Symbolic and numerical modeling of nonlinear dynamics of particles in accelerators

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Introduction

Symbolic methods

- Time-consuming calculations are carried out only once
- All particles of beam is considered simultaneously
- Storing results in the formulas
- The possibility of transition from coordinates and impulses to actually measured quantities
- Simplicity of interpretation of the results
- Parallel computing
- Simplify learning for the use of artificial intelligence

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Numerical methods

Software

- MAD - Methodical Accelerator Design
- COSY Infinity
- .....

Features of the approaches:

- The trajectory analysis
- Expensive by time
- Calculation only coordinate and momentum of particles
- The necessity of multiple complete recalculation in solving the optimization problem

Matrix formalism

Motion equation :

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X},t),\ \mathbf{F}(0,t) \equiv 0$$

$$\mathbf{F}(\mathbf{X},t) = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]} = \sum_{k=0}^{\infty} \frac{\partial^k \mathbf{F}(0,t)}{\partial \mathbf{X}^k} \frac{\mathbf{X}^k}{(k)!}$$

$$\mathbf{X}^{[k]} = \underbrace{\mathbf{X} \otimes \mathbf{X} \otimes \ldots \otimes \mathbf{X}}_{\text{k times}}$$

$$\frac{d\mathbf{X}}{dt} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]}$$

$$\mathbb{P}^{\infty} = \begin{pmatrix} \mathbb{P}^{11} & \mathbb{P}^{12} & \ldots & \mathbb{P}^{1k} & \ldots \\ \mathbb{O} & \mathbb{P}^{22} & \ldots & \mathbb{P}^{2k} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{P}^{2k} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

$$\mathbf{X}(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t|t_0)\mathbf{X}_0^{[k]},$$

$$\mathbb{R}^{ik}(t|t_0) = \sum_{j=i+1}^{k} \int_{t_0}^{t} \mathbb{R}^{ii}(t|\tau)\mathbb{P}^{ij}(\tau)\mathbb{R}^{jk}(\tau|t_0)d\tau,$$

$$\mathbb{R}^{ii}(t|t_0) = (\mathbb{R}^{11}(t|t_0))^{[i]}$$

$$\mathbb{R}^{\infty} = \begin{pmatrix} \mathbb{R}^{11} & \mathbb{R}^{12} & \ldots & \mathbb{R}^{1k} & \ldots \\ \mathbb{O} & \mathbb{R}^{22} & \ldots & \mathbb{R}^{2k} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{R}^{kk} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

Why we use Wolfram Mathematica ?

Creating libraries for various programming languages (C++, Java, Python)

Supports work with symbolic calculations

Manual

User friendly language

Great functionality for working with matrices

Supports work with Kronecker operations

Ability to create applications

Integrability with programming languages like C++, Java, Python.

Nonlinear harmonic oscillator

Solution of the equation of motion of a nonlinear harmonic oscillator in the framework of the matrix formalism.

$$H(t) = \frac{q^2(t) + p^2(t)}{2} - \alpha \frac{q^4(t)}{24}$$

$$\begin{cases} q'(t) = p(t) \\ p'(t) = -q(t) + \alpha \dfrac{q^3(t)}{6} \end{cases}$$

$$\mathbf{X} = \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\frac{d\mathbf{X}}{dt} = \mathbb{P}^{11}\mathbf{X} + \mathbb{P}^{12}\mathbf{X}^{[2]} + \mathbb{P}^{13}\mathbf{X}^{[3]}$$

$$\mathbf{X} = \mathbb{R}^{11}\mathbf{X}_0 + \mathbb{R}^{12}\mathbf{X}_0^{[2]} + \mathbb{R}^{13}\mathbf{X}_0^{[3]}$$

Symplectification procedure

Hamiltonian motion equation

$$\frac{d\mathbf{X}}{dt} = \mathbb{J}(\mathbf{X})\frac{\partial \mathcal{H}(\mathbf{X},t)}{\partial \mathbf{X}}, \qquad \mathbb{J}(\mathbf{X}) = \mathbb{J}_0 = \begin{pmatrix} \mathbb{O} & \mathbb{E} \\ -\mathbb{E} & \mathbb{O} \end{pmatrix},$$

