Generation and analysis of the second order difference scheme for the Korteveg-de Vries equation

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MMCP 2017, July 7, Dubna, Russia

Plan

Problem statement

Let ∂_x be the derivation operator and $\mathcal{R} := \mathbb{Q}(\alpha, \beta, \ldots)\{u\}$ be the ordinary differential polynomial ring over the field $\mathbb{Q}(\alpha, \beta, \ldots)$ of real constants (parameters).

Our goal is to apply the general algorithmic approach to generation of difference schemes (Gerdt, Blinkov,Mozzhilkin'2006) specified (Blinkov, Gerdt, Marinov'2017) to evolution equations of the form

$$u_t = (P + a\,u_{m-1})_x\ ,\ u_k := \partial_x^k u,\ P \in \mathcal{R},\ \mathrm{ord}_{\partial_x}(P) < m-1,\ a \in \mathbb{R},$$

to the Korteveg-de Vries equation (KdV)

$$u_t + (P + \beta u_{xx})_x = 0, \quad P = \frac{\alpha}{2}\,u^2, \quad \alpha, \beta \in \mathbb{R}$$

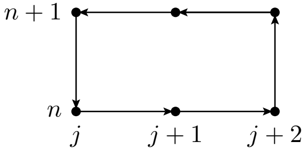
and investigate quality of the obtained scheme.

Discretization of KdV I

1. Choose the regular grid with $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$
2. Convert KdV into the integral conservation law form

$$u_t + (P + \beta u_{xx})_x = 0 \iff \oint_\Gamma -(P + \beta u_{xx})\,dt + u\,dx = 0$$

3. Select the rectangular integration contour



as a "control volume".

Discretization of KdV II

4. Add the (exact) integral relations

$$\int_{x_j}^{x_{j+2}} u_{xx}\, dx = u_x(t, x_{j+2}) - u_x(t, x_j)\,,$$
$$\int_{x_j}^{x_{j+1}} u_x\, dx = u(t, x_{j+1}) - u(t, x_j)\,.$$

5. Evaluate the contour integral numerically by the trapezoidal rule for integration over t and by the midpoint rule for integration over x.
6. Evaluate the integral relations numerically by the trapezoidal rule for the integration of u_x and by the midpoint rule for the integration of u_{xx}. This leads to the difference scheme which is outputted by the MAPLE code (Gerdt, Blinkov, Marinov'2017)

Maple code

```
> restart:
> with(LDA):
> L:=[(-(P(n,j)+P(n+1,j)-P(n,j+2)-P(n+1,j+2))-(beta*uxx(n,j)+beta*uxx(n+1,j)
  -beta*uxx(n,j+2)-beta*uxx(n+1,j+2)))*tau/2+(u(n+1,j+1)-u(n,j+1))*2*h,
  (ux(n,j+1)+ux(n,j))*h/2-(u(n,j+1)-u(n,j)),
   2*uxx(n,j+1)*h-(ux(n,j+2)-ux(n,j))]:
> JanetBasis(L,[n,j],[uxx,ux,u,F],2):
> collect(%[1,1]/(4*tau*h**3),[tau,h]);
```

$$\frac{\frac{1}{4}P(n+1,j+3)+\frac{1}{4}P(n,j+3)-\frac{1}{4}P(n+1,j+1)-\frac{1}{4}P(n,j+1)}{h}+\frac{1}{h^3}\Big(\frac{1}{4}\beta\,u(n+1,j+4)-\frac{1}{2}\beta\,u(n+1,j+3)+\frac{1}{4}\beta\,u(n,j+4)-\frac{1}{2}\beta\,u(n,j+3)+\frac{1}{2}\beta\,u(n+1,j+1)-\frac{1}{4}\beta\,u(n+1,j)+\frac{1}{2}\beta\,u(n,j+1)-\frac{1}{4}\beta\,u(n,j)\Big)+\frac{u(n+1,j+2)-u(n,j+2)}{\tau}$$

Computer algebra software used

To perform algebraically the difference elimination of the grid functions which correspond to partial derivatives of u, from the obtained discrete system we use the Maple package LDA (Linear Difference Algebra).

LDA created by D.Robertz (RWTH, Aachen) is freely available (http://wwwb.math.rwth-aachen.de/Janet/). It implements the involutive algorithm (Gerdt,Blinkov'98) specialized to difference ideals generated by linear difference polynomials.

Note that to apply LDA we "hide" the nonlinearity (caused by the presence of u^2 in the input difference equations) into the polynomial grid function $P_j^n := \alpha(u_j^n)^2/2$.

