i\right) \\ &\quad \exp\left\{\sum_{j=1}^N \int d^3x \left[i\pi_j(\phi_{j+1}-\phi_j) - \Delta\tau\left(\frac{1}{2}\pi_j^2 + \frac{1}{2}(\nabla\phi_j)^2 + \frac{1}{2}m^2\phi_j^2 + U(\phi_j)\right)\right]\right\}\ . \end{aligned}$$

The momentum integrations can be performed since they are just products of Gaussian integrals.

EXAMPLE: NEUTRAL SCALAR FIELD

Divide position space into M^3 little cubes with $V = L^3$, $L = aM$, $a \to 0$, $M \to \infty$, M an integer. For convenience, and to ensure that Z remains explicitely dimensionless at each step in the calculation, we write $\pi_j = A_j/(a^3\,\Delta\tau)^{1/2}$ and integrate A_j from $-\infty$ to $+\infty$. For each cube we obtain

$$\int_{-\infty}^{\infty} \frac{dA_j}{2\pi} \quad \exp\left[-\frac{1}{2}A_j^2 + i\left(\frac{a^3}{\Delta\tau}\right)^{1/2}(\phi_{j+1}-\phi_j)A_j\right] = (2\pi)^{-1/2}\exp\left[\frac{-a^3(\phi_{j+1}-\phi_j)^2}{2\Delta\tau}\right] .$$

Thus far we have

$$Z \;=\; \lim_{M,N\to\infty}\int\left[\Pi_{i=1}^{N}\frac{d\phi_i}{\sqrt{2\pi M^3}}\right]\exp\left\{\Delta\tau\sum_{j=1}^{N}\int d^3x\left[-\frac{1}{2}\left(\frac{(\phi_{j+1}-\phi_j)}{\Delta\tau}\right)^2-\frac{1}{2}(\nabla\phi_j)^2-\frac{1}{2}m^2\phi_j^2-U(\phi_j)\right]\right\}$$

Returning to the continuum limit, we obtain

$$Z = N'\int_{\text{periodic}}\mathcal{D}\phi\exp\left(\int_0^\beta d\tau\int d^3x\mathcal{L}\right) .$$

The Lagrangian is expressed as a functional of ϕ and its first derivatives.
Z is expressed as a functional integral over ϕ of the exponential of the action in imaginary time.
The normalization constant is irrelevant, since multiplication of Z by any constant does not change the thermodynamics.

Example: Neutral Scalar Field

Consider the case of noninteracting fields $U(\phi) = 0$. Interactions we discuss later. Define

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L} = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2 \right] .$$

Integrating by parts and taking note of the periodicity of ϕ, we obtain

$$S = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \phi \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 \right) \phi .$$

The field can be decomposed into a Fourier series according to

$$\phi(\vec{x},\tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_n(\vec{p}) ,$$

where $\omega_n = 2\pi n T$, due to the constraint of periodicity that $\phi(\vec{x},\beta) = \phi(\vec{x},0)$ for all $\vec{x}$.

$$S = -\frac{1}{2}\beta^2 \sum_n \sum_{\vec{p}} (\omega_n^2 + \omega^2)\phi_n(\vec{p})\phi_n^*(\vec{p}) \ , \ \ \omega = \sqrt{\vec{p}^2 + m^2}.$$

The phases can be integrated out to get

$$Z \ = \ N'\Pi_n\Pi_{\vec{p}} \left[\int_{-\infty}^{\infty} dA_n(\vec{p}) \exp\left[-\frac{1}{2}\beta^2(\omega_n^2+\omega^2)A_n^2(\vec{p}) \right] \right] = N'\Pi_n\Pi_{\vec{p}} \left[2\pi/(\beta^2(\omega_n^2+\omega^2)) \right]^{1/2} .$$

Example: Neutral Scalar Field

Ignoring an overall multiplicative factor independent of β and V,

$$Z = \Pi_n \Pi_{\vec{p}} \left[\beta^2(\omega_n^2 + \omega^2)\right]^{-1/2} \ .$$

More formally one can arrive at this result by using the general rules for Gaussian functional integrals over commuting (bosonic) variables,

$$Z = N' \int \mathcal{D}\phi \exp\left[-\frac{1}{2}(\phi, D\phi)\right] = N' \mathrm{constant}(\det \ D)^{-1/2} \ ,$$

where $D = \beta^2(\omega_n^2 + \omega^2)$ in $(\vec{p}, \omega_n)$ space and $(\phi, D\phi)$ the inner product on function space.

