Entanglement and quantum state transfer in spin chains with XY-Hamiltonian

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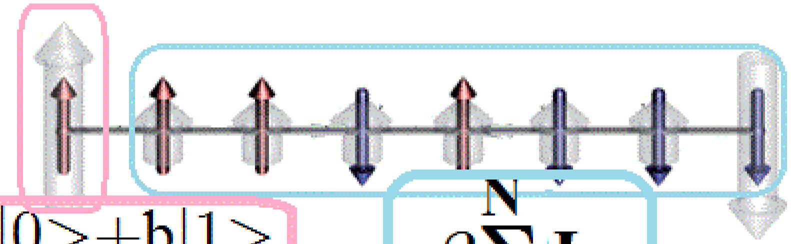
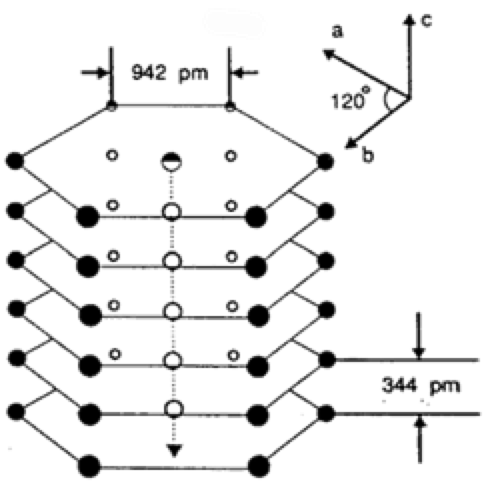
$$H=\sum_{j=1}^{N-1} d_{j,j+1}\left(I_j^+ I_{j+1}^- + I_j^- I_{j+1}^+\right),$$

$$d_{j,j+1}=d$$



$$a|0>+b|1>$$

$$\frac{e^{\beta\sum_{j=2}^{N} I_{j,z}}}{Z}$$



$$\rho(0)=\frac{\left(a|0\rangle+b|1\rangle\right)\left(a^*\langle 0|+b^*\langle 1|\right)\otimes \exp\left(\beta\sum_{j=2}^{N} I_{j,z}\right)}{Z}$$

$$\rho(\tau)=e^{-iH\tau}\rho(0)e^{iH\tau}$$

Density matrix of chain

$$\rho(t) =$$

$$\frac{Z}{2^N}\Big\{|b|^2 e^{\beta/2}\frac{e^{\beta I_z}}{\cosh^N\frac{\beta}{2}} + (|a|^2 e^{-\beta/2} - |b|^2 e^{\beta/2})$$

$$\times \sum_{j,j'} |f_N(t,j)|^2 \left(1+\tanh\frac{\beta}{2}\right)\left(1-\tanh\frac{\beta}{2}\right)^{\mathrm{sign}^2(j-j')} c^+_{j'}c_j$$

$$\times \prod_{l=1,l\neq j,j'}^N \left(\left(1+\tanh\frac{\beta}{2}\right)c_l^+c_l + \left(1-\tanh\frac{\beta}{2}\right)c_lc_l^+\right)$$

$$+ \left(\sum_{j=1}^N ab^*\left(1-\tanh\frac{\beta}{2}\right)e^{\beta/2}f_N^*(t,j)c_j^+\right.$$

$$\left.+\sum_{j=1}^N a^*b\left(1+\tanh\frac{\beta}{2}\right)e^{\beta/2}f_N(t,j)c_j\right)$$

$$\times \prod_{l=1,l\neq j}^N \left(\left(1+\tanh\frac{\beta}{2}\right)c_l^+c_l + \left(1-\tanh\frac{\beta}{2}\right)c_lc_l^+\right).\Big\}$$

$$f_N(t,j) = \frac{2}{N+1}\sum_k e^{i\varepsilon_k t}\sin k \sin jk,\ \ k=\frac{\pi}{N+1},\ \ \varepsilon_k = \cos k.$$

$$\rho_{1N}(\tau)=$$

$$\begin{pmatrix}
\frac{e^{\beta}}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)}+L\frac{|f|^2+|g|^2}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)} & -e^{\beta/2}\frac{ab^*}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}f^*\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} & e^{\beta/2}\frac{ab^*}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}g^* & 0 \\
-e^{\beta/2}\frac{a^*b}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}f\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} & \frac{1}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)}+L\frac{e^{-\beta}|g|^2-|f|^2}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)} & Le^{-\beta/2}\frac{g^*f}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} & e^{-\beta/2}\frac{ab^*}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}g^* \\
e^{\beta/2}\frac{a^*b}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}g & Le^{-\beta/2}\frac{f^*g}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} & \frac{1}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)}+L\frac{e^{-\beta}|f|^2-|g|^2}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)} & e^{-\beta/2}\frac{ab^*}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}f^*\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} \\
0 & e^{-\beta/2}\frac{a^*b}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}g & e^{-\beta/2}\frac{a^*b}{2\mathrm{ch}\left(\frac{\beta}{2}\right)}f\times\left(-\mathrm{th}\left(\frac{\beta}{2}\right)\right)^{N-2} & \frac{e^{-\beta}}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)}-Le^{-\beta}\frac{|f|^2+|g|^2}{4\mathrm{ch}^2\left(\frac{\beta}{2}\right)}
\end{pmatrix}$$

$$L=1-(1+e^{\beta})|b|^2$$

$$\rho_j(\tau) = \begin{pmatrix} e^{\beta}|b|^2 + \left(|a|^2 - e^{\beta}|b|^2\right)\left(\dfrac{1+\tanh\dfrac{\beta}{2}}{2}\right) + \dfrac{|f_N(t,j)|^2}{2}\left(1-\tanh\dfrac{\beta}{2}\right) & \left(1-\tanh\dfrac{\beta}{2}\right)^{N-1} ab^* f_N^*(t,N) \\ \left(1-\tanh\dfrac{\beta}{2}\right)^{N-1} a^*b\ f_N(t,N) & |b|^2 + \left(|a|^2 - e^{\beta}|b|^2\right)\left(\dfrac{1+\tanh\dfrac{\beta}{2}}{2}\right) - \dfrac{|f_N(t,j)|^2}{2}\left(1-\tanh\dfrac{\beta}{2}\right) \end{pmatrix}$$

Fidelity of transfer of quantum state in spin chain

For mixed states
$$F(\chi,\sigma)=\left(Tr\left(\sqrt{\sqrt{\chi}\sigma\sqrt{\chi}}\right)\right)^2$$

For pure initial state
$$F\left(|\psi\rangle\langle\psi|,\sigma\right)=Tr\left(|\psi\rangle\langle\psi|\sigma\right)=\langle\psi|\sigma|\psi\rangle=$$
$$=\left(a^*\langle 0|+b^*\langle 1|\right)\rho_j(\tau)\left(a|0\rangle+b|1\rangle\right),$$

$$|\psi\rangle=a|0\rangle+b|1\rangle,\quad \sigma=\rho_j(\tau).$$

Uhlmann, A., Rep. Math. Phys., 9, 273 (1976)
R.Jozsa, Journal of Modern Optics, 41, 2315-2323 (1994)

Fidelity for transfer of state between sender and receiver

$$F(\rho_1(0), \rho_N(t)) = \frac{e^\beta - e^\beta |b|^2 + |b|^2}{e^\beta + 1} + \frac{|f(t)|^2 \left(2e^\beta |b|^4 + 2|b|^4 - e^\beta |b|^2 - 3|b|^2 + 1\right)}{e^\beta + 1}$$

$$+ 2|a|^2 |b|^2 \operatorname{Re}\left(f_N(t, N)\right)\left(-\tanh\left(\frac{\beta}{2}\right)\right)^{n-1}$$

$$f_N(t, N) = \frac{2}{N+1} \sum_k \exp(i\tau \cos k) \sin(k) \sin(Nk),$$

$\beta -$ inverse temperature $\dfrac{\hbar\omega}{k_B T}$.

