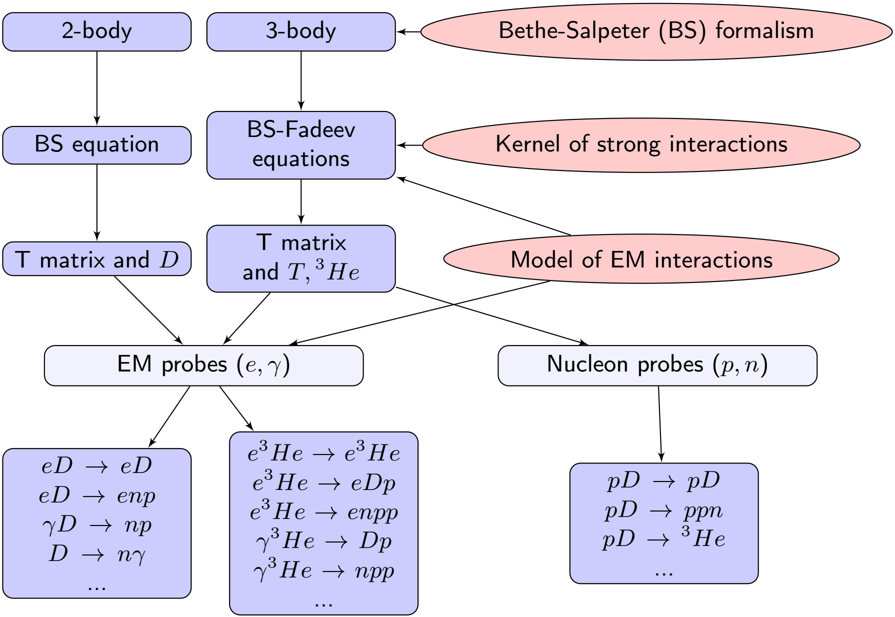
ELASTIC FORM FACTORS FROM SEPARABLE KERNEL

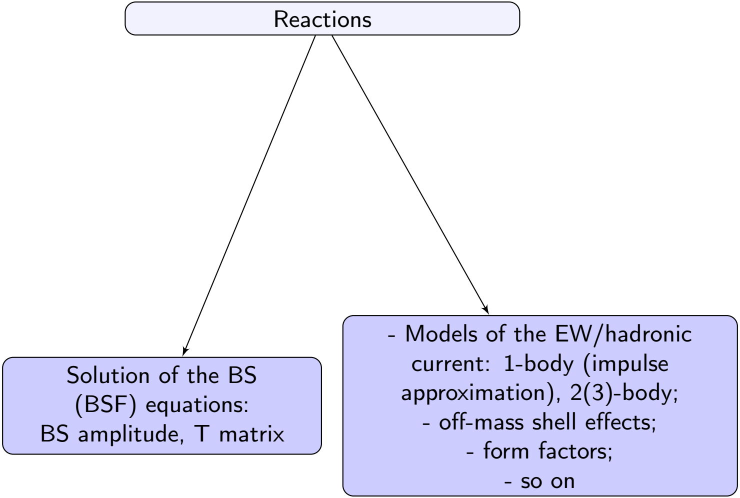
A. Bekzhanov, Serge Bondarenko, V. Burov
Joint Institute for Nuclear Research, Dubna, Russia

Mathematical Modeling and Computational Physics, JINR, Dubna, 3-7 July, 2017

The Bethe-Salpeter approach is a powerful tool to investigate few-body compounds such as the deuteron, unbound neutron-proton (np) system, three-nucleon system.

- Bethe-Salpeter equation and its solution for separable kernel of interaction
- Yamaguchi-type of kernel functions and Graz-II relativistic kernel
- elastic eD-scattering
- conclusion


2-body
3-body
Bethe-Salpeter (BS) formalism
BS equation
BS-Fadeev equations
Kernel of strong interactions
T matrix and D
T matrix and $T, {}^3He$
Model of EM interactions
EM probes (e, γ)
Nucleon probes (p, n)
$eD \to eD$
$eD \to enp$
$\gamma D \to np$
$D \to n\gamma$
...
$e^3He \to e^3He$
$e^3He \to eDp$
$e^3He \to enpp$
$\gamma^3He \to Dp$
$\gamma^3He \to npp$
...
$pD \to pD$
$pD \to ppn$
$pD \to {}^3He$
...