Jacobi matrix

$$\mathbb{M}(\mathbf{X}, t\,|\,t_0; \mathcal{M}) = \mathbb{M}(\mathbf{X}; t\,|\,t_0) = \frac{\partial \mathcal{M}(t\,|\,t_0; \mathcal{H}) \circ \mathbf{X}}{\partial \mathbf{X}^*},$$

The Jacobi identity

$$\mathbb{M}^*(\mathbf{X}; t\,|\,t_0)\mathbb{J}(\mathbf{X})\mathbb{M}(\mathbf{X}; t\,|\,t_0) = \mathbb{J}(\mathbf{X}).$$

For canonical Jacobi matrix

$$\mathbb{M}^*(\mathbf{X}; t\,|\,t_0)\mathbb{J}_0\mathbb{M}(\mathbf{X}; t\,|\,t_0) = \mathbb{J}_0, \qquad \det \mathbb{M}(\mathbf{X}; t\,|\,t_0) \equiv 1.$$

For any vector X we can evaluate

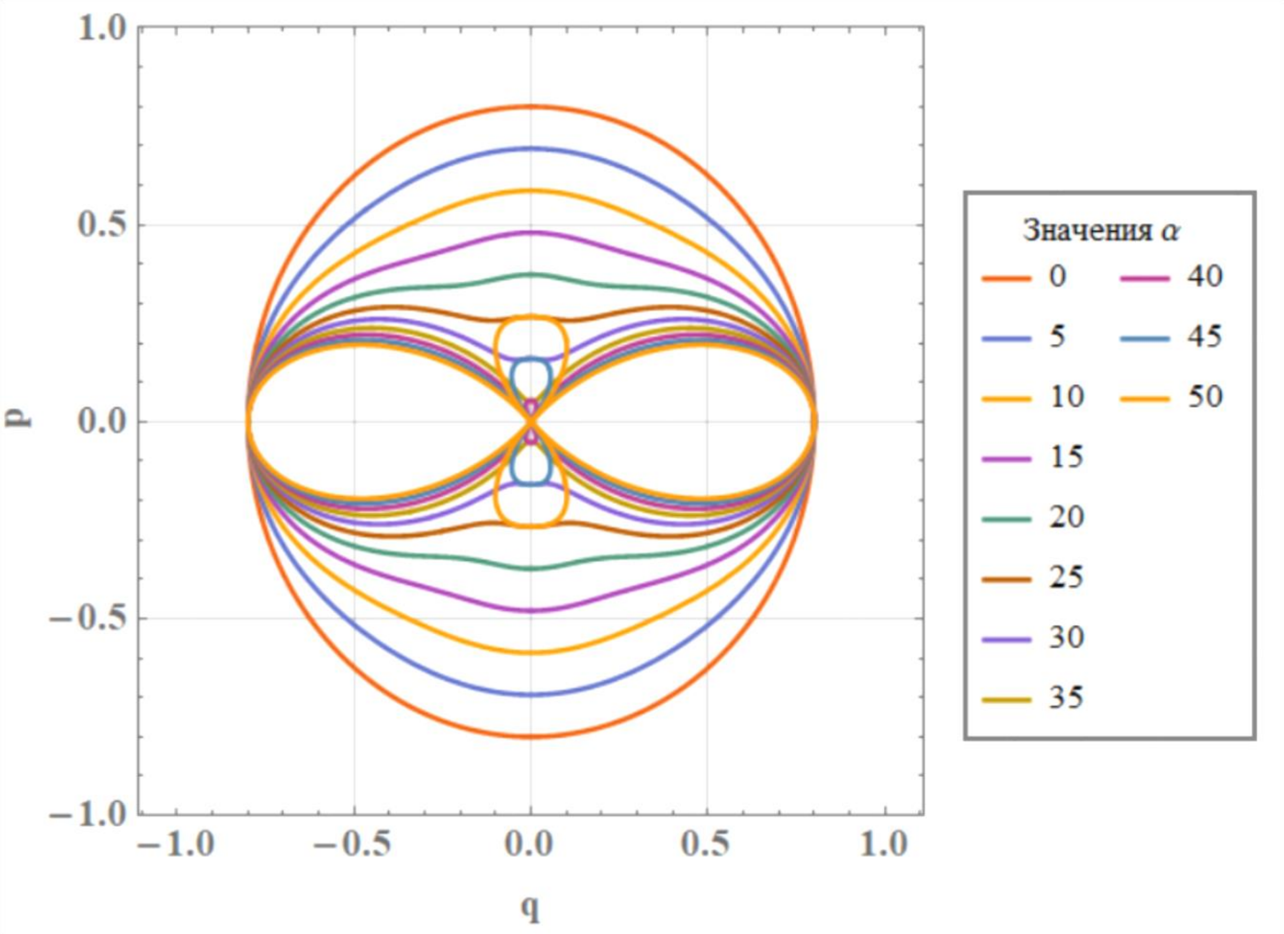
$$\boxed{\sum_{k+l=m} (\mathbf{X}^{\odot k})^* \left(\mathbb{M}^{1\,(k+1)}\right)^* \mathbb{J}_0 \mathbb{M}^{1\,(l+1)} \mathbf{X}^{\odot l} = 0, \quad m \geq 1.}$$