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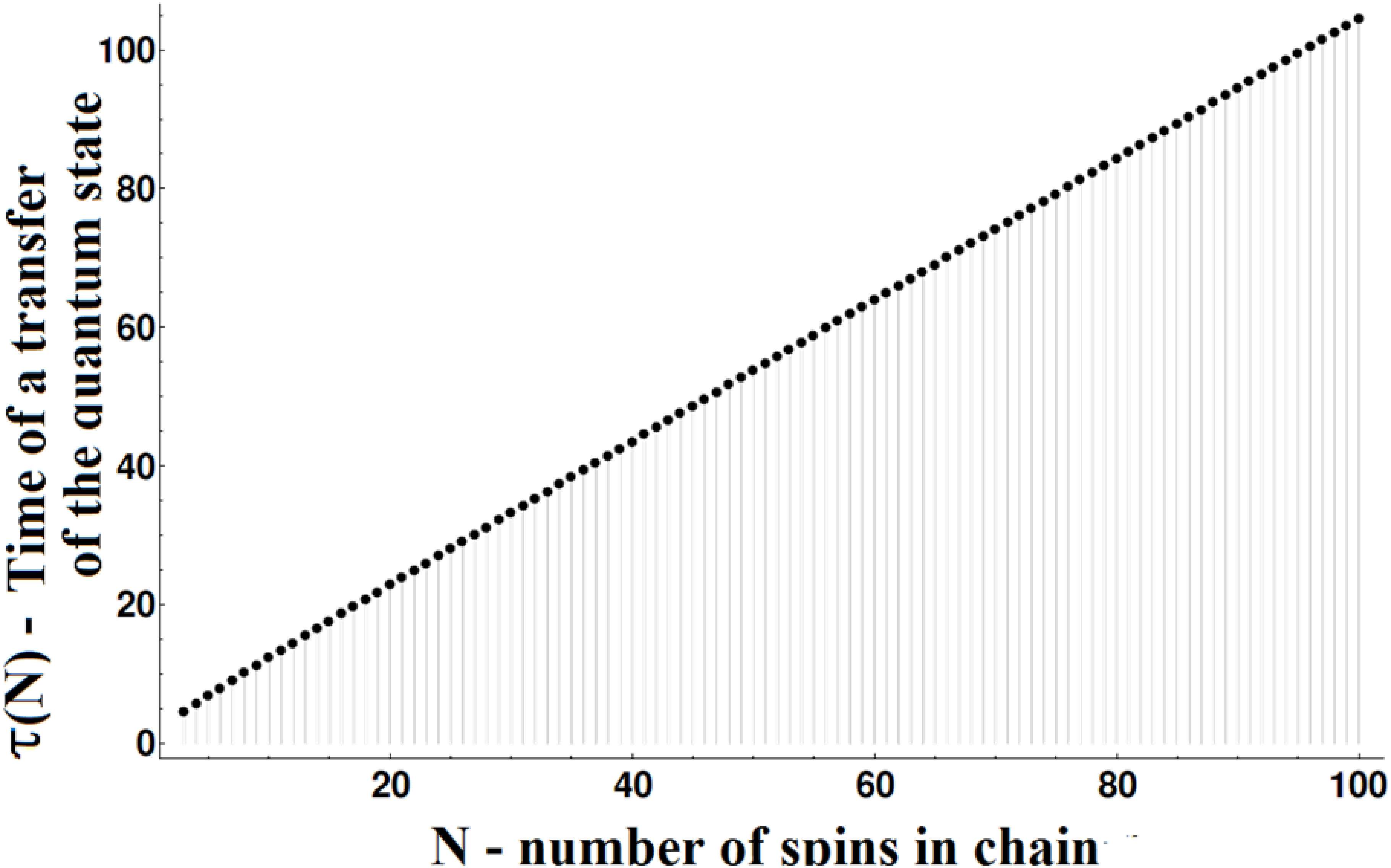
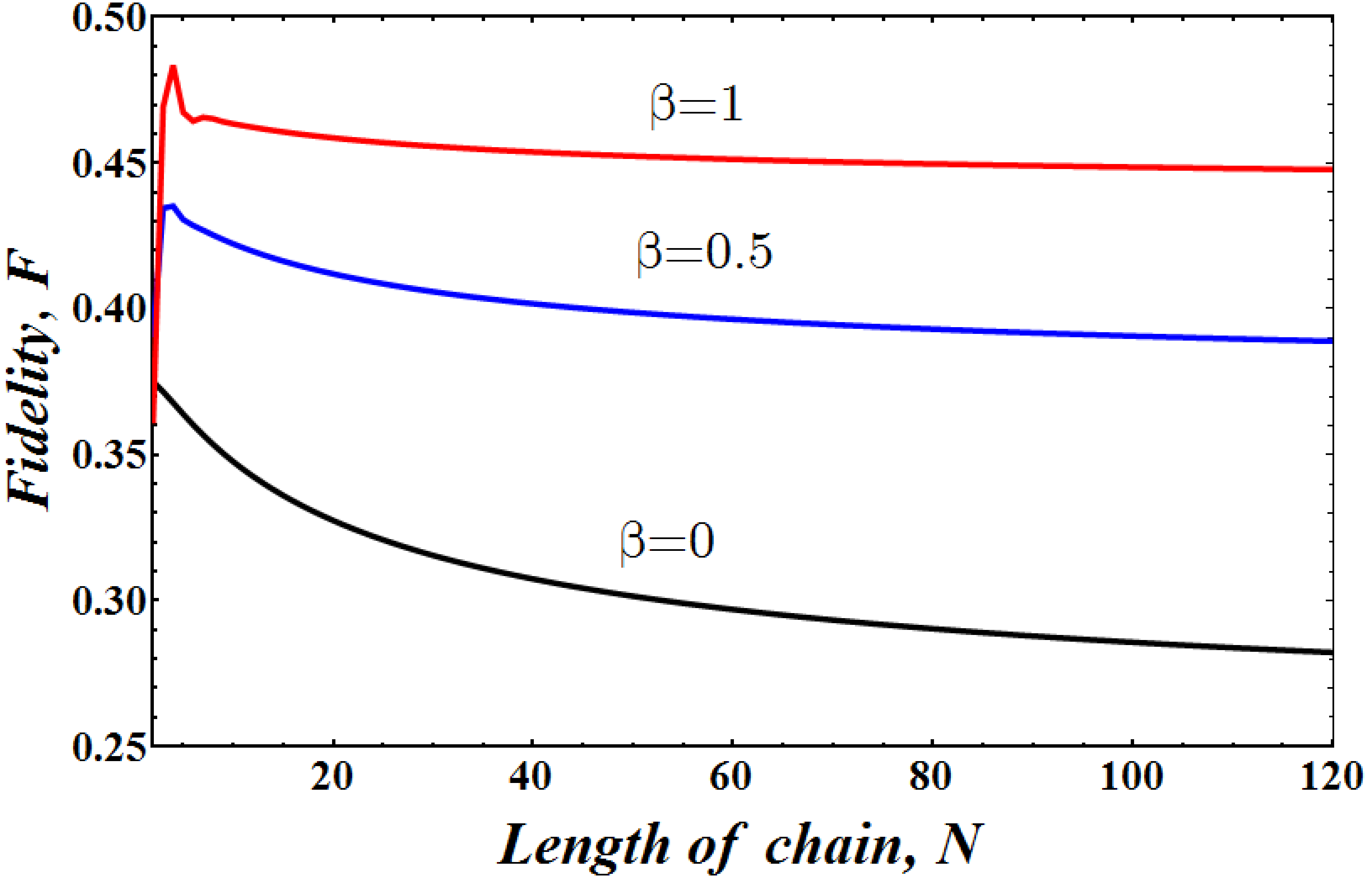
Transfer time of the quantum state





Fidelity, F
Length of chain, N
β=1
β=0.5
β=0
0.50
0.45
0.40
0.35
0.30
0.25
20
40
60
80
100
120