Reactions
Solution of the BS (BSF) equations: BS amplitude, T matrix
- Models of the EW/hadronic current: 1-body (impulse approximation), 2(3)-body;
- off-mass shell effects;
- form factors;
- so on

Bethe-Salpeter equation for the nucleon-nucleon T matrix

$$T(p',p;P) = V(p',p;P) + \frac{i}{4\pi^3}\int d^4k\, V(p',k;P)\, S_2(k;P)\, T(k,p;P)$$

p', p - the relative four-momenta
P - the total four-momentum

$V(p',p;P)$ - the interaction kernel

$S_2^{-1}(k;P) = \left(\frac{1}{2}\,P\cdot\gamma + k\cdot\gamma - m\right)^{(1)}\left(\frac{1}{2}\,P\cdot\gamma - k\cdot\gamma - m\right)^{(2)}$
free two-particle Green function

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable ansatz for the kernel

$$V_{a'a}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{m,n=1}^{N} \lambda_{mn}^{r[a'a]}(s) g_m^{[a']}(p_0', |\mathbf{p}'|) g_n^{[a]}(p_0, |\mathbf{p}|)$$

Solution for the T matrix

$$T_{a'a}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{i,j=1}^{N} \tau_{ij}(s)\, g_i^{[a']}(p_0', |\mathbf{p}'|)\, g_j^{[a]}(p_0, |\mathbf{p}|)$$

where

$$\left[\tau_{ij}(s)\right]^{-1} = \left[\lambda_{mn}^{r[a'a]}(s)\right]^{-1} + h_{ij}(s),$$

$$h_{ij}(s) = -\frac{i}{4\pi^3} \sum_a \int dk_0 \int \mathbf{k}^2 d|\mathbf{k}| \frac{g_i^{[a]}(k_0, |\mathbf{k}|) g_j^{[a]}(k_0, |\mathbf{k}|)}{(\sqrt{s}/2 - E_{\mathbf{k}} + i\epsilon)^2 - k_0^2},$$

$g_j^{[a]}$ - the model functions, $\lambda_{ij}^{[a'a]}(s)$ - a matrix of model parameters.

Separable kernel for Schrodinger equation with separable potential

Yoshio Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. I" Phys.Rev.95, 1628 (1954)
Yoshio Yamaguchi, Yoriko Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. II" Phys.Rev.95, 1635 (1954)

Nonlocal: $\langle \mathbf{r}|V|\mathbf{r}'\rangle \neq \delta^{(3)}(\mathbf{r}-\mathbf{r}')$
in configuration space

$$\langle \mathbf{r}|V|\mathbf{r}'\rangle = -(\lambda/m_N)v^*(\mathbf{r})v^*(\mathbf{r}')$$

in momentum space

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = (\lambda/m_N)g^*(\mathbf{p})g^*(\mathbf{p}')$$

for S-state: $g(p) = 1/(p^2+\beta^2)$
for D-state: $g(p) = p^2/(p^2+\beta^2)^2$
for the deuteron and scattering problem.

Separable nucleon-nucleon potential was widely uses for the two- and three-nucleon calculations in nonrelativistic nuclear physics

Willibald Plessas et al. Graz, Graz-II potentials, separable representation of the popular Bonn and Paris potentials

K. Schwarz, Willibald Plessas, L. Mathelitsch "Deuteron Form-factors And E D Polarization Observables For The Paris And Graz-II Potentials" Nuovo Cim. A76 (1983) 322-329.
J. Haidenbauer, Willibald Plessas "Separable Representation Of The Paris Nucleon Nucleon Potential" Phys.Rev. C30 (1984) 1822-1839.
Johann Haidenbauer, Y. Koike, Willibald Plessas "Separable representation of the Bonn nucleon-nucleon potential" Phys.Rev. C33 (1986) 439-446.

