Generalized Darcy's Law in Filtration Theory

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Introduction. Lattice approximation and continuous limit

The main difficulty in the filtration process seems to be the "near-wall effect", that is the anomalously large value of flow velocity near the wall due to more large value of the gap between the wall and the grains, the effectiveness of the filtration being decreasing. In order to take this effect into account, let us first consider the discrete variant of the mass conservation equation and number the lattice cells by the indices i, j (transverse to the flow direction) and k (along the flow), the corresponding Cartesian coordinates being x, y and z, respectively. Let us denote the local stream of the fluid by $G_{ijk} = S_0 u_{ijk}$, where u_{ijk} is the velocity of the flow and S_0 is the area of the gap between the grains, i. e. lattice spacing squared. Therefore, the conservation law reads

$$G_{ijk} = rG_{ijk-1} + p\left(G_{i-1jk-1} + G_{i+1jk-1}\right) + \\ + q\left(G_{ij-1k-1} + G_{ij+1k-1}\right), \quad (1)$$

where the branching coefficients r, p, q **are introduced and it is supposed that** $p = q$ **due to the** x, y **symmetry of the flow. Inserting (1) into the conservation equation**

$$\sum_{ij} G_{ijk} = \sum_{ij} G_{ijk-1}, \tag{2}$$

one immediately gets the following constraint on the branching coefficients:

$$r + 4p = 1. \tag{3}$$

Identifying now the lattice spacing with the size d **of the grain, it can be proved through (1), (2), (3) that in continuous limit the following differential equation holds:**

$$\mathbf{div}(r\vec{u}) = d\,\partial_x\left(p\,\partial_x u_z\right) + d\,\partial_y\left(p\,\partial_y u_z\right). \tag{4}$$

One can rewrite the equation (4) in the form of the stationary conservation law:

$$\mathbf{div}\vec{j} = 0, \tag{5}$$

where the components of the current $\vec{j}$ **in cylindrical coordinates** ρ, z **read:**

$$j_\rho = u_\rho - D\partial_\rho u_z, \quad j_z = r u_z, \tag{6}$$

and the transverse diffusion coefficient is introduced:

$$D = p\,d. \tag{7}$$

"Near-Wall" Effect: Velocity Profile for Two Geometries

Taking into account that $p = 0$ at the wall, one can easily deduce from (5), (6) and (7) the specific behavior of the fluid velocity near the wall. To find the velocity profile, let us first consider the flow in the cylindrical tube of the radius a. Suppose also that $u_\rho \ll u_z = u$ and $D \approx D_0 = const$ in the center of the tube $(\rho \to 0)$, but near the wall $(\rho \to a)$ one could expect the behavior of the form:

$$D = D_0(a - \rho)/d, \tag{8}$$

if $a - \rho \sim d$. Thus, the velocity $u(\rho, z)$ satisfies the equation

$$\partial_z(r\,u) - \frac{1}{\rho}\partial_\rho\left(D\,\rho\,\partial_\rho u\right) = 0, \tag{9}$$

where $D = D(\rho)$, $r = r(\rho)$. Let us solve the equation (9) by the separation of varibles:

$$u = \sum_{n=1}^{\infty} \exp(-\lambda_n z) R_n(\rho), \tag{10}$$

where $z > 0$ corresponds to the direction of

the flow. At $\rho \to 0$ **one easily finds**

$$R_n = R_n^- = C_n J_0(k_n \rho),\ k_n^2 = \lambda_n \frac{r_0}{D_0}, \qquad (11)$$

where $r_0 = r(0)$**,** $C_n =$ **const ,with** J_0 **standing for the Bessel function. However, at** $\rho \to a$ **one gets**

$$\partial_\rho \left[D_0 \rho (a - \rho) \partial_\rho R_n^+ \right] + d\lambda_n \rho r(\rho) R_n^+ = 0, \qquad (12)$$

where

$$r(\rho) = 1 - \frac{4D_0}{d^2}(a - \rho). \qquad (13)$$

In the first approximation one finds

$$R_n = R_n^+ \approx B_n \left(1 - \frac{dk_n^2}{r_0}(a - \rho) \right), \qquad B_n = \textbf{const}, \qquad (14)$$

where

$$B_n \approx C_n J_0(k_n a), \qquad J_1(k_n a) = 0. \qquad (15)$$

As follows from (10) and (15), the main contribution to the velocity profile is given by the first eigenfunction with $k_1 a \approx 3,8317$**.**

Let us now consider the radial flow, for which the components of the current $\vec{j}$ in cylindrical coordinates are given by the formulas similar to (6):

$$j_\rho = r u_\rho, \quad j_z = u_z - D\partial_z u_\rho, \tag{16}$$

with the stream conservation equation reading:

$$\partial_z \left(r u_z\right) + \frac{1}{\rho}\partial_\rho \left(r\rho u_\rho\right) - \partial_z \left(D\partial_z u_\rho\right) = 0. \tag{17}$$

If one supposes that $u_z \ll u_\rho = u$ and $D = D_0 =$ const at $z \to 0$, then at $z - l \sim d$ one can expect the form of $D(z)$ similar to (8):

$$D = D_0(l - z)/d, \tag{18}$$

with $u(\rho, z)$ satisfying the equation

$$\frac{1}{\rho}\partial_\rho \left(r(z)\rho u\right) - \partial_z \left(D(z)\partial_z u\right) = 0. \tag{19}$$

