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On the generalized Sundman transformations and integrable Lienard type equations

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Liénard–type equations

- We consider the following family of equations

$$y_{zz} + f(y)y_z^2 + g(y)y_z + h(y) = 0$$

- Liénard-type equations have a vast range of applications in physics, biology, mechanics e.t.c.
- Recently there has been a great interest in finding classes of integrable Liénard-type equations : the Jacobi last multiplier, the Prelle-Singer method, the classical Lie method, the Chiellini lemma:

W. Nakpim, S.V. Meleshko, SIGMA, (2010); V.K. Chandrasekar et al., Proc. R. Soc. A Math. Phys. Eng. Sci. (2005); S.N. Pandey et al., J. Math. Phys., (2009); S.C. Mancas, H.C. Rosu, Phys. Lett. A., (2013);

T. Harko et al., J. Eng. Math. (2014); A.K. Tiwari et al., Nonlin. Dynamics (2015)

Aim and approach

- Our main aim is to find new classes of integrable Liénard-type equations and to explain some of the previous results from a general perspective
- Our approach is based on the generalized Sundman transformations

$$y_{zz} + f(y)y_z^2 + g(y)y_z + h(y) = 0$$

$$w = F(y) \qquad d\zeta = G(y)dz$$

$$F_y G \neq 0$$

$$w_{\zeta\zeta} + \tilde{f}(w)w_\zeta^2 + \tilde{g}(w)w_\zeta + \tilde{h}(w) = 0$$

Canonical form

- We consider the following equation

$$y_{zz} + f(y)y_z^2 + g(y)y_z + h(y) = 0$$

- **Proposition 1.** This equation can be transformed into the Liénard equation by means of GST with

$$F(y) = y,\ G(y) = \exp\{-\int f dy\}.$$

As a result we get

$$y_{\zeta\zeta} + \tilde{g}(y)y_\zeta + \tilde{h}(y) = 0$$

$$\tilde{g}(y) = \exp\{\int f dy\}g(y),\ \tilde{h}(y) = \exp\{2\int f dy\}h(y)$$

- We assume that $g(y) \not\equiv 0$, i.e. we consider a dissipative case.

Linearization conditions

- **Proposition 2.** The mixed Liénard-type equation can be transformed into $w_{\zeta\zeta}+\sigma w_{\zeta}+2\sigma^2/9w=0$ via GST if

$$h(y)=\frac{2g(y)}{9}e^{-\int f(y)dy}\left[\int e^{\int f(y)dy}g(y)dy+\kappa\right].$$

- As a consequence we obtain criterion for existence of a non-standard Lagrangian (*Z.E. Musielak, J.Phys. A. (2008), CSF (2009), J.L.* Cieślinski, J.L., Nikiciuk, J.Phys.A. (2010), *A.K. Tiwari et al., Acta Mech. 2016*)

$$L=\frac{1}{w_{\zeta}+\frac{2\sigma}{3}w} \quad \overset{\text{GST}}{\Longleftrightarrow} \quad L=\frac{\sigma}{e^{\int f(y)dy}y_z+\frac{2}{3}[\int e^{\int f(y)dy}g(y)dy+\kappa]}$$

- Moreover, this class of linerizable mixed Liénard-type equations contains equations, which are completely integrable by the classical Lie approach, i.e. admit at least 2 dimensional Lie algebra and Chiellini integrable equations.

Connections with Painlevé–Gambier equations

- We study connections of the Liénard equations family with its subcases that are of the Painlevé–Gambier type: we consider Painlevé–Gambier equation of types I-III.

- We omit equations that can be linearized via the GST or are equivalent via the GST. Consequently, we need to consider equations II, V, VII, XV, XXIII, XXV, XXVII and XXVIII.

- As a result, we have obtained 9 criteria for the integrability of Liénard equations.