Entanglement

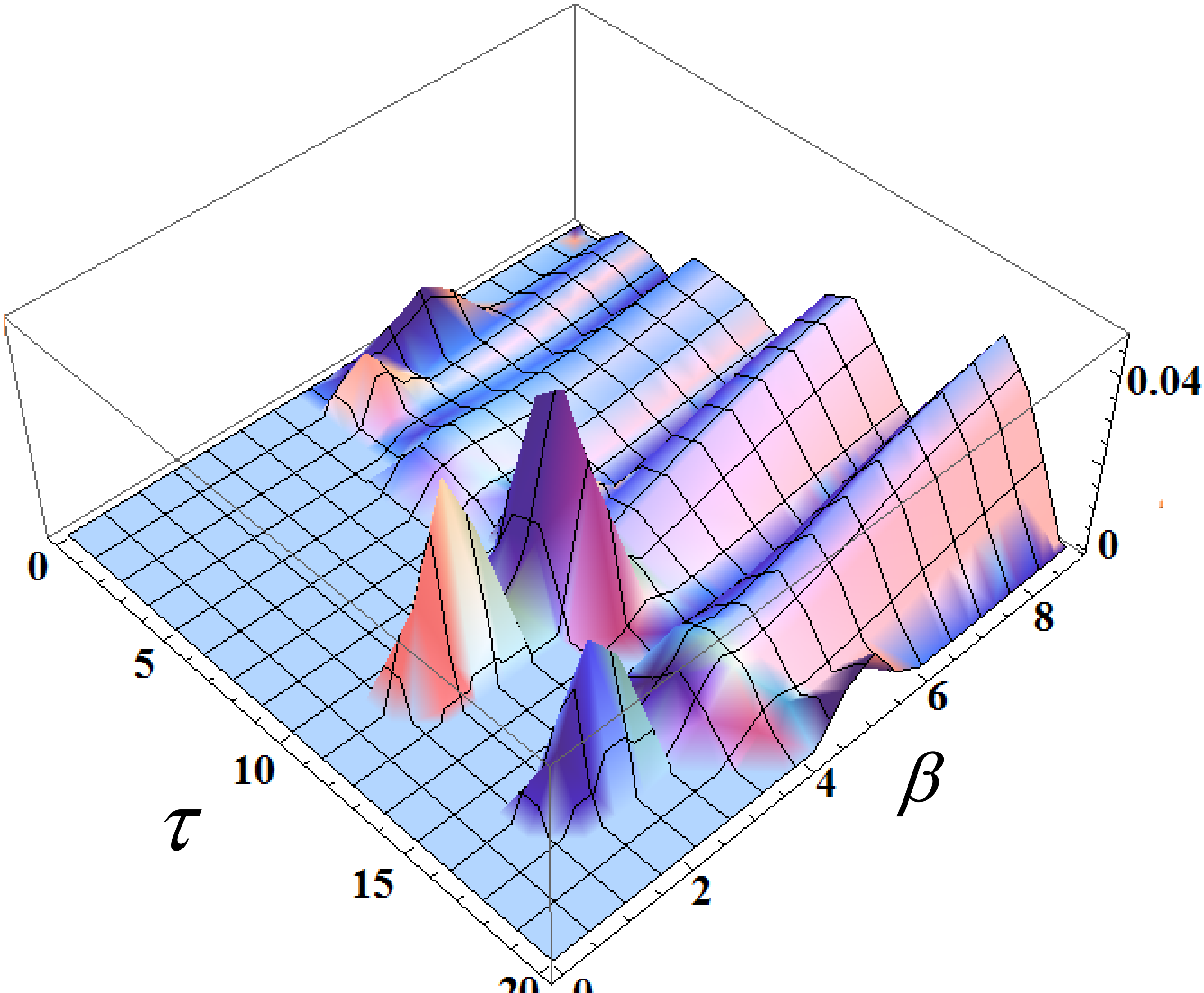
$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y);$$

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}};$$

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\};$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4.$$

William K. Wootters, Phys. Rev. Lett. 80, 2245 (1998)