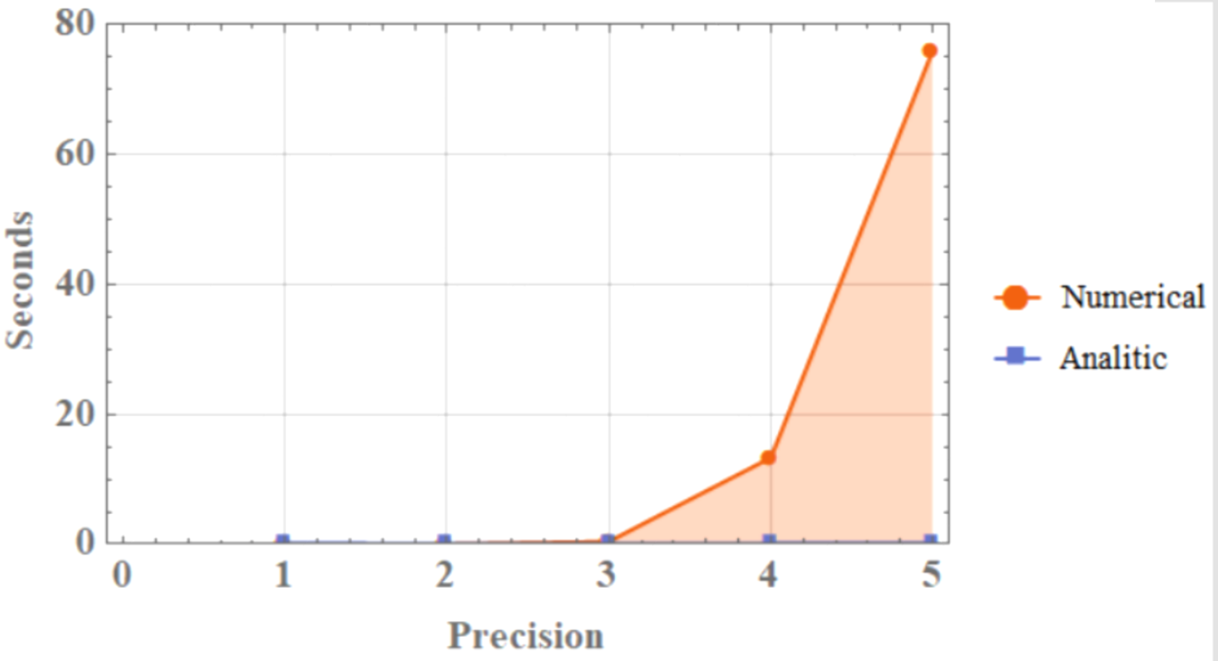
$$\mathbf{X}^{\odot k} = \sum_{j=0}^{k-1} \mathbf{X}^{[j]} \otimes \mathbb{E} \otimes \mathbf{X}^{[k-j-1]}$$

Nonlinear harmonic oscillator

$$\begin{cases} q(t) = 0.8\cos(t) + \alpha\dfrac{0.512}{24}\cos(t)(-1+\cos^4(t)) \\ p(t) = -0.8\sin(t) + \alpha\dfrac{0.512}{24}\sin(t)(-1+\cos^4(t)) \end{cases}$$