Example of integrability criterion (1)

- We consider non-canonical form of Painlevé–Gambier equation VII

$$w_{\zeta\zeta} + 3w_{\zeta} + w^3 + 2w = 0$$

$$w = e^{-(\zeta+\zeta_0)}\mathrm{cn}\{e^{-(\zeta+\zeta_0)} - C_1, 1/\sqrt{2}\}$$

- The Liénard-type equation can be transformed into equation VII of the Painlevé–Gambier type via GST if

$$h(y) = \frac{g(y)}{9}\mathcal{G}(\lambda^2\mathcal{G}^2 + 2)e^{-\int f(y)dy}$$

- The corresponding GST have the form

$$F = \lambda\mathcal{G},\; G = \frac{1}{2}g,\; \mathcal{G} = \int e^{\int f(y)dy} g(y)dy + \kappa$$

Example of integrability criterion (2)

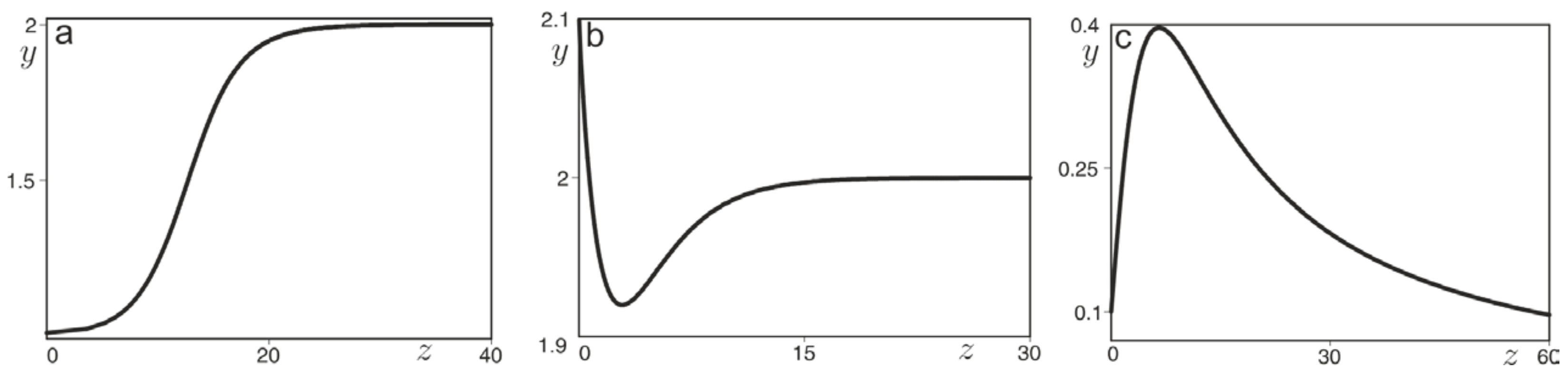
- Traveling wave reduction of the generalized Burgers-Huxley equation

$$y_{zz} + (\alpha y + \beta) y_z + \frac{y}{18}(\alpha y + \beta)(\alpha y + 2\beta)(y^2[\alpha y + 2\beta]^2 + 2) = 0$$

- The general solution has the form

$$y = \frac{1}{\alpha}\left[\pm\left(\beta^2 + \alpha e^{-(\zeta-\zeta_0)} \mathrm{cn}\left\{e^{-(\zeta-\zeta_0)} - C_1, \frac{1}{\sqrt{2}}\right\}\right)^{1/2} - \beta\right],$$

$$z = \int \frac{3}{\alpha y + \beta} d\zeta.$$



The Liénard equation: Ince XXVII (1)

- We consider equation XXVII of the Painlevé–Gambier type

$$w_{\zeta\zeta} - \frac{4}{3}\frac{w_\zeta^2}{w} + \frac{5}{3}\frac{w_\zeta}{w} + 3\epsilon w - \frac{1}{3w} = 0$$

$$w = \frac{\sinh\{\sqrt{\epsilon}(\zeta-\zeta_0)\}(\cosh^2\{\sqrt{\epsilon}(\zeta-\zeta_0)\}+2)+C_1}{3\sqrt{\epsilon}\cosh^3\{\sqrt{\epsilon}(\zeta-\zeta_0)\}}$$

