On the Wigner quasiprobability function for N-level quantum system

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Overview

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   - The Stratonovich-Weyl correspondence
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The quasi-probability distributions

Quasi-Probabilty Distributions (QPD)'s – "quantum analogue" of the statistical distribution on the phase space of a classical system

We are interested in QPD's for finite dimensional quantum systems

THE AIM OF TALK:

To build the s-parametric family of Wigner functions $W_\varrho(\Omega; s)$ over the classical phase space (Ω) for a generic N-level quantum system with density matrix ϱ.

The standard form of the Wigner function

For a given state, describing by the density operator ϱ, the Wigner function $W(\boldsymbol{q}, \boldsymbol{p})$ defined on a classical $2n$-dimensional phase space spanned by the canonical coordinates $\boldsymbol{q}$ and momentum $\boldsymbol{p}$ reads

$$W(\boldsymbol{q}, \boldsymbol{p}) := \int \mathrm{d}^n \boldsymbol{z} e^{\frac{\imath}{\hbar} \boldsymbol{z}\boldsymbol{p}} \langle q + \frac{z}{2} \, |\varrho| \, q - \frac{z}{2} \rangle \,,$$

Representation for W via displacement operator D

$$W = \mathrm{Tr}\left[\varrho D \Pi D^\dagger\right]$$

where D and Π are the displacement and parity operators respectively.

Quasiprobability distributions

- For any operator A on the Hilbert space $\mathcal{H}$ of quantum system one can define a family of functions $F_A(\Omega; s)$ onto the phase space Ω. The index s labels functions in the family.
- When the operator A represents the density matrix, $A = \varrho$, the corresponding phase-space functions $F_\varrho(\Omega; s) := P(\Omega; s)$ are named as **Quasiprobability Distributions**.

The Stratonovich-Weyl correspondence

The physically motivated properties of $P(\Omega; s)$ were formulated by R.L.Stratonovich more than sixty years ago (1955) and are usually referred to as the Stratonovich-Weyl correspondence

The Stratonovich-Weyl Correspondence

Clauses of SW correspondence:

- **Mapping** • For a density matrix ϱ the Wigner function W_ρ on the classical phase-space (Ω) is given by the map:

$$W_\rho(\Omega) = \mathrm{tr}\left(\rho\Delta(\Omega)\right)$$

defined by the Hermitian kernel $\Delta(\Omega) = \Delta(\Omega)^\dagger$, with a unit norm

$$\int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) = 1$$

- **Reconstruction** • The state ρ can be reconstructed as

$$\rho = \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) W_\rho(\Omega)\,.$$

- **Covariance** • The unitary symmetry $\rho' = U(\alpha)\rho U^\dagger(\alpha)$ induces the kernel transformation:

$$\Delta(\Omega') = U(\alpha)\Delta(\Omega)U^\dagger(\alpha)$$

The Wigner distribution kernel

The Wigner distribution $W_\rho(\Omega)$ over a phase space parametrized by the set Ω is determined by the kernel $\Delta(\Omega)$:

$$W_\rho(\Omega) = \mathrm{tr}\left(\rho\Delta(\Omega)\right)$$

Since the kernel $\Delta(\Omega)$ is Hermitian it can be written as

$$\Delta(\theta_1, \dots \theta_{d_F} | \pi_1, \dots, \pi_N) = U(\theta) P U^\dagger(\theta)\,.$$

Here $P = \kappa N\,\Pi$, with κ - real normalization factor, a diagonal matrix $\Pi = \mathrm{diag}||\pi_1, \pi_2, \dots, \pi_N||$ with a unit trace, $\mathrm{tr}(\Pi) = 1$.

In accordance with the SU(n)-covariance of kernel we identify:

d_F - parameters of unitary matrix $U(\theta) \in SU(N)$ with the coordinates of classical phase-space, $\Omega = (\theta_1, \dots \theta_{d_F})$.

Constructing the kernel $\Delta(\Omega)$

- **Step 1** • The SU(N) symmetry allows to define the "reconstruction" integral for ϱ over the SU(N) group with the Haar measure:

$$\varrho = \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega) W_\rho(\Omega)\,.$$

- **Step 2** • Substitute decomposition $\Delta = U(\theta) P U^\dagger(\theta)$ into the identity

$$\varrho = \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega) \mathrm{tr}\left[\Delta(\Omega)\varrho\right].$$

and evaluate the integral using the Weingarten formula:

$$\int d\mu U_{i_1 j_1} U_{i_2 j_2} U^\dagger_{k_1 l_1} U^\dagger_{k_2 l_2} = \frac{1}{N^2-1}\left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_1}\delta_{j_2 l_2} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_2}\delta_{j_2 l_1}\right)$$

$$-\frac{1}{N(N^2-1)}\left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_2}\delta_{j_2 l_1} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_1}\delta_{j_2 l_2}\right).$$

Constructing the kernel

- **Step 3** • Derive the algebraic equation for kernel

The equation for $n \times n$ matrix P reads:

$$P_{ki}P_{sj} - \frac{1}{N}P_{si}P_{kj} = \frac{N^2-1}{2}\delta_{is}\delta_{jk}\,.$$