$$\mathbf{X}_0 = \begin{pmatrix} 0.8 \\ 0 \end{pmatrix}$$





Conclusion

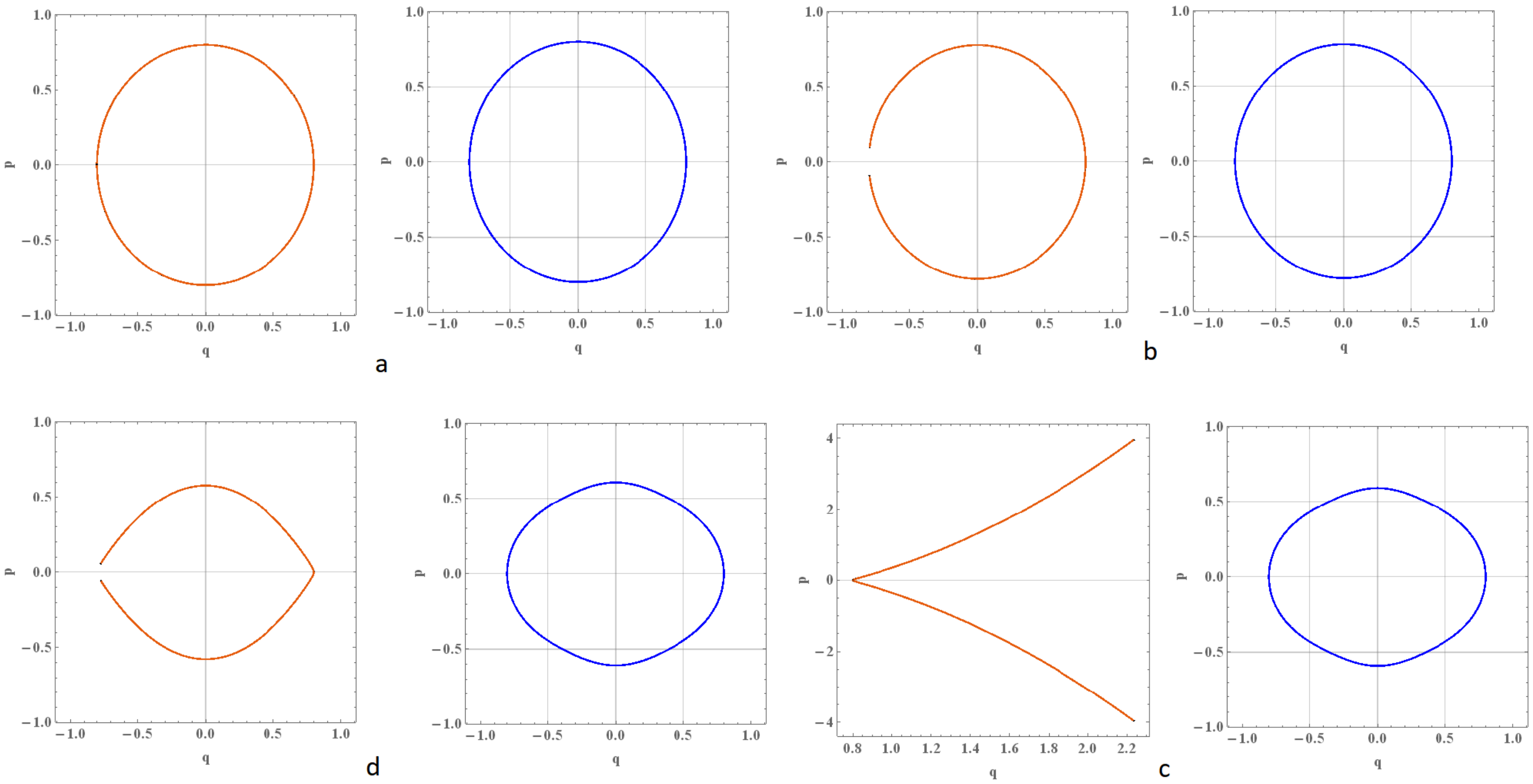
Symbolic methods

- Time-consuming calculations are carried out only once
- Storing results in the formulas
- All particles of beam is considered simultaneously
- Useful and speed of the procedure for selecting parameters
- The possibility of choosing the order of computations

Thank you for your attention !

Nonlinear harmonic oscillator

The images show the phase portraits with a) α = 0, b) α = 1, c) α = 9, d) α = 9.8.
Orange indicates the graphs with solutions obtained by the method Runge-Kutta of the 4th order
Blue – with matrix formalism



Accelerator scheme - order of arrangement - order of nonlinearity of control fields and so on…

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Mathematical and computer models of accelerator

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Control elements representation in symbolical and numerical forms up to necessary orders of nonlinearities

LEGO objects - a modular accelerator design codes