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right] \ .$$

$$\mathrm{Trick:} \quad \ln\left[(2\pi n)^2 + \beta^2\omega^2)\right] \ = \int_1^{\beta^2\omega^2} \frac{d\Theta^2}{\Theta^2 + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right] \ ,$$

The $\beta-$ independent term can be ignored. Furthermore,

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + (\Theta/2\pi)^2} = \frac{2\pi^2}{\Theta}\left(1 + \frac{2}{e^{\Theta} - 1}\right) \ ,$$

EXAMPLE: NEUTRAL SCALAR FIELD

Hence we arrive at

$$\ln Z = -\sum_{\vec{p}} \int_1^{\beta\omega} d\Theta \left(\frac{1}{2} + \frac{1}{e^{\Theta} - 1} \right) .$$

Carrying out the Θ integral, and throwing away a $\beta-$ independent piece, we finally arrive at

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[-\frac{1}{2}\beta\omega - \ln(1 - e^{-\beta\omega}) \right] ,$$

from which we obtain immediately the well-known expression for the ideal Bose gas ($\mu = 0$), once we subtract the divergent expressions for the zero-point energy

$$E_0 = -\frac{\partial}{\partial\beta} \ln Z_0 = V \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{2} ,$$

and for the zero-point pressure

$$P_0 = T \frac{\partial}{\partial V} \ln Z_0 = -\frac{E_0}{V} ,$$

which are typical for the quantum field-theoretical treatment. With this subtraction the vacuum is defined as the state with zero energy and pressure.

Partition Function for Fermionic Fields

Dirac fermions are descibed by a four-spinor field ψ with a Lagrangian density

$$\mathcal{L} \;=\; \bar{\psi}(i\partial\!\!\!/ - m)\psi = \psi^\dagger\gamma^0\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi\;.$$

The momentum conjugate to this field is

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^\dagger\;,$$

because $\gamma^0\gamma^0 = 1$. Thus, ψ and $\psi^\dagger$ must be treated as independent entities in the Hamiltonian formulation.

$$\mathcal{H} = \Pi\frac{\partial\psi}{\partial t} - \mathcal{L} = \psi^\dagger\left(i\frac{\partial}{\partial t}\right)\psi - \mathcal{L} = \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi\;,$$

and the partition function is

$$Z = \mathrm{Tr}\; \mathrm{e}^{-\beta(\hat{H}-\mu\hat{Q})}\;,$$

with the conserved charge $Q = \int d^3x\psi^\dagger\psi$. The path integral representation is

$$Z = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi\exp\left[\int_0^\beta d\tau\int d^3x\psi^\dagger\left(-\gamma^0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla} - m + \mu\gamma^0\right)\psi\right]$$

Partition Function for Fermionic Fields

As with bosons, it is most convenient to work in $(\vec{p}, \omega_n)$ space instead of $(\vec{x}, \tau)$ space, i.e.,

$$\psi_\alpha(\vec{x}, \tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n \tau)} \tilde{\psi}_{\alpha;n}(\vec{p}) ,$$

where now $\omega_n = (2n+1)\pi T$ due to the antiperiodicity of the (Grassmannian) Fermion field at the borders of the fundamental strip $0 \leq \tau \leq \beta$ in the imaginary time, $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$.
Now we are ready to evaluate the fermionic partition function (2),

$$\begin{aligned} Z &= \left[\Pi_n \Pi_{\vec{p}} \Pi_\alpha \int i d\tilde{\psi}^\dagger_{\alpha;n}(\vec{p}) d\tilde{\psi}_{\alpha;n}(\vec{p})\right] \mathrm{e}^S , \quad S = \sum_n \sum_{\vec{p}} i\psi^\dagger_{\alpha;n}(\vec{p}) D_{\alpha\rho} \psi_{\rho;n}(\vec{p}) , \\ D &= -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0\right] , \end{aligned} \tag{2}$$

using Grassmannian integration of Gaussian functional integrals, we obtain

$$Z = \det \ D .$$

Employing the identity $\ln \det D = \mathrm{Tr} \ln D$, and evaluating the determinant in Dirac space explicitly (Exercise !), one finds

$$\ln Z = 2 \sum_n \sum_{\vec{p}} \ln \left\{ \beta^2 \left[(\omega_n + i\mu)^2 + \omega^2 \right] \right\} .$$

Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] = \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\}$

Partition Function for Fermionic Fields

Since both positive and negative frequencies have to be summed over, the latter expression can be put in a form analogous to the above expression in the bosonic case,

