# IVC Calculation Problem for Josephson Junction Stacks. On Asymptotic Construction near the Breakpoint .

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A detailed investigation of the IVC breakpoint and the breakpoint region width gives important information [1],[2] concerning the peculiarities of stacks with a finite number of intrinsic Josephson junctions. In [3] IVC for a stack of $n$ Josephson junctions is defined numerically using the fourth- order Runge-Kutta method. The current voltage characteristic has the shape of a Hysteresis loop. On the back branch of the Hysteresis loop, near the breakpoint $I_b$, voltage $V(I)$ decreases to zero rapidly. In addition, in numerical modelling (non-periodic boundary condition) IVC branching is observed near $I_b$. It is interesting to study this phenomenon analytically developing asymptotic methods. A numerical-analytical method was proposed in [4]. The general scheme of suggested numerical-analytical method of the hysteresis loop calculation is following: the right branch of the hysteresis loop and the back branch (not nearing some finite distance to $I_b$) are calculated using the "asymptotic" formulas. The rest points $(I, V(I))$ of the hysteresis loop are calculated numerically using the fourth- order Runge-Kutta method. This method showed good results in IVC branching calculation in particular. I succeeded to calculate analytically the whole hysteresis loop in the case of periodic boundary conditions. The approximate solution at the breakpoint region had been developed using the Bogolyubov-Krylov method.

1. *Zappe H.H. Minimum current and related topics in Josephson tunnel junction devices//*
   Journal of Applied Physics, Vol.44, No.3, 1371-1377, 1973.

2. *Matsuda Y., Gaifullin M.B., Kumagai K., Kadowaki K. and Mochiku T. Collective Josephson Plasma Resonance in the Vortex State of $Bi_2Sr_2CaCu_2O_{8+\delta}$//*
   Vol.75, No.24,4512-4515,1995.

3. *Shukrinov Yu.M., Mahfousi F. and Pedersen N.F. Investigation of the Breakpoint Region in Stacks with a Finite Number of Intrinsic Josephson Junctions* //Phys.Rev. B 75,104508, 2007.

4. *Serdyukova S.I. Numerical-Analytical Method for Computing the Current-Voltage Characteristics for a Stack of Josephson Junctions*//Computational Mathematics and Mathematical Physics,2012, Vol.52, No.11, pp. 1590-1596.

5. *Н.Н.Боголюбов и Ю.А.Митропольский "Асимптотические методы в теории нелинейных колебаний", ФМ, Москва 1963, стр.48.*

Solving the system

$$\ddot{\phi}_l = \sum_{l'=1}^{n} A_{l,l'}(I - sin(\phi_{l'}) - \beta\dot{\phi}_{l'}),\ l = 1, ..., n, \tag{1}$$

for different $I : I = I_0 + k\Delta I \leq I_{max}$; $\quad I = I_{max} - k\Delta I$, the current-voltage characteristics of stacks as hysteresis loops are found [3]. For initial value of the current ($I = I_0$) the system (1) is solved with zero initial data on an interval $[0, T_{max}]$. For each next $I : I = I_{k+1}$, found $\phi_l(I_k, T_{max}), \dot{\phi}_l(I_k, T_{max})$ are used as initial data.

In the case of periodic boundary conditions $A$ matrix is

$$\begin{pmatrix} 1+2\alpha & -\alpha & 0 & ... & 0 & -\alpha \\ -\alpha & 1+2\alpha & -\alpha & 0 & ... & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & -\alpha & 1+2\alpha & -\alpha \\ -\alpha & 0 & ... & 0 & -\alpha & 1+2\alpha \end{pmatrix}, \tag{2}$$

square matrix of order n. The parameter $\alpha$ gives the coupling between junctions, $\beta$ is the dissipation parameter.