$$g(p) = \sum_n p^{2m}/(p^2+\beta_n^2)^n,$$

m corresponds to angular momentum

Lippmann-Schwinger equation $\rightarrow$ Bethe-Salpeter equation

G. Rupp and J. A. Tjon "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)

$$\mathbf{p}^2 \to -p^2 = -p_0^2 + \mathbf{p}^2$$

$$g_p(p,P) = \frac{1}{-p^2+\beta^2} \xrightarrow{\text{c.m.}} \frac{1}{-p_0^2+\mathbf{p}^2+\beta^2+i\epsilon}$$

singularities: $p^0 = \pm\sqrt{\mathbf{p}^2+\beta^2} \mp i\epsilon$

Graz-II covariant kernel, rank III ($J = 1 :^3 S_1 -^3 D_1$ partial-wave states)

$$g_1^{(\mathrm{S})}(p_0, |\mathbf{p}|) = \frac{1 - \gamma_1(p_0^2 - \mathbf{p}^2)}{(p_0^2 - \mathbf{p}^2 - \beta_{11}^2)^2},$$
$$g_2^{(\mathrm{S})}(p_0, \mathbf{p}) = -\frac{(p_0^2 - \mathbf{p}^2)}{(p_0^2 - \mathbf{p}^2 - \beta_{12}^2)^2},$$
$$g_3^{(\mathrm{D})}(p_0, |\mathbf{p}|) = \frac{(p_0^2 - \mathbf{p}^2)(1 - \gamma_2(p_0^2 - \mathbf{p}^2))}{(p_0^2 - \mathbf{p}^2 - \beta_{21}^2)(p_0^2 - \mathbf{p}^2 - \beta_{22}^2)^2},$$
$$g_1^{(\mathrm{D})}(p_0, |\mathbf{p}|) = g_2^{(\mathrm{D})}(p_0, |\mathbf{p}|) = g_3^{(\mathrm{S})}(p_0, |\mathbf{p}|) \equiv 0. \quad (1)$$

Table: Deuteron and low-energy scattering properties

	$p_{\mathrm{D}}(\%)$	ϵ_{D} (MeV)	Q_{D} (Fm^{-2})	μ_{D} ($e/2m$)	$\rho_{\mathrm{D/S}}$	r_0 (Fm)	a (Fm)
Covariant Graz-II	4	2.225	0.2484	0.8279	0.02408	1.7861	5.4188
Experimental data		2.2246	0.286	0.8574	0.0263	1.759	5.424

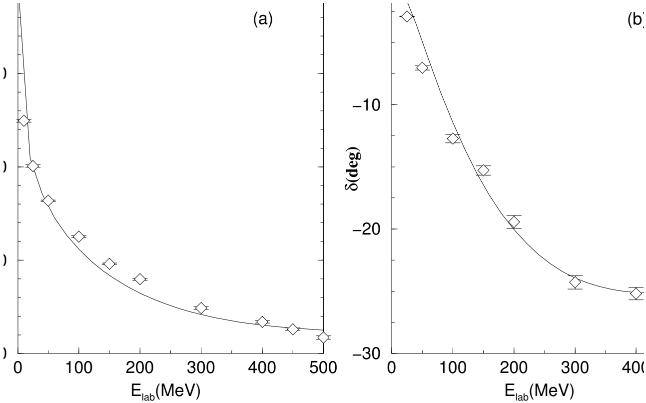


Figure: Phase shifts of the $^3S_1$ and $^3D_1$ partial states

Elastic eD scattering cross section

$$\frac{d\sigma}{d\Omega'_{\rm e}} = \left(\frac{d\sigma}{d\Omega'_{\rm e}}\right)_{\rm Mott}\left[A(q^2) + B(q^2)\tan^2\frac{\theta_{\rm e}}{2}\right],$$

$$\left(\frac{d\sigma}{d\Omega'_{\rm e}}\right)_{\rm Mott} = \frac{\alpha^2\cos^2\theta_{\rm e}/2}{4E_{\rm e}^2(1+2E_{\rm e}/M_d\sin^4\theta_{\rm e}/2)},$$

where $\theta_{\rm e}$ is the electron scattering angle, M_d is the deuteron mass, E_e is the incident electron energy.