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln \left[\beta^2 \left(\omega_n^2 + (\omega - \mu)^2 \right) \right] + \ln \left[\beta^2 \left(\omega_n^2 + (\omega + \mu)^2 \right) \right] \right\} \ .$$

In the further evaluation we can go similar steps as in the bosonic case, with two exceptions: (1) the presence of a chemical potential, splitting the contributions of particles and antiparticles; (2) the Matsubara frequencies are now odd multiples of πT, so that the infinite sum to be exploited reads

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2 \pi^2 + \Theta^2} = \frac{1}{\Theta} \left(\frac{1}{2} - \frac{1}{\mathrm{e}^{\Theta} + 1} \right) \ .$$

Integrating over the auxiliary variable Θ, and dropping terms independent of β and μ, we finally obtain

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega + \ln(1 + \mathrm{e}^{-\beta(\omega - \mu)}) + \ln(1 + \mathrm{e}^{-\beta(\omega + \mu)}) \right] \ .$$

Notice that the factor 2 corresponding to the spin-$\frac{1}{2}$ nature of the fermions comes out automatically. Separate contributions from particles (μ) and antiparticles (-μ) are evident. Finally, the zero-point energy of the vacuum also appears in this formula.

INTERACTIONS: HUBBARD-STRATONOVICH TRICK

A general class of interactions for which the Hubbard-Stratonovich (HS) transformation is immediately applicable, are four-fermion couplings of the current-current type

$$\mathcal{L}_{int} = G(\bar{\psi}\psi)^2 \; . \tag{3}$$

A Fermi gas with this typ of interaction serves as a model for electronic superconductivity (Bardeen-Cooper-Schrieffer (BCS) model, 1957) or for chiral symmetry breaking in quark matter (Nambu–Jona-Lasinio (NJL) model, 1961).

The HS-transformation for (3) reads

$$\exp\left[G(\bar{\psi}\psi)^2\right] = \mathcal{N}\int \mathcal{D}\sigma \exp\left[\frac{\sigma^2}{4G} + \bar{\psi}\psi\sigma\right]$$

and allows to bring the functional integral over fermionic fields into a quadratic (Gaussian) form so that fermions can be integrated out.

This is also called *Bosonization* procedure.

Bose-Einstein Condensation: Charged Scalar Field

Consider a complex scalar field (two real components ϕ_1, ϕ_2):
$\Phi = (\phi_1 + i\phi_2)/\sqrt{2}\ ,\quad \Phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda(\Phi^*\Phi)^2\ ,$$

with U(1) symmetry: $\Phi \longrightarrow \Phi \mathrm{e}^{-i\alpha}$, where α is a real constant.
Noether theorem: continuous symmetry $\rightarrow$ conserved current

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}' &= \partial_\mu(\Phi^* \mathrm{e}^{i\alpha(x)})(\partial^\mu \Phi \mathrm{e}^{-i\alpha(x)}) - m^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2\ , \\ &= \mathcal{L} + \Phi^*\Phi\partial_\mu\alpha\partial^\mu\alpha + i\partial_\mu\alpha(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*) \end{aligned}$$

Equation of motion for the "field" $\alpha(x)$

$$\partial^\mu \frac{\partial \mathcal{L}'}{\partial(\partial^\mu \alpha)} = \frac{\partial \mathcal{L}'}{\partial \alpha}$$

Since $\partial\mathcal{L}'/\partial\alpha = 0$ follows a conserved "current": $\partial\mathcal{L}'/\partial(\partial^\mu\alpha) = \Phi^*\Phi\partial_\mu\alpha - i\Phi\partial_\mu\Phi^* + i\Phi^*\partial_\mu\Phi$.
Recover original field theory by setting $\alpha =$constant. Then

$$j_\mu = i(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*)\ ,\quad \partial^\mu j_\mu = 0$$

Full current: $J_\mu = \int d^3x j_\mu(x)$; conserved charge: $Q = \int d^3x j_0(x)$

Bose-Einstein Condensation: Charged Scalar Field (2)

Decompose the complex $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ into real and imaginary parts: ϕ_1, ϕ_2.
Conjugate momenta: $\pi_1 = \partial\phi_1/\partial t$, $\pi_2 = \partial\phi_2/\partial t$
Hamiltonian density and charge:

$$\mathcal{H} = \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\right] + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2 \,, \quad Q = \int d^3x(\phi_2\pi_1 - \phi_1\pi_2)$$