- Lagrangian, first integral and Jacobi multiplier

$$L = -\frac{2}{3}\frac{9\epsilon w^2 - 2(w_\zeta - 1)^2}{\sqrt{9\epsilon w^2 - (w_\zeta - 1)^2}}$$

$$I = \frac{2\left(81\epsilon^2 w^4 + 27\epsilon(w_\zeta - 1)w^2 - 2(w_\zeta - 1)^3\right)}{3(9\epsilon w^2 - (w_\zeta - 1)^2)^{3/2}}$$

$$M = \frac{162\epsilon^2 w^4}{(9\epsilon w^2 - (w_\zeta - 1)^2)^{5/2}}$$

The Liénard equation: Ince XXVII (2)

- Criterion of equivalence between the Liénard equation and Painlevé–Gambier XXVII

$$h(y) = -\frac{4}{25} g(y)\mathcal{G}\left[9\epsilon\lambda^2\mathcal{G}^{-3/2} - 1\right] e^{-\int f(y)dy}$$

$$F = \lambda\mathcal{G}^{-3/4},\ G(y) = \frac{3\lambda}{5} g(y)\mathcal{G}^{-3/4},\ \mathcal{G} = \int e^{\int f(y)dy} g(y)dy + \kappa$$

- The corresponding Lagrangian, first integral and Jacobi multiplier

$$L = -\frac{2}{3}\frac{9\epsilon\lambda^2\mathcal{G}^{-3/2} - 2\left[\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right]^2}{\left(9\epsilon\lambda^2\mathcal{G}^{-3/2} - \left[\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right]^2\right)^{1/2}}$$

$$M = \frac{2025\epsilon^2\lambda^4\mathcal{G}^{-5}e^{2\int f dy}}{8\left(9\epsilon\lambda^2\mathcal{G}^{-3/2} - \left[\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right]^2\right)^{5/2}}$$

$$I = \frac{2}{3}\left[81\epsilon^2\lambda^4\mathcal{G}^{-3} - 27\epsilon\lambda^2\mathcal{G}^{-3/2}\left(\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right) + 2\left(\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right)^3\right]$$
$$\left(9\epsilon\lambda^2\mathcal{G}^{-3/2} - \left[\frac{5}{4}\mathcal{G}^{-1}e^{\int f dy} y_z + 1\right]^2\right)^{-3/2}$$

The Liénard equation: Ince XXVIII (1)

- We consider equation XXVIII of the Painlevé–Gambier type

$$w w_{\zeta\zeta} - \frac{1}{2} w_\zeta^2 + w^2 w_\zeta - \frac{1}{2} w^4 + 72H = 0$$

$$w = \frac{6(\wp^2\{\zeta - \zeta_0, 12H, g_3\} - H)}{\wp_z(\zeta - \zeta_0, 12H, g_3)}$$

- Lagrangian, first integral and Jacobi multiplier

$$L = -\frac{(w_\zeta + w^2)^2}{3w^2}\left[(w_\zeta + w^2)^2 - 864H\right] + \frac{20736H^2}{w^2}$$

$$I = p w_\zeta - L = -\frac{(3w_\zeta - w^2)(w_\zeta + w^2)^3}{3w^2} + \frac{288H(w_\zeta^2 - w^4 - 72H)}{w^2}$$

$$M = 4[144H - (w_\zeta + w^2)^2]w^{-2}$$

The Liénard equation: Ince XXVIII (2)

- Criterion of equivalence between the Liénard equation and Painlevé–Gambier XXVIII

$$h(y) = (3/4)g(y)\lambda^{-4}\mathcal{G}^{-5/3}\left[144H - \lambda^4\mathcal{G}^{8/3}\right]$$

$$F(y) = \lambda\mathcal{G}^{2/3},\ G = g(y)\lambda^{-1}\mathcal{G}^{-2/3},\ \mathcal{G} = \int e^{\int f(y)dy} g(y)dy + \kappa$$