The dynamics of phase differences $\phi_l(t)$ had been simulated by solving the equation system (1) using the fourth order Runge-Kutta method. After simulation of the phase differences dynamics the voltages on each junction were calculated as

$$\partial\phi_l/\partial t = \sum_{l'=1}^{n} A_{l,l'}V_{l'}. \tag{3}$$

The total voltage $\mathbf{V}$ of the stack is obtained by

$$\mathbf{V} = \sum_{l=1}^{n} \bar{V}_l, \qquad \bar{V}_l = \frac{1}{T_{max} - T_{min}} \int_{T_{min}}^{T_{max}} V_l \partial t. \tag{4}$$

The calculation can be simplified using specific properties of the matrix $A$. This matrix has complete system of orthonormal eigenvectors $E_l$ with real eigenvalues $\lambda_l$ . The fundamental matrix $D$ (whose columns are $E_l$) reduces $A$-matrices to the diagonal form.

After changing the variables

$$\Phi = D\Psi, \quad \phi_l = \sum_{l'=1}^{n} d_{l,l'}\psi_{l'}, \qquad V = DW, \quad V_l = \sum_{l'=1}^{n} d_{l,l'}W_{l'}$$

we get a system:

$$\ddot{\psi}_l = -\lambda_l\beta\dot{\psi}_l + \lambda_l * I * S_l - \lambda_l \sum_{l'=1}^{n} d_{l',l} sin(\phi_{l'}),\ l = 1, ..., n$$

where $S_l$ is the sum of $E_l$ elements: $S_l = d_{1,l} + d_{2,l} + ... + d_{n,l}$.

Relations (3),(4) are transformed in

$$\partial\psi_l/\partial t = \lambda_l W_l, \qquad \bar{W}_l = \frac{\psi_l(T_{max}) - \psi_l(T_{min})}{\lambda_l(T_{max} - T_{min})} \tag{5}$$

respectively. As a result, we get the total voltage of the stack as

$$\mathbf{V} = \sum_{l=1}^{n} S_l * \bar{W}_l. \tag{6}$$

$$S_1 = \sqrt{n}, \qquad S_l = 0, \quad l = 2, ..., n.$$

As result calculating IVC for a stack of $n$ Josephson junctions of given zero initial data is reduced to solving a unique equation

$$\ddot{\eta}(t) = -\beta\dot{\eta}(t) + I - sin(\eta(t)).$$

Solution of this equation with initial data $\eta(0) = \xi_1, \ \dot{\eta}(0) = \xi_2$ . is equvalent to solution of an integral eauation

$$\eta(t) = \xi_1 + \frac{(\xi_2 - \omega)}{\beta}(1 - e^{-\beta t}) + \omega t - \frac{1}{\beta}\int_0^t (1 - e^{-\beta(t-s)}) sin(\eta(s))ds. \tag{7}$$

Solving this equation we find $\psi_1(t) = \sqrt{n}\eta(t)$. The rest components $\psi_j(t), \ j = 2, ..., n$ are equal zeros. The equation (7) is solved by the simple iterations method. Starting from $\eta_0 = 0$, we obtain $\eta_1(t) = \xi_1 + (\xi_2 - \omega)(1 - e^{-\beta t})/\beta + \omega t$,

$$\eta_2 = \omega t + A + \theta + \frac{sin(\omega t + A + arctg(\beta/\omega))}{\omega\sqrt{\beta^2 + \omega^2}} + O(\omega^{-3} + e^{-\beta t}), \tag{8}$$

Here $\omega = I/\beta, \quad A = \xi_1 + (\xi_2 - \omega)/\beta, \quad \theta = -cos(A)/(\omega\beta)$.

Remark that $V(I, n) = \sqrt{n}\bar{W}_1(I)$ (see (5),(6)) and

$$\bar{W}_1(I) = \sqrt{n}\frac{\eta(I, T_{max}) - \eta(I, T_{min})}{T_{max} - T_{min}}, \quad V(I, n) = n\frac{\eta(I, T_{max}) - \eta(I, T_{min})}{T_{max} - T_{min}}.$$

In Fig.1 the pictures of the back way of the hysteresis loop for 9 Josephson junctions are shown. The solid line refers to numerical calculation and the circles on this line refers to "asymptotic" (using (8)) calculation.