Deuteron structure functions $A(q^2)$ and $B(q^2)$

$$A(q^2) = F_{\rm C}^2(q^2) + \frac{8}{9}\eta^2 F_{\rm Q}^2(q^2) + \frac{2}{3}\eta F_{\rm M}^2(q^2)$$

$$B(q^2) = \frac{4}{3}\eta(1+\eta)F_{\rm M}^2(q^2)$$

where $\eta = -q^2/4M_d^2 = Q^2/4M_d^2$

Relativistic impulse approximation (RIA)

Deuteron current matrix element

$$\langle D'\mathcal{M}'|J_\mu^{RIA}|D\mathcal{M}\rangle =$$

$$ie\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left\{\bar{\chi}^{1\mathcal{M}'}(P',k')\Gamma_\mu^{(\mathrm{S})}(q)\chi^{1\mathcal{M}}(P,k)(P\cdot\gamma/2-k\cdot\gamma+m)\right\}$$

$\chi^{1\mathcal{M}}(P,k)$ - the BS amplitude of the deuteron, $P'=P+q$ and $k'=k+q/2$.

The vertex of γNN interaction

$$\Gamma_\mu^{(\mathrm{S})}(q)=\gamma_\mu F_1^{(\mathrm{S})}(q^2)-\frac{\gamma_\mu q\cdot\gamma-q\cdot\gamma\gamma_\mu}{4m}F_2^{(\mathrm{S})}(q^2)$$

is chosen to be the form factor on mass shell.

The isoscalar form factors of the nucleon

$$F_{1,2}^{(\mathrm{S})}(q^2)=(F_{1,2}^{(\mathrm{p})}(q^2)+F_{1,2}^{(\mathrm{n})}(q^2))/2$$

with normalization condition

$$F_1^{(\mathrm{S})}(0)=1/2,\quad F_2^{(\mathrm{S})}(0)=(\varkappa_p+\varkappa_n)/2$$

with $\varkappa_p=\mu_p-1$ and $\varkappa_n=\mu_n$ being anomalous parts of the proton μ_p and neutron μ_n magnetic moments, respectively.

Analytic structure

After the partial-wave decomposition the matrix element of the deuteron current has the following form

$$\langle D'\mathcal{M}'|j_\mu|D\mathcal{M}\rangle = \mathcal{I}_{1\ \mu}^{\mathcal{M}'\mathcal{M}}(q^2)\ F_1^{(S)}(q^2) + +\mathcal{I}_{2\ \mu}^{\mathcal{M}'\mathcal{M}}(q^2)\ F_2^{(S)}(q^2),$$

$$\mathcal{I}_{1,2\ \mu}^{\mathcal{M}'\mathcal{M}}(q^2) = i\int dp_0\ |\mathbf{p}|^2\ d|\mathbf{p}|\ d(\cos\theta) \sum_{L\prime,L=0,2} \phi_{L\prime}(p_0', |\mathbf{p}'|)\phi_L(p_0, |\mathbf{p}|)$$
$$\times I_{1,2\ \mathcal{M}'\mathcal{M}\ \mu}^{L',L}(p_0, |\mathbf{p}|, \cos\theta, q^2),$$

where the function $I_{1,2\ \mathcal{M}'\mathcal{M}\ \mu}^{L',L}(p_0, |\mathbf{p}|, \cos\theta, q^2)$ is the result of the trace calculations. The radial part of the amplitude is

$$\phi_L(p_0, |\mathbf{p}|) = S_{++}(p_0, |\mathbf{p}|) g_L(p_0, |\mathbf{p}|), \tag{2}$$

with $g_L(p_0, |\mathbf{p}|)$ being the radial part of the vertex function and

$$S_{++}(p_0, |\mathbf{p}|) = \frac{1}{(M_d/2 + p_0 - E_{\mathbf{p}})(M_d/2 - p_0 - E_{\mathbf{p}})}, \tag{3}$$

being the positive energy part of the propagators and the energy $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$.