Solution to (19) can be found by the separation of variables:

$$u = \sum_{n=1}^{\infty} Z_n(z) \frac{1}{\rho} \exp(-\lambda_n \rho), \qquad (20)$$

where $\rho \geq \rho_0 > 0$**,** $z \in [-l, l]$**. If** $z \to 0$**, one easily finds that**

$$Z_n = Z_n^- = C_n \cos(k_n z), \quad k_n^2 = \lambda_n \frac{r_0}{D_0}. \qquad (21)$$

However, at $z \to l$ **one obtains the equation**

$$\left[D(z) Z_n'\right]' = -\lambda_n Z_n$$

with the following behavior of its solution:

$$Z_n = Z_n^+ = B_n \left[1 - \lambda_n \frac{d}{D_0}(l - z)\right]. \qquad (22)$$

The smooth matching of the functions Z_n^- **and** Z_n^+ **gives the conditions:**

$$\sin(k_n l) = 0, \quad B_n = (-1)^n C_n.$$

In particular, for the first mode ($n = 1$**) one gets** $k_1 = \pi/l$ **and** $B_1 = -C_1$**. Thus, such a behavior of the velocity profile again illustrates the "near-wall effect".**

Effectiveness of the Filtration Process

Finally, let us estimate the so-called filtration coefficient, which is given by the ratio of the impurity concentration n for the outgoing flow to that for the incoming one. Taking into account that n satisfies the balance equation of the form

$$\partial_t n + \mathbf{div}(n\,\vec{u}) = -\beta\,n, \tag{23}$$

where $\vec{u}$ stands for the velocity of the fluid and β — for the absorption coefficient of the porous medium, one gets for the stationary process of filtration the following relation:

$$\mathbf{div}(n\,\vec{u}) = -\beta\,n, \tag{24}$$

For the rude estimation of the solution to (24) one can neglect the contribution of the diffusion term to the current $\vec{j}$, since $D \sim d$ and $\lambda_n\,l \sim ld/a^2 \sim d/l \ll 1$ for both kinds of filters. Therefore, for the cylindrical filter one derives from (24), in supposition that $u = u_0 =$ const, the more simple relation:

$$\mathbf{div}(n\,\vec{u}) \approx \partial_z n\,u_0 = -\beta\,n, \tag{25}$$

whence

$$n(z) = n_0 \exp\left(-\frac{\beta z}{u_0}\right). \tag{26}$$

Taking into account that for the radial flow the velocity appears to be inverse proportional to the radial coordinate ρ:

$$u = u_0\, \rho_0/\rho,$$

one easily finds

$$\mathbf{div}(n\,\vec{u}) \approx \frac{1}{\rho}\partial_\rho(\rho n\, u) = \frac{u_0\,\rho_0}{\rho}\partial_\rho\, n = -\beta\, n,$$

whence

$$n(\rho) = n_0 \exp\left[\frac{\beta}{2u_0\,\rho_0}(\rho_0^2 - \rho^2)\right]. \tag{27}$$

Comparing the formulas (26) and (27) giving the expressions for the filtration coefficient, one can easily see that for $\rho \gg \rho_0$ the radial filter proves to be more effective than the cylindrical one.

To find the profiles of the velocity $\vec{u}$ and the pressure P, it is necessary to solve also the Euler equation

$$(\vec{u}\bigtriangledown)\vec{u} + \bigtriangledown P = \vec{f} = \vec{g} - k_D\vec{u}, \tag{28}$$

where the force density $\vec{f}$ includes the gravity acceleration $\vec{g}$ and the Darcy force $\vec{f}_D = -k_D \vec{u}$ In the simplest approximation the Darcy coefficient k_D appears to be constant: $k_D \approx k_0 =$ const, but in general it should be some function of the velocity and pressure. We suggest a generalization of the Darcys law by including in k_D the natural invariant $I = (\vec{u} \bigtriangledown P)$ in the simplest linear form:

$$k_D = k_0 + k^* I. \tag{29}$$

We analyze the dependence of the filtration process on the coefficient k^*.

If one supposes that $w \ll u$, one can admit the following substitution in the first approximation:

$$u = u_0 + A \exp(-\lambda z) J_0(\alpha\rho); \quad w = B \exp(-\lambda z) J_1(\alpha\rho), \tag{30}$$

where J_0, J_1 stand for the Bessel functions. Finally, one gets the following algebraic equations for the constant parameters A, B, λ, α:

$$J_1(\alpha a) = 0; \quad \lambda A = \alpha B + \alpha^2 A\, D_0; \tag{31}$$

$$(\lambda u_0 - k_0)(A\alpha - \lambda B) = \lambda B\, k^* u_0(\lambda u_0^2 - g); \tag{32}$$

$$\triangle P(1 + k^* u_0^2) = (g - k_0 u_0)l - A(u_0 - k_0/\lambda)[e^{-\lambda l} - 1], \tag{33}$$

where $\triangle P = P(z=0) - P(z=l)$**,** $z \in [0, l]$**.**

It is natural to use the following small parameters:

$$\epsilon = k^* u_0^2 \ll 1; \quad \mu = \alpha D_0 \ll 1. \tag{34}$$

In the linear approximation with respect to ϵ **and** μ **one gets from (31), (32), (33):**

$$\lambda \approx \alpha \left[1 + k^* u_0 \frac{g - \alpha u_0^2}{2(\alpha u_0 - k_0)} + \frac{\alpha}{2} D_0 \right], \tag{35}$$

where $\alpha a = 3,8317$ **is the first nontrivial root of** J_1**.**