In Fig.2 the solid line is the same as in Fig.1, while the circles on this line refer to calculation performed by the following mixed analytical and numerical method. The right way of the hysteresis loop and the back way on the interval

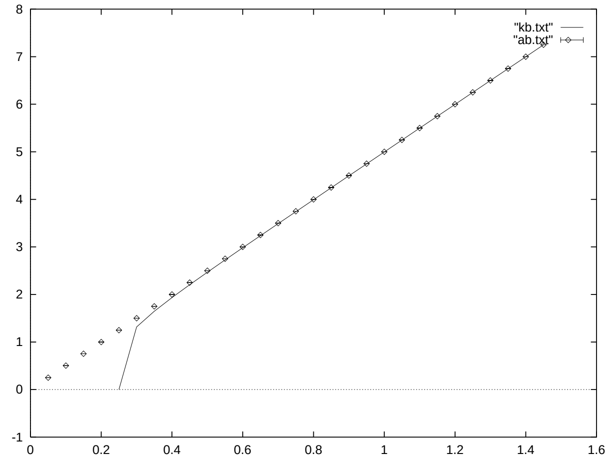


Fig.1,

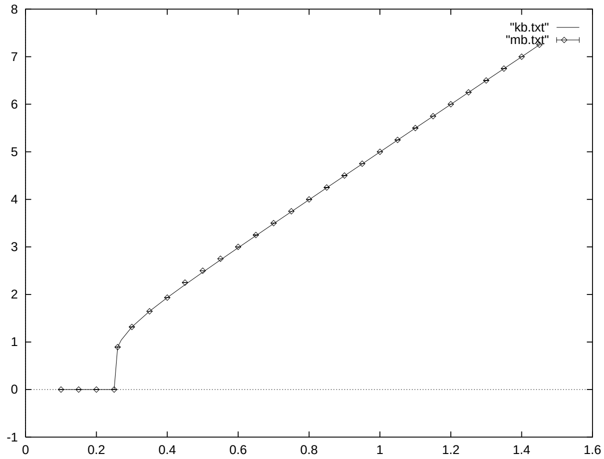


Fig.2

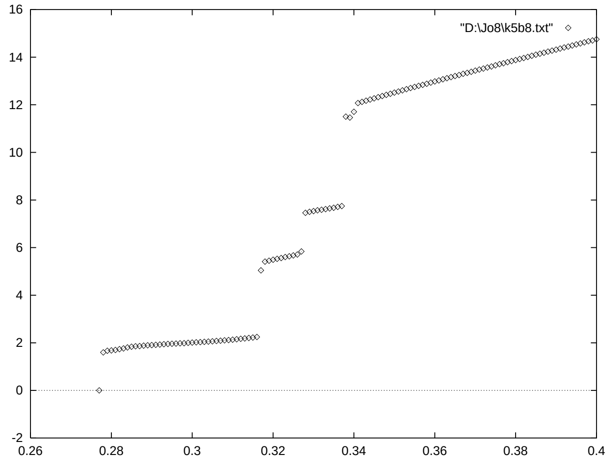


Fig.3

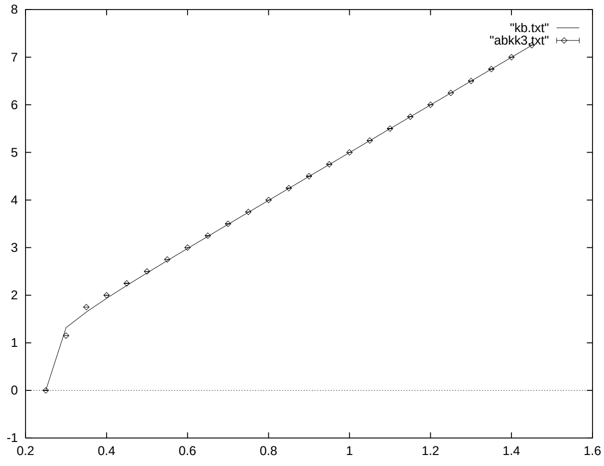


Fig.4

$0.45 \geq I < 1.45$ are computed using (8). The rest points of the hysteresis loop are computed numerically.