N=4

$|b|=1/\sqrt{20}$

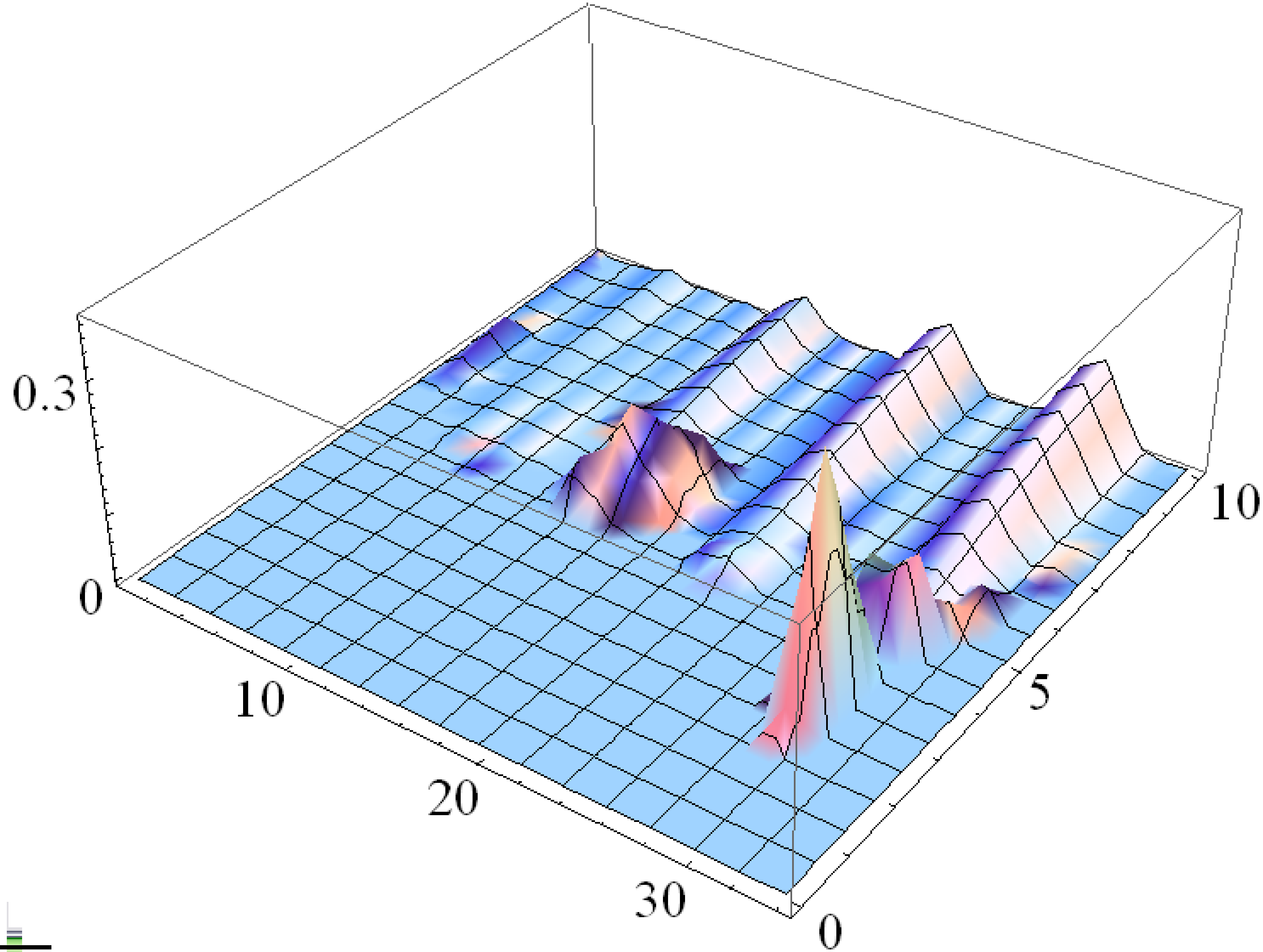
$(\rho_1(0) = a|0\rangle + b|1\rangle)$

Border for two regions

$\beta < 2\ln(|a/b|)$ – Entanglement arises at small parameter |b|

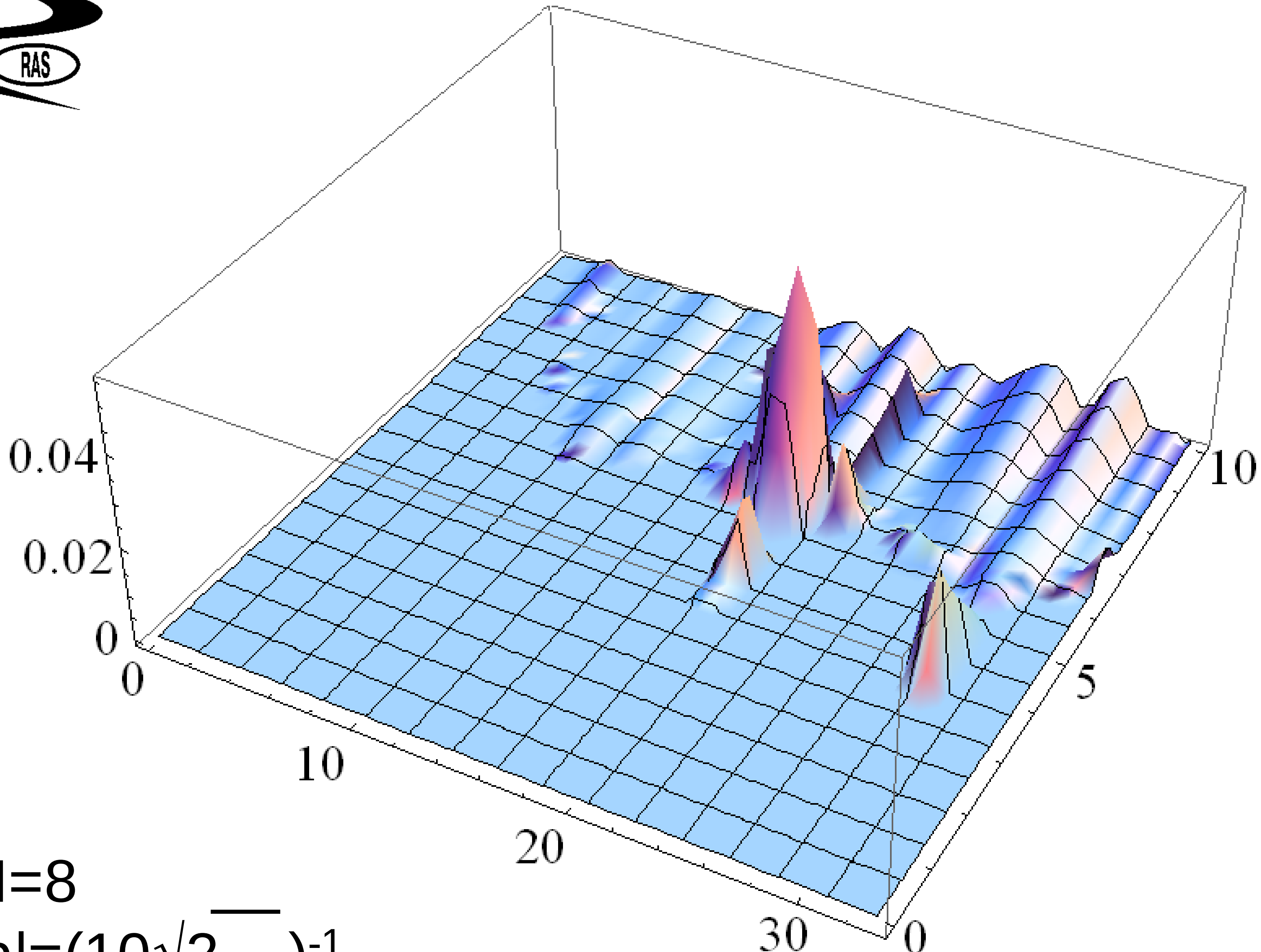
$\beta = 2\ln(|a/b|)$ – Entanglement between sender and receiver is zero.