Analyzing the analytic structure of expressions (2) and (3) we can write the following expression for the poles in the p_0 complex plane:

- initial deuteron
  for propagator $S_{++}(p_0, |\mathbf{p}|)$:

$$\bar{p}_0 = \pm M_d/2 \mp E_{\mathbf{p}} \pm i\epsilon, \qquad (4)$$

  for functions $g_L(p_0, |\mathbf{p}|)$:

$$\bar{p}_0 = \pm E_{\beta_k} \mp i\epsilon, \qquad (5)$$

- final deuteron
  for propagator $S_{++}(p'_0, |\mathbf{p}'|)$:

$$\bar{p}_0 = -(1+4\eta)M_d \pm \pm\sqrt{E_{\mathbf{p}}^2 + 4\xi M_d|\mathbf{p}|\cos\theta + 4\xi^2 M_d^2} \mp i\epsilon, \qquad (6)$$

  for functions $g_{L'}(p'_0, |\mathbf{p}'|)$:

$$\bar{p}_0 = -\eta M_d \pm \sqrt{E_{\beta_k}^2 + 2\xi M_d|\mathbf{p}|\cos\theta + \xi^2 M_d^2} \mp i\epsilon, \qquad (7)$$

with the energy $E_{\beta_k} = \sqrt{\beta_k^2 + \mathbf{p}^2}$, $\eta = Q^2/4M_d^2$ and $\xi = \sqrt{\eta(1+\eta)}$.
To calculate the matrix elements (2) we should perform the Wick rotation procedure.

During the Wick rotation procedure some poles can get into the contour of the p_0 integration. Additionally, the residue in these poles should be taken into account with the folowing threshold value on Q^2:
for the propagator $S_{++}(p'_0, |\mathbf{p}'|)$:

$$Q_0^2 = M_d(2m - M_d),$$

for the functions $g_{L'}(p'_0, |\mathbf{p}'|)$:

$$Q_k^2 = 4M_d\beta_k.$$

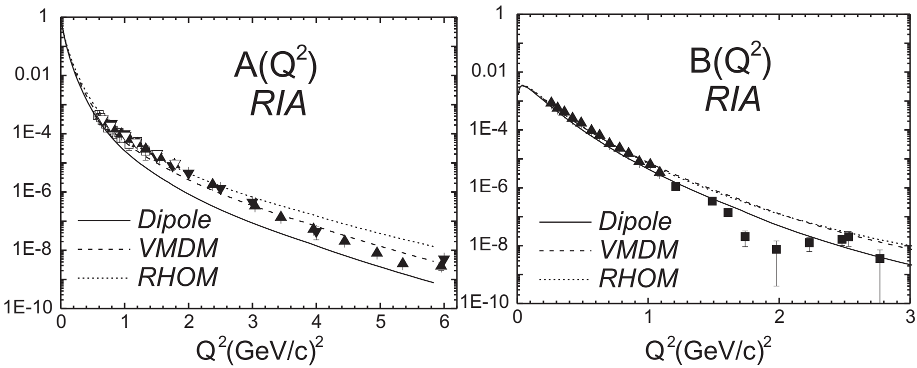
The Wick rotation procedure can be written as:

$$i\int\limits_{-\infty}^{\infty} f dp_0 = \int\limits_{-\infty}^{\infty} f dp_4 - -2\pi \sum_k \theta(Q^2 - Q_k^2)\ Res_k(f, p_0 = \bar{p}_0^k), \tag{8}$$

where the threshold values Q_k^2 for the Graz II kernel are in table.

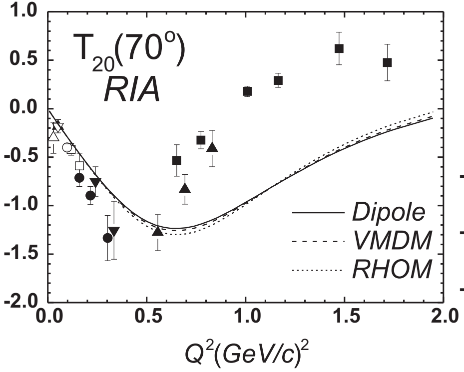
k	Q_k^2 (GeV/c)^2
0	0.004
1	1.182
2	1.736
3	3.915

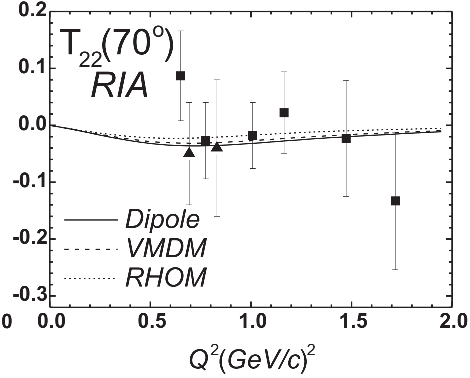
Structure functions $A(q^2)$ and $B(q^2)$



Long and short dashes represent calculations with the VMDM and RHOM nucleon form factors, respectively. The solid curve corresponds to the dipole fit.