The partition function is

$$Z = \int [d\pi_1][d\pi_2]\int_{\text{periodic}}[d\phi_1][d\phi_2]\exp\left[\int^\beta d^4x\left(i\pi_1\frac{\partial\phi_1}{\partial\tau} + i\pi_2\frac{\partial\phi_2}{\partial\tau} - \mathcal{H}(\pi_1,\pi_2,\phi_1,\phi_2) + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\right]$$

Integration over conjugate field momenta can be done with the result:

$$\begin{aligned} Z \;=\; (N')^2\int_{\text{periodic}}[d\phi_1][d\phi_2]\exp\Big\{\int^\beta d^4x\Big[-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 \\ -\frac{1}{2}(\nabla\phi_1)^2 - \frac{1}{2}(\nabla\phi_2)^2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2\Big]\Big\} \,. \end{aligned}$$

Differs from naïve expectation $\mathcal{L}(\phi_1,\phi_2,\partial_\mu\phi_1,\partial_\mu\phi_2;\mu = 0) + \mu j_0(\phi_1,\phi_2,\partial\phi_1/\partial\tau,\partial\phi_2/\partial\tau)$ by $\mu^2\Phi^*\Phi$

Bose-Einstein Condensation: Charged Scalar Field (3)

In the following: ideal gas ($\lambda = 0$). For $\lambda \neq 0$, perform HS-transformation! (Exercise)
Expand components of $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ in Fourier series:

$$\phi_1(\vec{x},\tau) = \sqrt{2}\zeta\cos\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_{1;n}(\vec{p}) ,$$

$$\phi_2(\vec{x},\tau) = \sqrt{2}\zeta\sin\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_{2;n}(\vec{p}) .$$

Infrared character of Φ carried by ζ and θ, independent of $(\vec{x},\tau)$, so $\phi_{1;0}(\vec{0}) = \phi_{2;0}(\vec{0}) = 0$. Possibility of condensation of bosons into the zero-momentum state: finite fraction of particles in $n = 0, \vec{p} = \vec{0}$ state.

$$Z = (N')^2 \left[\Pi_n \Pi_{\vec{p}} \int i d\phi_{1;n}(\vec{p}) d\phi_{2;n}(\vec{p})\right] e^S ,$$

$$S = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\sum_n \sum_{\vec{p}} (\phi_{1;-n}(-\vec{p}), \phi_{2;-n}(-\vec{p})) D \begin{pmatrix} \phi_{1;n}(\vec{p}) \\ \phi_{2;n}(\vec{p}) \end{pmatrix} ,$$

$$D = \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix} .$$

Bose-Einstein Condensation: Charged Scalar Field (4)

Carrying out integrations yields:

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \ln(\det D)^{-1}$$

Second term can be handled as:

$$\begin{aligned}\ln\det D &= \ln\left\{\Pi_n\Pi_{\vec{p}}\beta^4\left[(\omega_n^2+\omega^2-\mu^2)^2+4\mu^2\omega_n^2\right]\right\}\\ &= \ln\left\{\Pi_n\Pi_{\vec{p}}\beta^2\left[\omega_n^2+(\omega-\mu)^2\right]\right\}+\ln\left\{\Pi_n\Pi_{\vec{p}}\beta^2\left[\omega_n^2+(\omega-\mu)^2\right]\right\}\end{aligned}$$

Putting all together and evaluating the Matsubara sums, we obtain

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - V\int\frac{d^3p}{(2\pi)^3}[\beta\omega + \ln(1-\mathrm{e}^{-\beta(\omega-\mu)}) + \ln(1-\mathrm{e}^{-\beta(\omega+\mu)})]$$

Result independent of θ because of $U(1)$ symmetry. Parameter ζ from variation of $\ln Z$.

$$\frac{\partial \ln Z}{\partial\zeta} = 2\beta V(\mu^2 - m^2)\zeta = 0$$

implies that $\zeta = 0$ unless $|\mu| = m$, when ζ is determined from $\rho = Q/V$

$$\rho = \frac{T}{V}\left(\frac{\partial \ln Z}{\partial\mu}\right)_{\mu=m} = 2m\zeta^2 + \rho^*(\beta,\mu=m)\ ;\quad \rho^* = \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{\mathrm{e}^{\beta(\omega-\mu)}-1} - \frac{1}{\mathrm{e}^{\beta(\omega+\mu)}-1}\right)$$

Critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.