$\beta > 2\ln(|a/b|)$ – Entanglement arises at any polarization



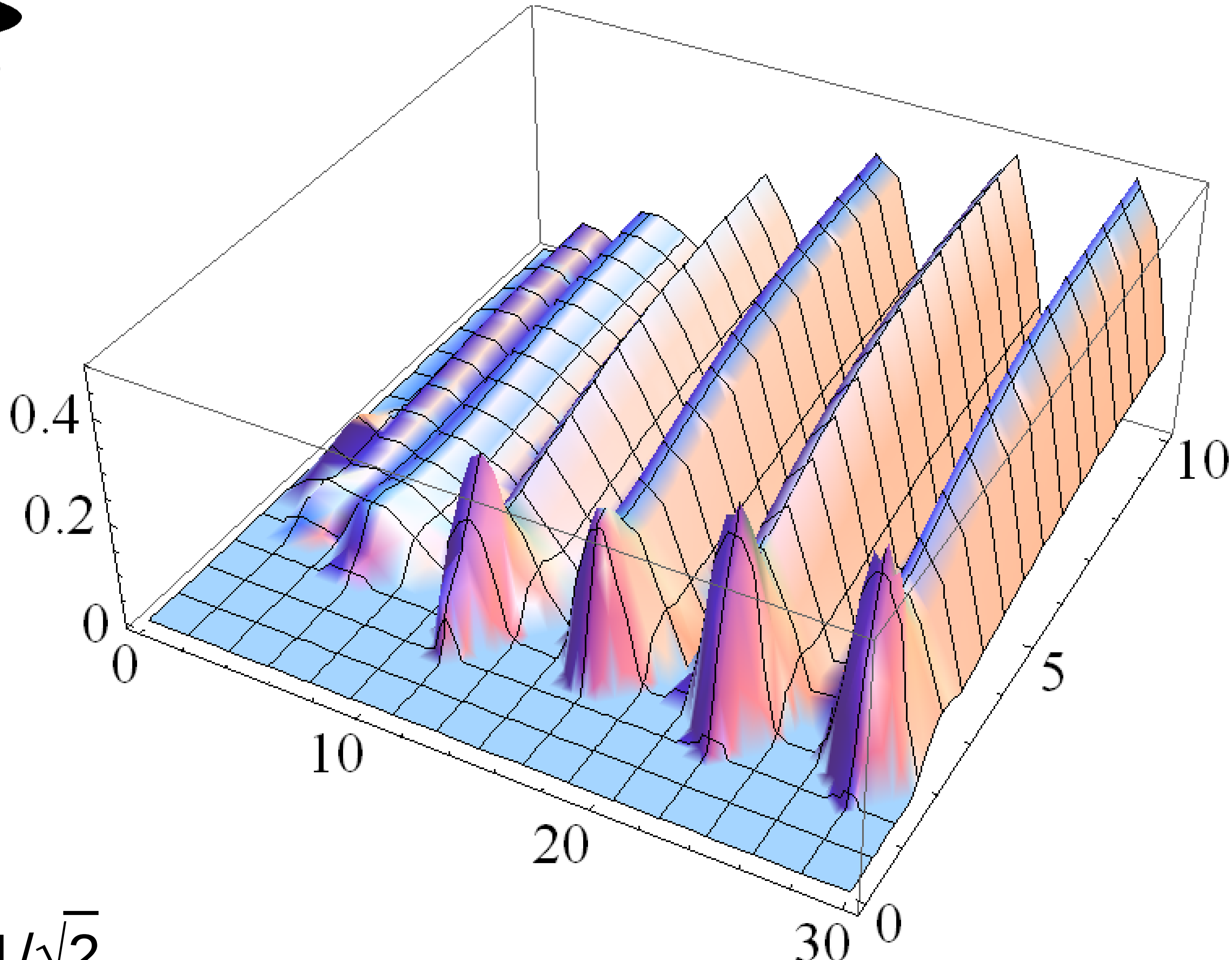
N=6

$|b|=1/\sqrt{50}$



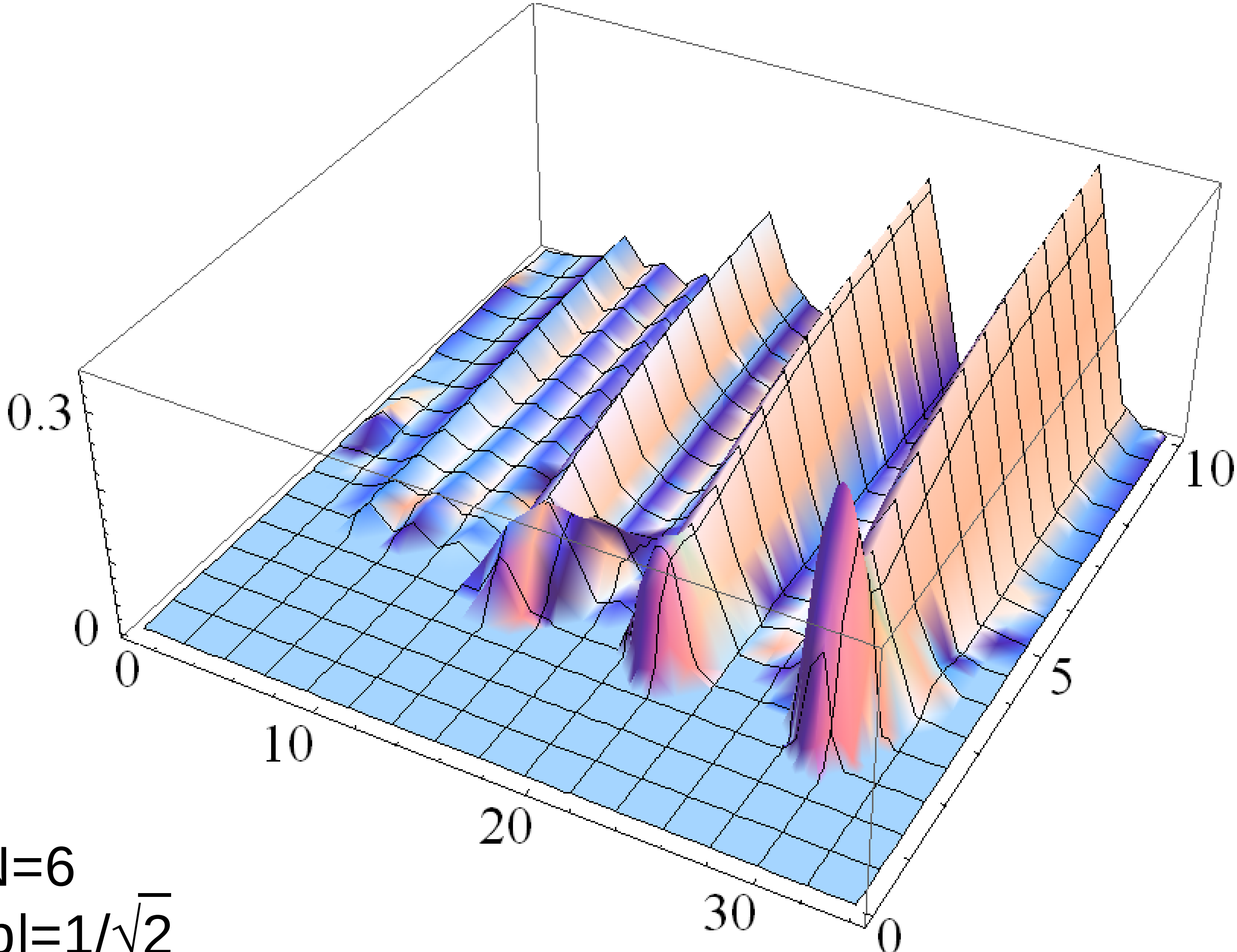
N=8

$|b|=(10\sqrt{2})^{-1}$



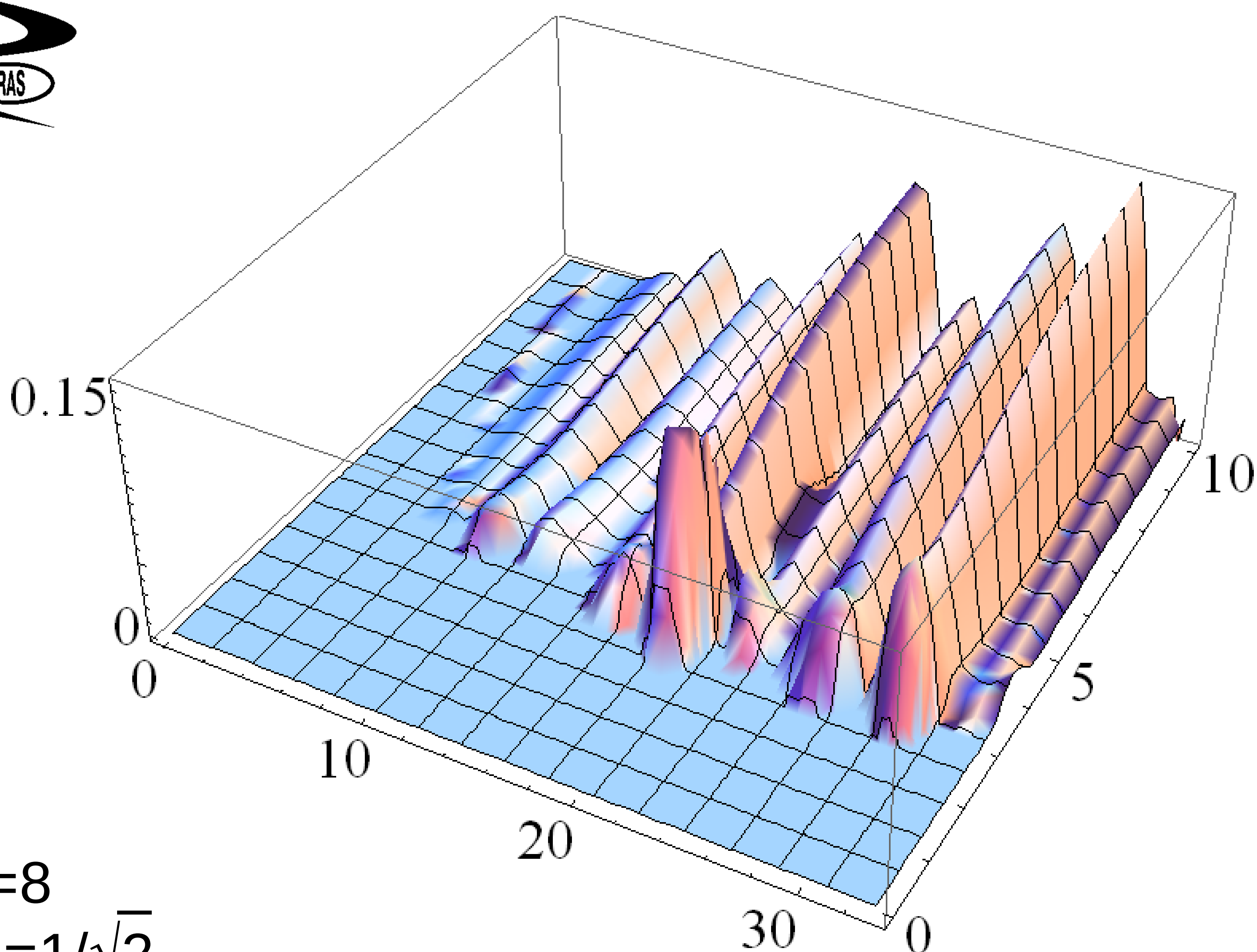
N=4

$|b|=1/\sqrt{2}$



N=6

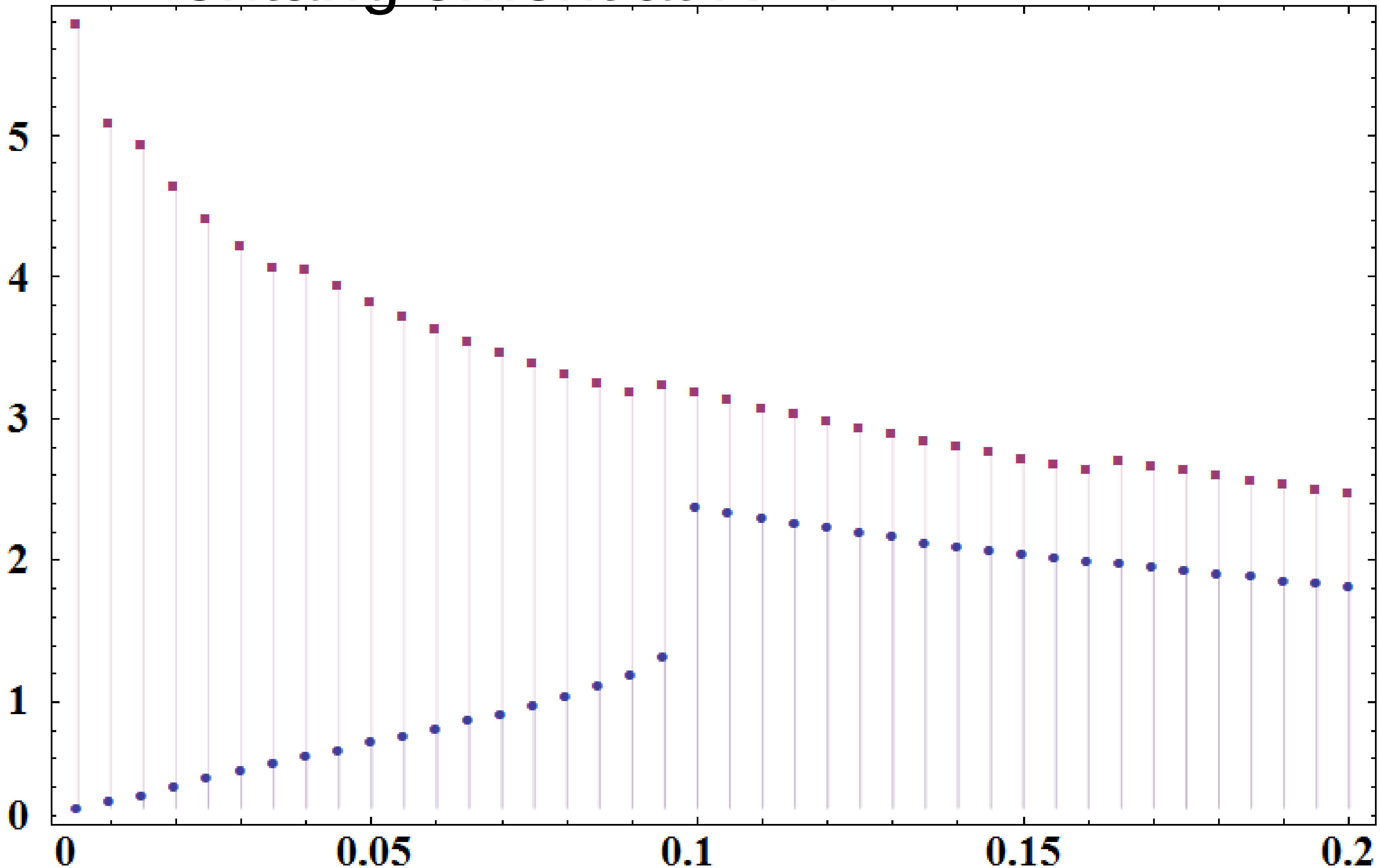
$|b|=1/\sqrt{2}$



N=8

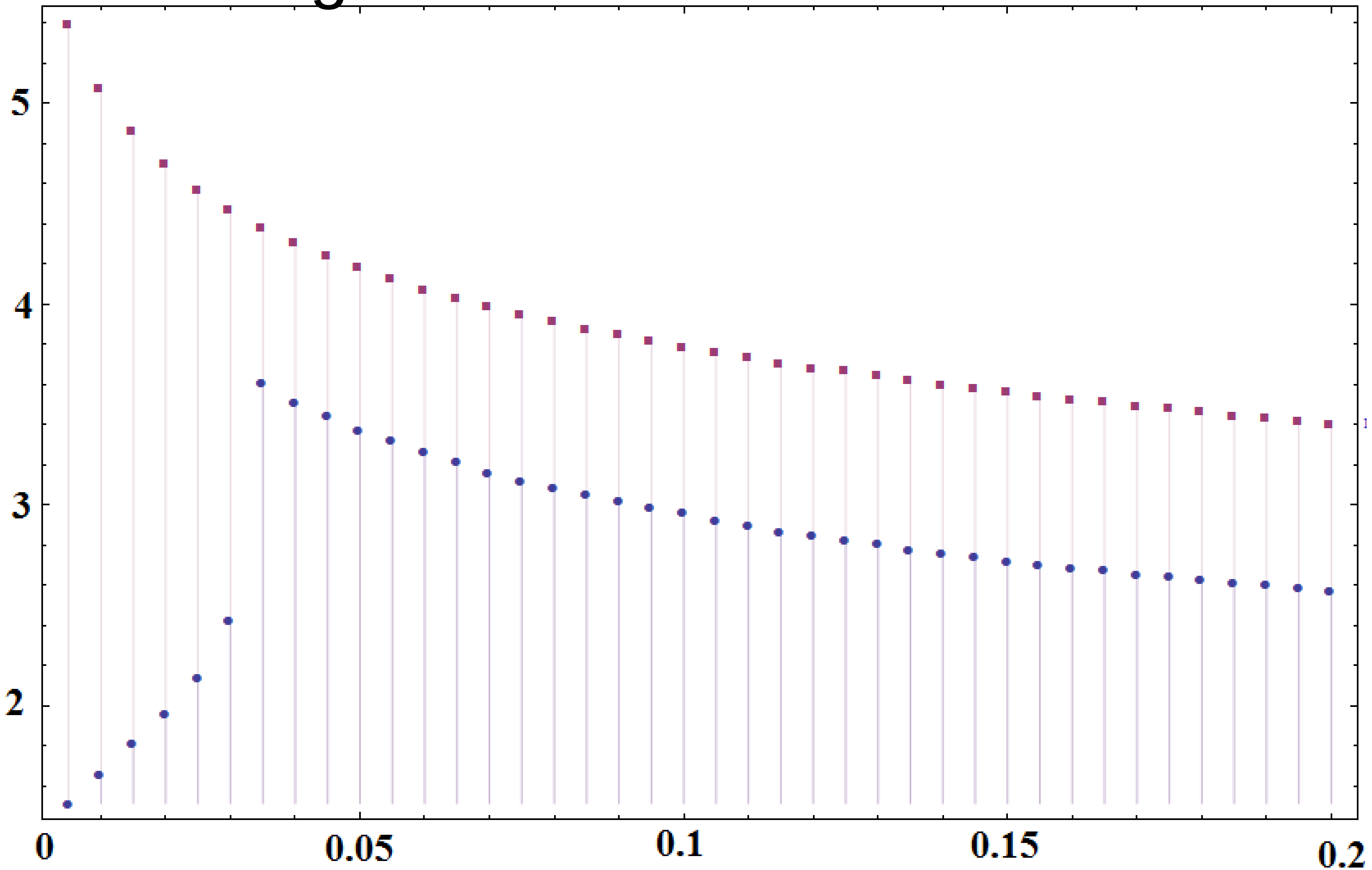
$|b|=1/\sqrt{2}$

Critical temperature for emergence of entanglement at N=4



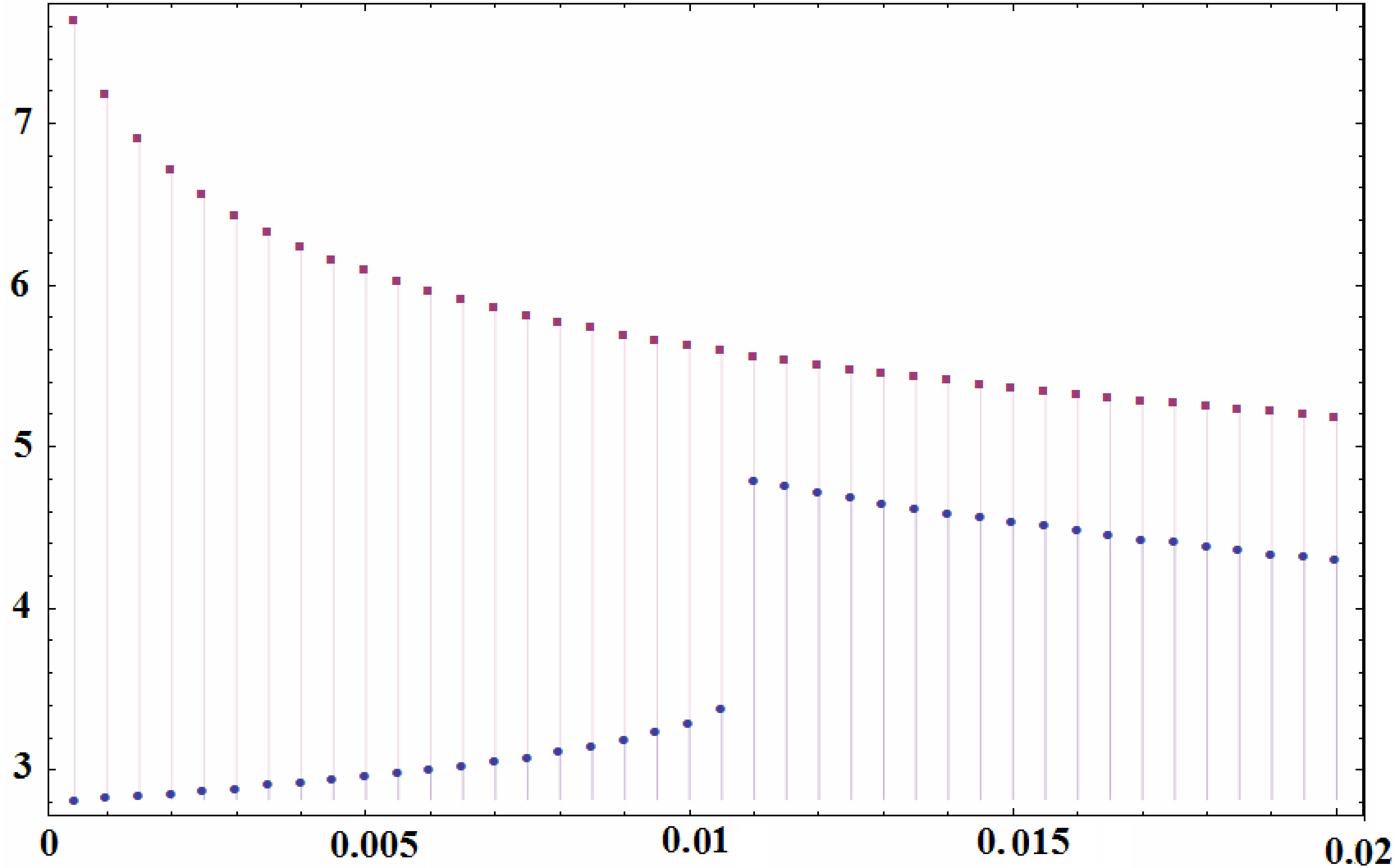
N=4

Critical temperature for emergence of entanglement at N=6



N=6

Critical temperature for emergence of entanglement at N=8

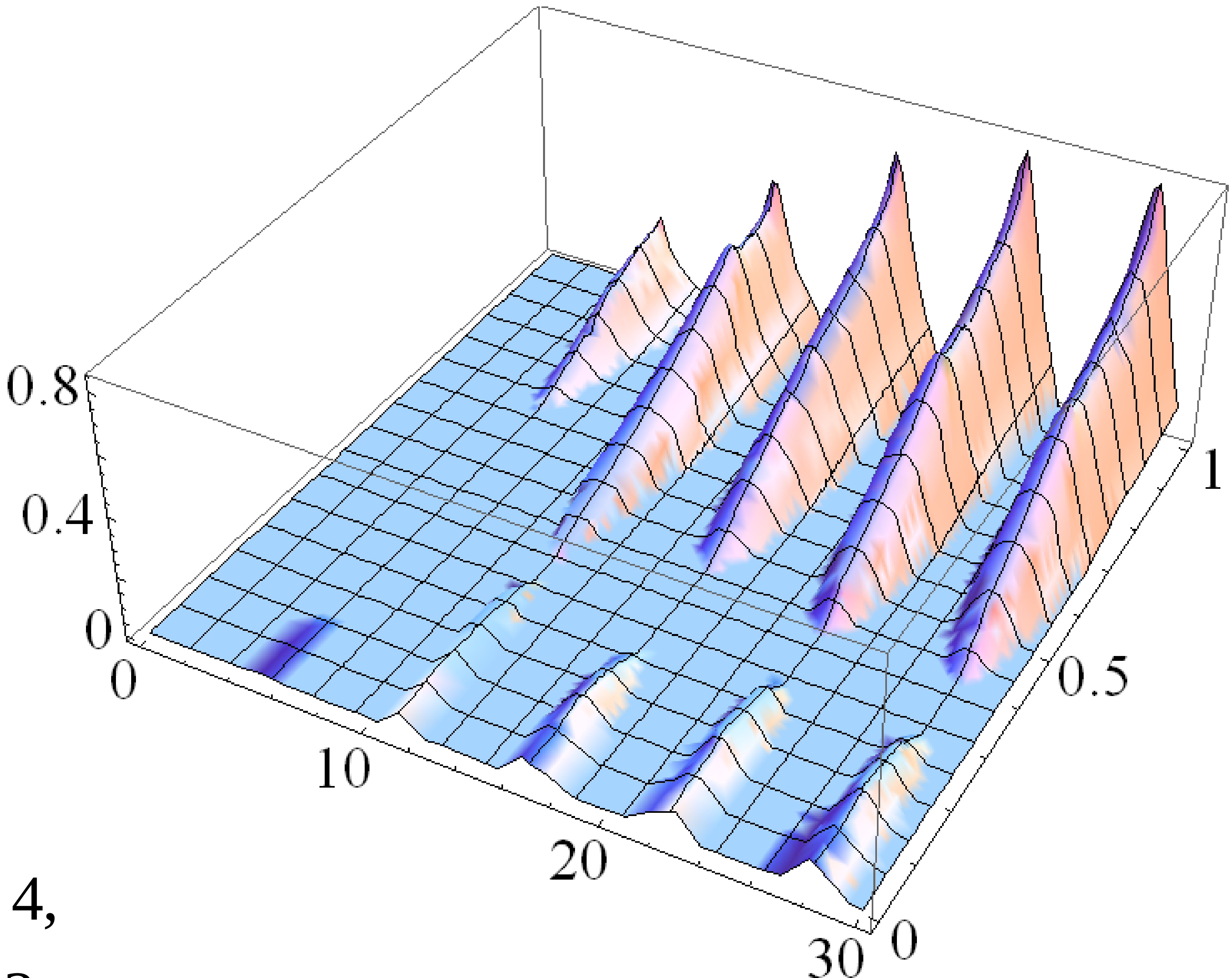


Conclusions

1. The analytical solution for evolution of the density matrix is obtained and the reduction is performed over all spins besides the first spin and an arbitrary second one.
2. A quantum state transfer is investigated in the one-dimensional spin chain.
3. It is shown that entanglement between the sender and the receiver emerges in the course of the system evolution.
4. Entanglement between the sender and the receiver is studied and conditions of appearing entanglement are found.