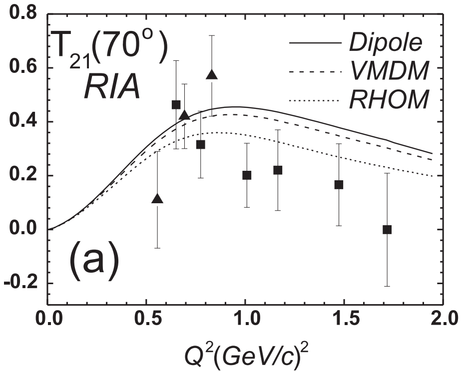
Tensor polarization components $T_{20}(q^2)$ and $T_{22}(q^2)$ calculated at $\theta_e = 70°$.





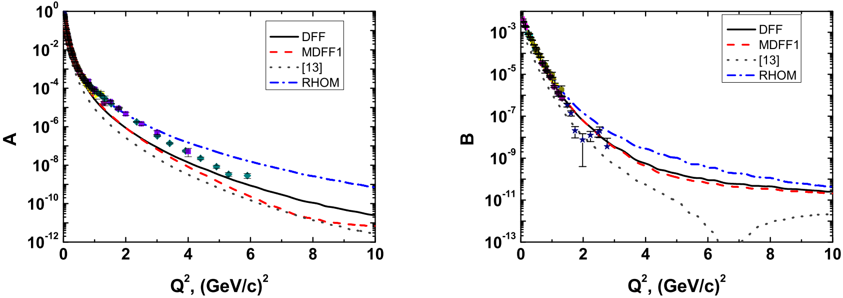
Long and short dashes represent calculations with the VMDM and RHOM nucleon form factors, respectively. The solid curve corresponds to the dipole fit.

Tensor polarization component $T_{21}(q^2)$ calculated at $\theta_e = 70°$



Long and short dashes represent calculations with the VMDM and RHOM nucleon form factors, respectively. The solid curve corresponds to the dipole fit.

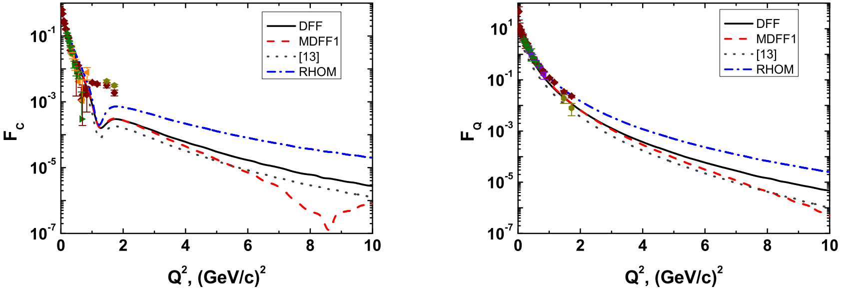
Structure functions $A(q^2)$ and $B(q^2)$ at high momentum transfer



Calculations with DFF (black solid line), MDFF1 (dashed red line), [13] (gray dotted line) and RHOM (blue dashed dot- ted line) nucleon form factors are shown.

[13] C. Adamuscin et al., Nucl. Phys. Proc. Suppl. 245, 69 (2013).

Elastic deuteron form factors at high momentum transfer



Calculations with DFF (black solid line), MDFF1 (dashed red line), [13] (gray dotted line) and RHOM (blue dashed dot- ted line) nucleon form factors are shown.

[13] C. Adamuscin et al., Nucl. Phys. Proc. Suppl. 245, 69 (2013).

Conclusion

- the covariant separable kernel of the nucleon-nucleon can be used to describe the properties of the two-nucleon system and reactions with it
- moving singularities are taken into account
- investigated the contribution of the different models of EM nucleon form factors.