Bose-Einstein Condensation: Exercises

1. Find the critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.
   Show that the nonrelativistic (NR, $\rho \ll m^3$) and ultrarelativistic (UR, $\rho \gg m^3$) limits are given by

$$T_{c,\mathrm{NR}} = \frac{2\pi}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3} \;\; ; \;\; T_{c,\mathrm{UR}} = \left(\frac{3\rho}{m}\right)^{1/2}$$

2. For $T < T_c$, the value of ζ is the order parameter of the 2nd order condensation phase transition

$$\zeta^2 = \left[\rho - \rho^*(\beta, \mu = m)\right]/(2m) \;\; , \;\; T < T_c \, .$$

   At the critical temperature, the expansion $\zeta \sim t^\nu$ for small $t = T - T_c$ determines a critical exponent ν. Find the value of ν in both cases.

3. Consider the interacting case $\lambda \neq 0$. Perform the Hubbard-Stratonovich transformation by introducing a collective scalar field σ. Discuss the effect of $\lambda \neq 0$ on the thermodynamic potential in the mean-field approximation for σ, i.e. by neglecting the path integral over the σ-fields and determining σ in the thermodynamical equilibrium from a gap equation $\partial \ln Z/\partial\sigma = 0$.

Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Solution: 1. Use explicit form of gamma matrices (and Pauli matrices)

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. Write down the determinant

$$\|\gamma_\mu p_\mu - m^*\| = \begin{vmatrix} (p_0 - m^*) & 0 & p_3 & (p_1 - ip_2) \\ 0 & (p_0 - m^*) & (p_1 + ip_2) & -p_3 \\ -p_3 & (-p_1 + ip_2) & (-p_0 - m^*) & 0 \\ (-p_1 - ip_2) & p_3 & 0 & (-p_0 - m^*) \end{vmatrix}$$

3. Determine the subdeterminants

$$\begin{aligned} D_{11} &= -(p_0 + m^*) \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{13} &= p_3 \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{14} &= -(p_1 + ip_2) \left(\vec{p}^2 + m^{*2} - p_0^2\right) \end{aligned}$$

4. Calculate the determinant according to standard rules

$$\begin{aligned} \|\gamma_\mu p_\mu - m^*\| &= (p_0 - m^*) D_{11} + p_3 D_{13} - (p_1 - ip_2) D_{14} \\ &= \left(-p_0^2 + p_1^2 + p_2^2 + p_3^2 + m^{*2}\right) \left(\vec{p}^2 + m^{*2} - p_0^2\right) \\ &= \left(\vec{p}^2 + m^{*2} - p_0^2\right)^2 \end{aligned}$$

5. Result:

$$= \left[\omega^2 + (\omega_n + i\mu)^2\right]^2, \qquad \omega^2 = \vec{p}^2 + m^{*2}$$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] = \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega-\mu)^2] + \ln \beta^2[\omega_n^2 + (\omega+\mu)^2]\right\}$

Solution: 1. Consider an analytic function $F(z_n)$ where $z_n = (i\omega_n - \mu)$, with $\omega_n = (2n+1)\pi T$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n + i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=-\infty}^{-1} F\left((\omega_n + i\mu)^2\right)$$

$$\sum_{n=-\infty}^{-1} F\left((\omega_n + i\mu)^2\right) = \sum_{n=1}^{\infty} F\left((\omega_{-n} + i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((\omega_{-n-1} + i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((-\omega_{-n-1} - i\mu)^2\right)$$

2. For the fermionic Matsubara frequencies holds $-\omega_{-n-1} = -\pi T(2(-n-1)+1) = \pi T(2n+1) = \omega_n$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n + i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=0}^{+\infty} F\left((\omega_n - i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n + i\mu)^2\right) + \sum_{n=0}^{+\infty} F^*\left((\omega_n + i\mu)^2\right)$$

$$= 2\sum_{n=0}^{+\infty} Re\, F\left((\omega_n + i\mu)^2\right)$$

3. Using this relationship based on the symmetry of the Matsubara frequencies, transform:

$$\begin{aligned}
2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] &= 4\sum_{n=0}^{+\infty} Re\, \ln \beta^2[(\omega^2 + \omega_n^2 - \mu^2) + i(2\omega_n\mu)] \\
&= 2\sum_{n=0}^{+\infty} \ln \beta^2[(\omega^2 + \omega_n^2 - \mu^2)^2 + (2\omega_n\mu)^2] \\
&= 2\sum_{n=0}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega-\mu)^2] + \ln \beta^2[\omega_n^2 + (\omega+\mu)^2]\right\} \\
&= \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega-\mu)^2] + \ln \beta^2[\omega_n^2 + (\omega+\mu)^2]\right\}
\end{aligned}$$