- The corresponding Lagrangian, first integral and Jacobi Multiplier

$$L = -(\lambda^2/3)\left[\tfrac{2}{3}y_z + \mathcal{G}\right]^2 \mathcal{G}^{-2/3}\left[\mathcal{G}^{2/3}\lambda^4\left[\tfrac{2}{3}y_z + \mathcal{G}\right]^2 - 864H\right] + \frac{20736H^2}{\lambda^2\mathcal{G}^{4/3}}.$$

$$I = -\frac{\lambda^6(2y_z - \mathcal{G})}{3}\left(\tfrac{2}{3}y_z + \mathcal{G}\right)^3 + \frac{288H\lambda^2}{\mathcal{G}^{2/3}}\left(\tfrac{4}{9}y_z^2 - \mathcal{G}^2\right) - \frac{20736H^2}{\lambda^2\mathcal{G}^{4/3}}.$$

$$M = \frac{576H}{\lambda^2\mathcal{G}^{4/3}} - 4\lambda^2\mathcal{G}^{-2/3}\left(\tfrac{2}{3}y_z + \mathcal{G}\right)^2$$

Example: the modified Emden equation

- We assume that $f(y)=0,\ g(y)=\alpha y,\ \kappa=0,\ \lambda^2=(25/36)(2\alpha)^{-1/2}$

$$y_{zz}+\alpha y y_z+\frac{2\alpha^2}{25}y^3-\epsilon=0$$

- Lagrangian, first integral and Jacobi multiplier

$$L=-\frac{2(25\epsilon y-(5y_z+2\alpha y^2)^2)}{3\alpha y^2\sqrt{50\epsilon y-(5y_z+2\alpha y^2)^2}}$$

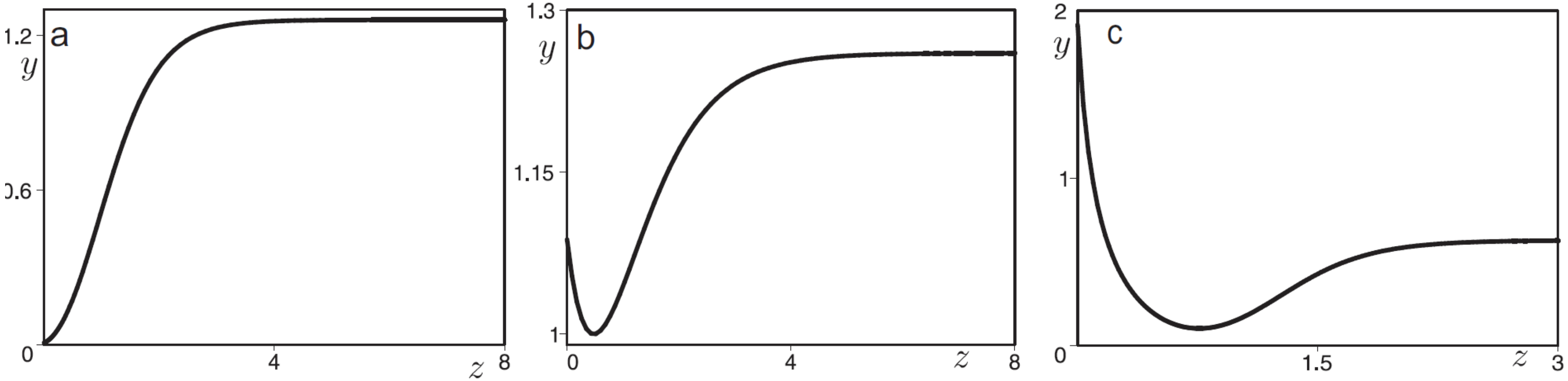
$$I=\frac{2500\epsilon^2-300\epsilon\alpha y(5y_z+2\alpha y^2)+4\alpha(5y_z+2\alpha y^2)^3}{3\alpha(50\epsilon y-(5y_z+2\alpha y^2)^2)^{3/2}}$$

$$M=\frac{62500\epsilon^2}{(50\epsilon y-(5y_z+2\alpha y^2)^2)^{5/2}}$$

Example: the modified Emden equation

- The general solution has the form

$$y = \left[\frac{5\sqrt{2\epsilon}}{2\alpha} \frac{\cosh^3\{\sqrt{\epsilon}(\zeta-\zeta_0)\}}{\sinh\{\sqrt{\epsilon}(\zeta-\zeta_0)\}(\cosh^2\{\sqrt{\epsilon}(\zeta-\zeta_0)\}+2)+C_1} \right]^{2/3}, \; z = \int \sqrt{2y} d\zeta.$$