Difference scheme

In the conventional notations the obtained difference scheme reads

$$\tilde{f} = 0\,, \quad \text{where} \quad \tilde{f} := \frac{u_j^{n+1} - u_j^n}{\tau} + \frac{(P_{j+1}^{n+1} - P_{j-1}^{n+1}) + (P_{j+1}^{n} - P_{j-1}^{n})}{4h}$$

$$+ \frac{\beta(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + \beta(u_{j+2}^{n} - 2u_{j+1}^{n} + 2u_{j-1}^{n} - u_{j-2}^{n})}{4h^3}$$

Properties of the derived scheme I

Using the library SymPy (http://www.sympy.org/en/) written in Python we computed the modified equation of the scheme

$$
\begin{aligned}
&u_t + \alpha u u_1 + \beta u_3 + h^2 \left[\alpha \left(\frac{1}{6} u u_3 + \frac{1}{2} u_1 u_2 \right) + \frac{\beta}{4} u_5 \right] + \\
&+ \tau^2 \left[\alpha^3 \left(\frac{1}{12} u_3 + \frac{3}{4} u^2 u_1 u_2 + \frac{1}{2} u u_1^3 \right) + \alpha^2 \beta \left(\frac{1}{4} u^2 u_5 + \frac{7}{4} u u_1 u_4 + \right. \right. \\
&\quad \left. + \frac{11}{4} u u_2 u_3 + \frac{9}{4} u_1^2 u_3 + \frac{11}{4} u_1 u_2^2 \right) + \alpha \beta^2 \left(\frac{1}{4} u u_7 + u_1 u_6 + \right. \\
&\quad \left. \left. + \frac{9}{4} u_2 u_5 + \frac{7}{2} u_3 u_4 \right) + \frac{\beta^3}{12} u_9 \right] + \mathcal{O}(\tau^4, \tau^2 h^2, h^4)
\end{aligned}
$$

where

$$
u_k := u_{\underbrace{xx\cdots x}_{k \text{ times}}}, \quad k \geq 1 .
$$

Properties of the derived scheme II

1. The modified equation shows that the scheme has the 2-nd order in τ and in h.
2. This also implies that the scheme is (strongly) consistent.
3. The scheme is implicit, and hence it is unconditionally stable.
4. Because of universally adopted condition for convergency of difference schemes (rigorously proved for linear Cauchy problem - the Lax(-Richtmyer) equivalence theorem):

   $$\text{convergence} = \text{consistency} + \text{stability}$$

   the obtained scheme is convergent.

Other schemes with $\mathcal{O}(\tau^2, h^2)$ approximation

Scheme I. The explicit scheme (Belashov,Vladimirov'05, Eq.1.80)

$$u_i^{n+1} = u_i^{n-1} - \frac{\alpha\tau}{h} u_i^n \left(u_{i+1}^n - u_{i-1}^n\right) - \frac{\beta\tau}{h^3}\left(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n\right)$$

stable for $\tau \le \dfrac{2h^3}{3\sqrt{3}\beta} \cong 0.384\dfrac{h^3}{\beta}$. The modified equation

$$u_t + \alpha u u_1 + \beta u_3 + h^2\left(\frac{\alpha}{6} u u_3 + \frac{\beta}{4} u_5\right) -$$

$$- \tau^2\Big[\alpha^3\left(\frac{1}{6}u^3 u_3 + \frac{3}{2}u^2 u_1 u_2 + u u_1^3\right) + \alpha^2\beta\Big(\frac{1}{2}u^2 u_5 + \frac{7}{2}u u_1 u_4 +$$

$$+ \frac{11}{2}u u_2 u_3 + \frac{9}{2}u_1^2 u_3 + \frac{11}{2}u_1 u_2^2\Big) + \alpha\beta^2\Big(\frac{1}{2}u u_7 + 2u_1 u_6 +$$

$$+ \frac{9}{2}u_2 u_5 + 7u_3 u_4\Big) + \frac{\beta^3}{6}u_9\Big] + \mathcal{O}(\tau^4, h^4)$$

Scheme II. The implicit scheme (Belashov,Vladimirov'05, Eq.1.96)

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{\alpha}{4h}\left[u_j^n\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)+u_j^{n+1}\left(u_{j+1}^n-u_{j-1}^n\right)\right]+ + \frac{\beta}{4h^3}\left(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1}+ + u_{j+2}^n-2u_j^{n+1}+2u_{j-1}^n-u_{j-2}^n\right)=0\,.$$