- **Step 4** • Solve these equations classifying its solution by the values of adjoint SU(n)-invariants constructed from P ,

Solutions are characterized by the Casimir invariants obeying relations:

$$\mathfrak{C}_r + \mathfrak{C}_{r+1} = 0\,, \qquad r = 2, 3, \ldots, N-1\,.$$

Table: Invariant characteristics of kernel Δ for low N

Group	$\mathfrak{C}_2$	κ	**Spectrum** $\boldsymbol{\sigma}(\Pi)$
SU(3)	2	$\sqrt{6}$	$\{-\frac{1}{3}\,;\ \frac{2}{3}, \frac{2}{3}\}$
SU(4)	$\frac{3}{4}$	$2\sqrt{10}$	$\{-\frac{1}{8}\,;\ \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\}$
SU(5)	$\frac{4}{9}$	$3\sqrt{15}$	$\{-\frac{1}{15}\,;\overbrace{\frac{4}{15}, \dots, \frac{4}{15}}^{4}\}$
	6	$\sqrt{10}$	$-\frac{2}{5}, -\frac{2}{5}\,;\overbrace{\frac{3}{5}, \dots, \frac{3}{5}}^{3}$
SU(6)	$\frac{5}{16}$	$4\sqrt{21}$	$\{-\frac{1}{24}\,;\overbrace{\frac{5}{24}, \dots, \frac{5}{24}}^{5}\}$
	2	$\frac{\sqrt{210}}{2}$	$\{-\frac{1}{6}, -\frac{1}{6}\,;\overbrace{\frac{1}{3}, \dots, \frac{1}{3}}^{4}\}$

Table II: Spectrum of kernel Δ for given N

$N^* = \left[\frac{N-1}{2}\right]$ **types degeneracy of spectrum**		
	$\mathfrak{C}_2$	$\sigma(\Pi)$
1	$\frac{N-1}{(N-2)^2}$	$[\overbrace{-\frac{1}{N(N-2)}, \dots, -\frac{1}{N(N-2)}}^{1}; \overbrace{\frac{N-1}{N(N-2)}, \dots, -\frac{N-1}{N(N-2)}}^{N-1}]$
2	$\frac{2(N-2)}{(N-4)^2}$	$[\overbrace{-\frac{2}{N(N-4)}, \dots, -\frac{2}{N(N-4)}}^{2}; \overbrace{\frac{N-2}{N(N-4)}, \dots, -\frac{N-2}{N(N-4)}}^{N-2}]$
$\vdots$	$\vdots$	$\vdots \quad \vdots$
N^*	$\frac{N^*(N-N^*)}{(N-2N^*)^2}$	$[\overbrace{\frac{-N^*}{N(N-2N^*)}, \dots, \frac{-N^*}{N(N-2N^*)}}^{N^*}; \overbrace{\frac{N-N^*}{N(N-2N^*)}, \dots, \frac{N-N^*}{N(N-2N^*)}}^{N-N^*}]$

Few examples of the Wigner distributions

- Qubit – 2-level system;
- Qubit-Qubit– binary system composed from two 2-level. subsystems

Qubit

Paramentrization of the qubit state by Bloch Sphere

The state of a qubit may be represented as a point on/in a unit sphere of coefficients in decomposition $\rho = \frac{1}{2}\left(I + \vec{r}\,\vec{\sigma}\right)$, where the points on the sphere correspond the pure states, while the interior points represent the mixed states.

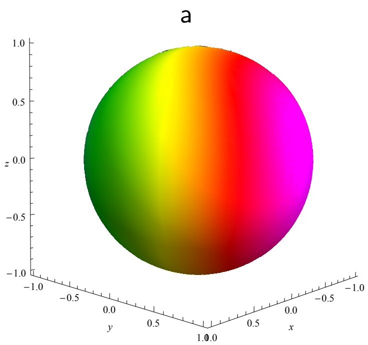
Qubit kernel

$$\Delta(\alpha_1, \alpha_2) = U(\alpha_1, \alpha_2)\frac{1}{2}\left(I - \sqrt{3}\sigma_z\right)U^\dagger(\alpha_1, \alpha_2)$$

Qubit Wigner function

$$W(\alpha_1, \alpha_2\,;\; r, \theta, \phi) = tr\left[\rho\,\Delta(\alpha_1, \alpha_2)\right] =$$
$$\frac{1}{2}\left(1 + \sqrt{3}r\left(\sin\left(\alpha_2\right)\sin(\theta)\cos\left(\alpha_1 + \phi\right) - \cos\left(\alpha_2\right)\cos(\theta)\right)\right).$$

Qubit



The Wigner function for the qubit on the unit sphere of "phase angles" (a) for a singlet (b) half way to the singlet along z axes.