For studying branching IVC analytically we hope to develope "asymptotics"proper for all $I_j$. This moment a little step had been made.

We developed approximate solution for the equation

$$\ddot{\eta}(t) = -\beta\dot{\eta}(t) - \sin(\eta(t)) + I$$

with given initial data $\eta(0) = d_1,\ \dot{\eta}(0) = d_2$, in the case of small $\beta,\ I$, using the Bogolyubov-Krylov method.

We consider $\beta = 0.2,\ 0.2 < I < 0.4$. Let $\eta_0 = \arcsin(I)$,$I = sin(\eta_0)$,

$$sin(\eta) - I = \cos(\eta_0)(\eta - \eta_0) - \frac{\sin(\eta_0)}{2}(\eta - \eta_0)^2 - \frac{\cos(\eta_0)}{6}(\eta - \eta_0)^3 + ...$$

New variable $\phi = \eta - \eta_0$ satisfying equation $\ddot{\phi} = -\beta\dot{\phi} - \cos(\eta_0)\phi + \frac{\sin(\eta_0)}{2}\phi^2 + \frac{\cos(\eta_0)}{6}\phi^3$, which can be rewritten as

$$\ddot{\phi} + \beta\dot{\phi} + \omega^2(\phi - \frac{\phi^3}{6} - \mathrm{tg}(\eta_0)\frac{\phi^2}{2}) = 0, \qquad \ddot{\phi} + \omega^2\phi = \beta f(\phi, \dot{\phi}),$$

$\omega^2 = \cos(\eta_0) = (1 - I^2)^{1/4}$. This equation has solution

$$\phi = a\cos(\psi), \quad \dot{\phi} = -a\omega\sin(\psi), \quad \psi = \omega t + \theta,$$

where $a$ and $\psi$ are determined of the first approximation equations

$$\frac{da}{dt} = -\frac{\beta}{2}\ a, \qquad \frac{d\psi}{dt} = \omega\ (1 - \frac{a^2}{16}).$$

Following step by step to *Н.Н.Боголюбов и Ю.А. "Митропольский "Асимптотические методы в теории нелинейных колебаний ФМ, Москва 1963, стр.48.*

we obtain the approximate solution

$$\eta = \eta_0 + a_0\exp(-\beta t/2)\cos\left(\omega(t + \frac{a_0^2(\exp(-\beta t) - 1)}{16\beta}) + \theta\right). \qquad (9)$$

$a_0$ and $\theta$ are determined of given initial data:

$$d_1 = \eta_0 + a_0\cos(\theta),\ d_2 = -a_0\beta/2 - a_0\omega(1 - a_0^2/16)\sin(\theta)$$

On such way we could to correct Fig.1. In Fig.4 the back branches of the hysteresis loop for the case of periodic boundary conditions are presented.

The solid line refers to numerical calculation. The circless on this line were found analytically using mixed analytical method: all points of the hysteresis loop were calculated using

$$\eta_2 = \omega t + A + \theta + \frac{sin(\omega t + A + arctg(\beta/\omega))}{\omega\sqrt{\beta^2+\omega^2}} + O(\omega^{-3} + e^{-\beta t}), \tag{8}$$

excepting two points (I= 0.3, 0.25) calculated using

$$\eta = \eta_0 + a_0 \exp(-\beta t/2) \cos\left(\omega(t + \frac{a_0^2(\exp(-\beta t) - 1)}{16\beta}) + \theta\right). \tag{9}$$

**THANK YOU VERY MUCH FOR YOUR ATTENTION**