Thank you for attention!

Critical temperature for emergence of entanglement



$N = 4,$

$\beta = 2.$

$$\beta \approx 2\left(\frac{\chi + |b|}{|f\,g|}\right)^{1/(N-2)} .$$

Perturbation theory method

$$R^2 = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) = M_0(\tau, B, \beta) + \varepsilon(B) M_1(\tau, B, \beta)$$
$$\varepsilon^2(B) M_2(\tau, B, \beta)$$

$$\lambda_1(\tau, B, \beta) \geq \lambda_2(\tau, B, \beta) \geq \lambda_3(\tau, B, \beta) \geq \lambda_4(\tau, B, \beta),$$

$$\lambda_i(\tau, B, \beta) \approx \lambda_{0,i}(\tau, B, \beta) + \varepsilon^2 \lambda_{2,i}(\tau, B, \beta).$$

$$\varepsilon = |ab|, \quad \varepsilon \leq \frac{1}{2}$$

$$C = \mathrm{Max}(0,\ \lambda_1(\tau, B, \beta) - \lambda_2(\tau, B, \beta)$$
$$- \lambda_3(\tau, B, \beta) - \lambda_4(\tau, B, \beta)).$$

$$R^2 = M_0 + \varepsilon M_1 + \varepsilon^2 M_2,$$

$$M_0 = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & A_3 & 0 \\ 0 & A_4 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} c_1 & 0 & 0 & e^{\beta} c_2 \\ 0 & c_1 & c_3 & 0 \\ 0 & -c_3^* & c_1 & 0 \\ -e^{-\beta} c_2^* & 0 & 0 & c_1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0 & e^{\beta/2}\sigma_1 & e^{\beta/2}\sigma_2 & 0 \\ e^{-\beta/2}\sigma_3 & 0 & 0 & e^{\beta/2}\sigma_2 \\ e^{-\beta/2}\sigma_4 & 0 & 0 & e^{\beta/2}\sigma_1 \\ 0 & e^{-\beta/2}\sigma_4 & e^{-\beta/2}\sigma_3 & 0 \end{pmatrix}.$$

$$L = \frac{1}{\operatorname{ch}\frac{\beta}{2}} - e^{\beta}|b|^2 << \frac{1}{\varepsilon} \qquad \Rightarrow \beta << -6\ln|b|.$$

$$\{\lambda_{0i}\} - \text{Eigenvalue of matrix } M_0,$$

$$M_0 = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & A_3 & 0 \\ 0 & A_4 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix}$$

$$\lambda_{01} = A_2 + \sqrt{A_3 A_4}, \quad \lambda_{02} = A_2 - \sqrt{A_3 A_4}, \quad \lambda_{03} = \lambda_{04} = A_1$$

$$\{\lambda_{0i}\} - \text{Eigenvalue of matrix } M_0,$$

$$\lambda_{01} = A_2 + \sqrt{A_3 A_4} = \left(\frac{1}{2} \left| Lf\, g\, \left(\text{th} \frac{\beta}{2} \right)^{N-2} \right| + \frac{1}{4} R_1 \right)^2,$$

$$\lambda_{02} = A_2 - \sqrt{A_3 A_4} = \left(\frac{1}{2} \left| Lf\, g\, \left(\text{th} \frac{\beta}{2} \right)^{N-2} \right| - \frac{1}{4} R_1 \right)^2$$

$$R_1 = e^{-\beta} \sqrt{(e^{\beta}(1 - L|f|^2) + L|g|^2)(e^{\beta}(1 - L|g|^2) + L|f|^2)} \,/\, \text{ch} \frac{\beta}{2}$$