Qubit-qubit

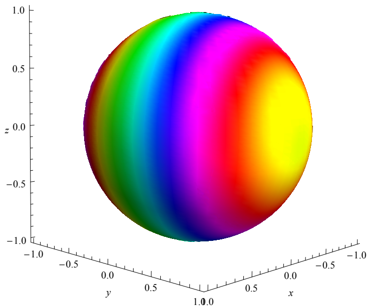
Wigner function for the qubit-qubit pair in the Werner state

For the $U(2) \otimes U(2)$ invariant Werner state $\rho_{Werner} = \frac{1}{4}(I + p\vec{\sigma}\,\vec{\sigma})$, choosing an appropriate parametrization we get the following form for the Wigner function

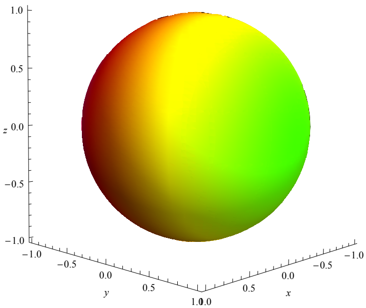
$$W(\alpha_0, \alpha_2, \alpha_4, \alpha_6\,;\, p) = \frac{1}{2}\sqrt{\frac{5}{2}}(-p + 8 + p(\cos(\alpha_6)\cos(\alpha_4) - \cos(\alpha_4)$$
$$- \quad 3\cos(\alpha_6) - 2\,(1 - \cos(\alpha_6))\,(\frac{1}{2}\cos(\alpha_2)\,(\cos(\alpha_4) + 1)$$
$$+ \quad 2\sin\left(\frac{\alpha_2}{2}\right)\sin(\alpha_4)\cos\left(\frac{\alpha_0}{2}\right))).$$

Qubit-qubit

a



b



The Wigner function for the qubit-qubit pair an Werner maximally ($p = -1$) entangled an separable ($p = -\frac{1}{4}$), (a) and (b) respectively. The slice $\alpha_4 = \frac{3\pi}{4}$, $\alpha_6 = \frac{5\pi}{6}$ has been taken.

Concluding remarks

1. The exact form of the kernel of Wigner-Weyl transformation yielding a quasi-probability distribution function for general N-level quantum system has been constructed.
2. For an N-level quantum system a $N^* = \lfloor \frac{N-1}{2} \rfloor$ fold family of transformation kernels fulfilling the Stratonovich-Weyl restrictions has been found.
3. The family members are classified by the corresponding values of the Casimir invariants.

Thank you!

Technical details of calculations

$$\boxed{\Pi - \frac{1}{N} I = \frac{\alpha}{N}\Pi^{-1}} \quad \boxed{\operatorname{tr}\left(\Pi^{-1}\right) = 0}$$

$$\alpha = (N^2 - 1)/2\kappa^2 \,.$$

Expressing the traces of powers as polynomials in α

$$\operatorname{tr}\left(\Pi^2\right) = \frac{1}{N} + \alpha$$

$$\operatorname{tr}\left(\Pi^3\right) = \frac{1}{N^2} + \frac{2}{N}\alpha$$

....

$$\operatorname{tr}\left(\Pi^k\right) = C_{Q(k)}^{\frac{k+Q(k)}{2}} \left(\frac{1}{n}\right)^{Q(k)-1} \left(\frac{\alpha}{n}\right)^{\frac{k}{2}-\frac{Q(k)}{2}} \times$$

$${}_3F_2 \left(1, \frac{Q(k)-k}{2}, \frac{Q(k)+k+2}{2}; \frac{Q(k)+1}{2}, \frac{Q(k)}{2}+1; -\frac{1}{4n\alpha}\right)$$

Classification via Casimir invariants

$$\det[\lambda - \Pi] = \lambda^N - S_1\lambda^{N-1} + S_2\lambda^{N-2} + \cdots + (-1)^N S_N = 0$$

$S_{N-1} = 0$ polynomial equation provides $N^* = \left[\frac{N-1}{2}\right]$ values for α.

Cartan subalgebra decomposition of P

$$P = \kappa \left(I + \sqrt{\frac{N(N-1)}{2}} \sum_{\lambda_a \in h} \xi_a \cdot \lambda_a \right).$$

Casimir invariants

$$\mathfrak{C}_2 = (N-1)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} \quad \mathfrak{C}_3 = (N-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi})\cdot\boldsymbol{\xi}$$

$$\mathfrak{C}_4 = (N-1)\,(\boldsymbol{\xi}\vee\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\vee\boldsymbol{\xi}) \quad \mathfrak{C}_5 = (N-1)\Big((\boldsymbol{\xi}\vee\boldsymbol{\xi})\vee(\boldsymbol{\xi}\vee\boldsymbol{\xi})\Big)\cdot\boldsymbol{\xi}$$

where $(\boldsymbol{U}\vee\boldsymbol{V})_a := c\, d_{abc} U_a V_b$. d_{abc} are symmetric structure constants for $\mathfrak{su}(N)$ and $c = \sqrt{N(N-1)/2}$

Realization of the construction

$$S_k = \frac{1}{k!}\det\begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & \cdots & 1 \\ t_3 & t_2 & t_1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & k-1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{pmatrix}.$$

$$\det[\lambda - \Pi] = (\lambda - \lambda_1)^{N_1}(\lambda - \lambda_2)^{N_2}$$