The modified equation

$$u_t+\alpha uu_1+\beta u_3+h^2\left(\frac{\alpha}{6}uu_3+\frac{\beta}{4}u_5\right)+ + \tau^2\left[\alpha^3\left(\frac{1}{12}u^3u_3+\frac{1}{4}u^2u_1u_2\right)+\alpha^2\beta\left(\frac{1}{4}u^2u_5+\frac{5}{4}uu_1u_4+ + \frac{9}{4}uu_2u_3+\frac{7}{4}u_1^2u_3+\frac{11}{4}u_1u_2^2\right)+\alpha\beta^2\left(\frac{1}{4}uu_7+u_1u_6+ + \frac{9}{4}u_2u_5+3u_3u_4\right)+\frac{\beta^3}{12}u_9\right]+\mathcal{O}(\tau^4,\tau^2h^2,h^4)$$

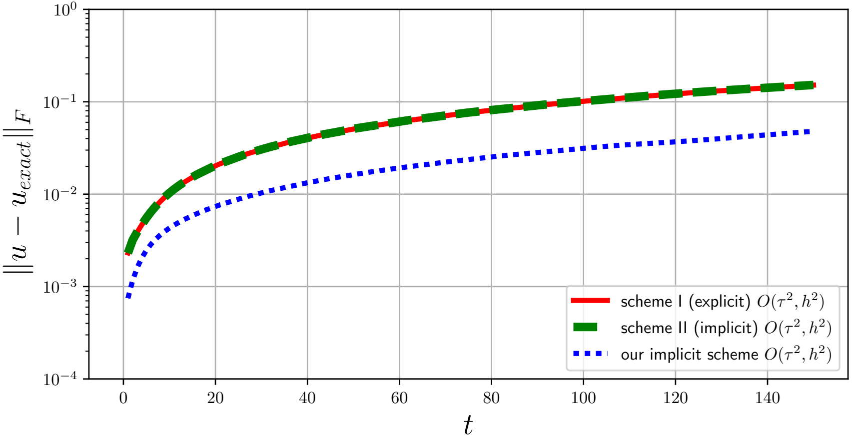
Computational experiment

Our numerical comparison of the above difference schemes was done with the Python package SciPy (http:\scipy.org). As a benchmark, we used the exact one-soliton solution

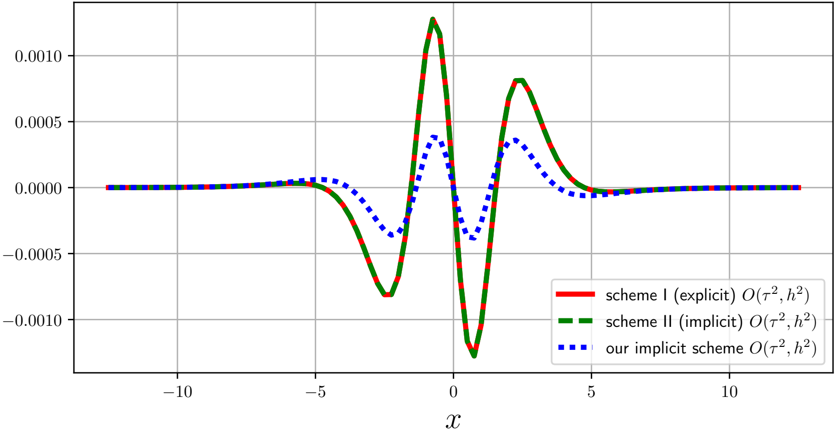
$$u_{\text{exact}}(t,x) = \frac{2k_1^2}{\cosh(k_1(x-4k_1^2t))^2}$$

to the KdV with $\alpha = 6$, $\beta = 1$ and $k_1 = 0.4$. In so doing, we fixed $h = 0.25$, $\tau = 0.37h^3\beta$ and considered the solution in interval $-50 \le x \le 50$ with periodic boundary conditions (cf. Belashov, Vladimirov'05, p.49). The numerical inaccuracy was estimated by the Frobenius norm.

Numerical discrepancy



The left-hand side of modified equation for $u_{\text{exact}}(0, x)$



Conclusions

- By applying algorithmic methods of difference algebra we generated implicit difference scheme for KdV with $\mathcal{O}(\tau^2, h^2)$ accuracy.
- The obtained scheme is consistent and stable.
- We compared, on the exact one-soliton solution, the numerical behavior of our scheme with two other schemes of the same accuracy known in the literature.
- Our scheme reveals numerical superiority.

References

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