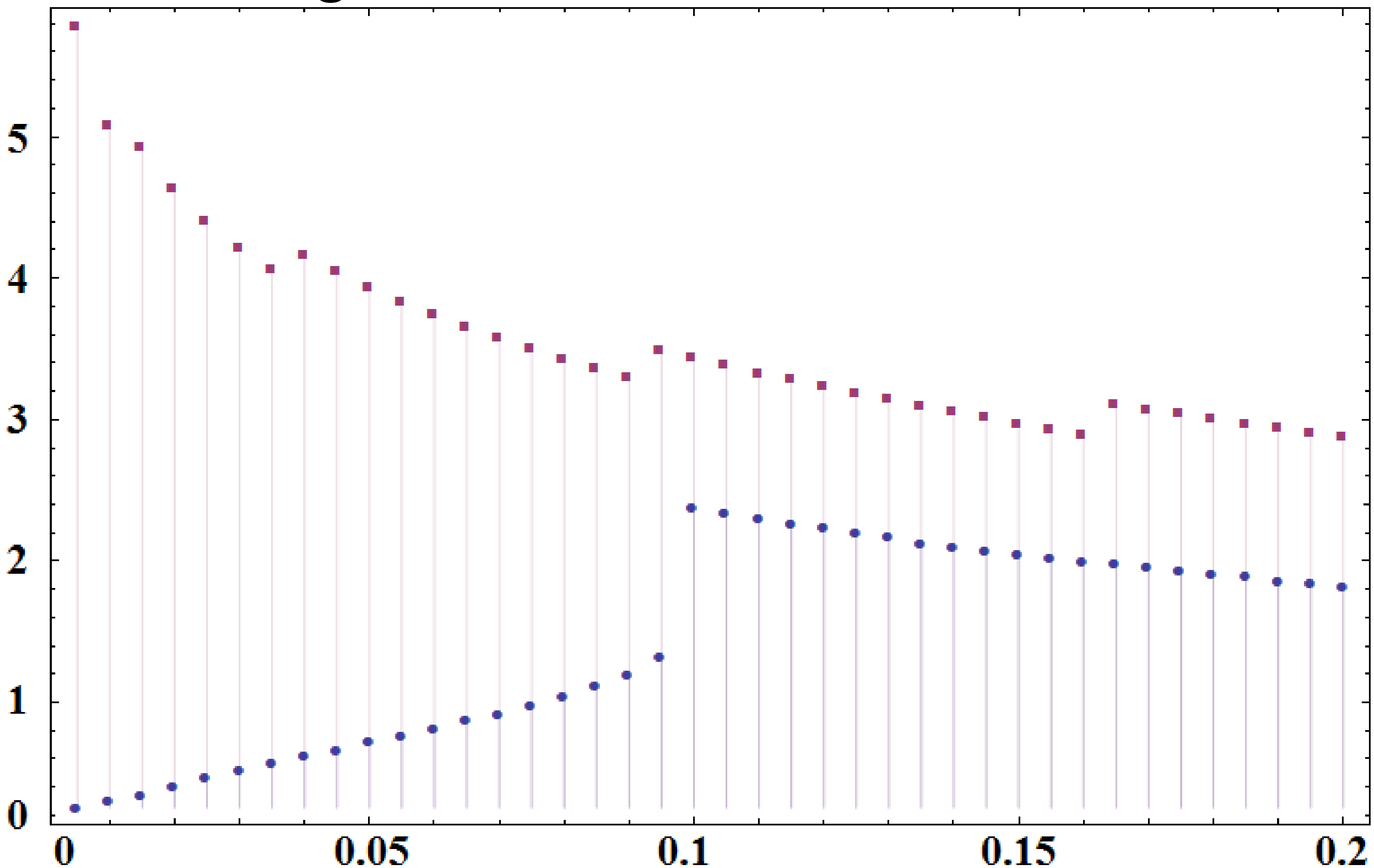
$$\lambda_{03} = \lambda_{04} = A_1 = \frac{(L\,(|f|^2 + |g|^2\,)\text{-}1)\,(e^{\beta} + L\,(|f|^2 + |g|^2\,)}{e^{\beta}\,\text{ch}^2 \frac{\beta}{2}}$$

$$P(\lambda) = P(\lambda_0 + \varepsilon^2 \lambda_2) = P_0(\lambda_0) + P_2(\lambda_2)$$

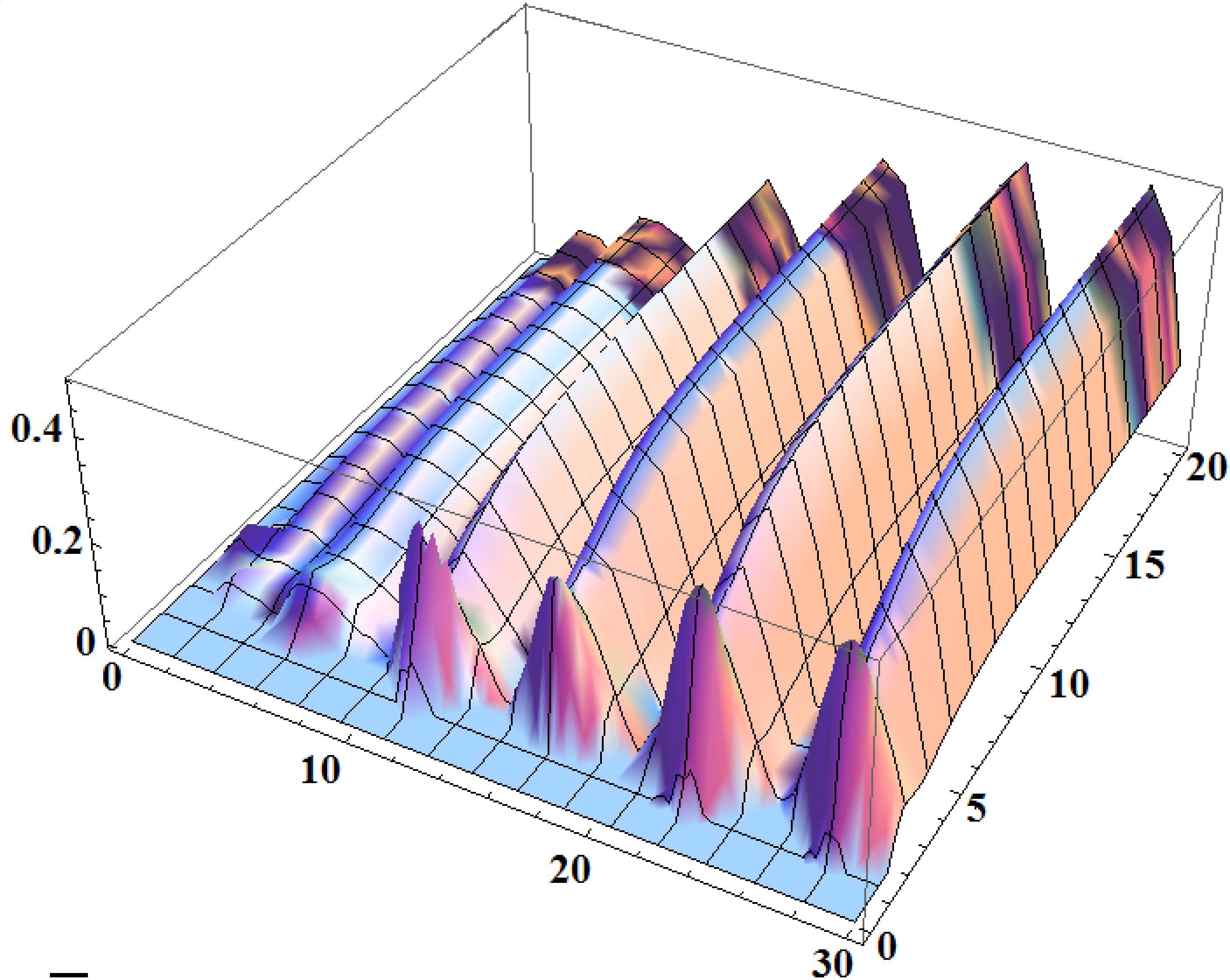
$$X = \mathrm{Max}\left(\frac{1}{2}\left| Lf\, g\, \left(\mathrm{th}\frac{\beta}{2}\right)^{N-2} \right|,\ \frac{1}{4}R_1 \right),$$

$$Y = \mathrm{Min}\left(\frac{1}{2}\left| Lf\, g\, \left(\mathrm{th}\frac{\beta}{2}\right)^{N-2} \right|,\ \frac{1}{4}R_1 \right).$$

Critical temperature for emergence of entanglement at N=4



N=4



N=4

$|b|=1/\sqrt{2}$