Example: trigonometric nonlinearity

- We suppose that $f(y) = -\tanh(y),\ g(y) = y^3\cosh(y),\ \kappa = 0$

$$y_{zz} - \tanh(y)y_z^2 + y^3\cosh(y)y_z + \frac{1}{25}(y^6 - 72\epsilon\lambda^2)y\cosh^2 y = 0$$

- Lagrangian, first integral and Jacobi multiplier

$$L = \frac{4}{3}\frac{(5y_z+y^4\cosh(y))^2-36\epsilon\lambda^2y^2\cosh^2(y)}{\cosh(y)y^4\sqrt{72\epsilon\lambda^2y^2\cosh^2(y)-(5y_z+y^4\cosh(y))^2}}$$

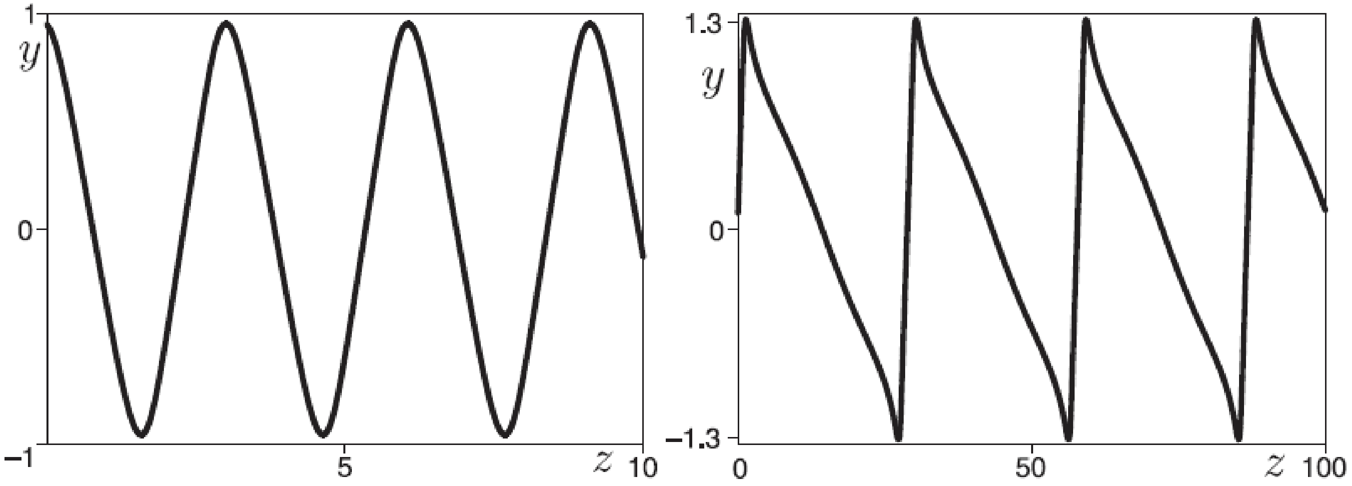
$$M = \frac{259200\epsilon^2\lambda^4\cosh^3(y)}{(72\epsilon\lambda^2y^2\cosh^2(y)-(5y_z+y^4\cosh(y))^2)^{5/2}}$$

$$I = \frac{4}{3}\left[2592\epsilon^2\lambda^4\cosh^3(y) - 108\epsilon\lambda^2y^2\cosh^2(y)(5y_z + y^4\cosh(y)) + \right.$$
$$\left. +(5y_z + y^4\cosh(y))^3\right](72\epsilon\lambda^2y^2\cosh^2(y) - (5y_z + y^4\cosh(y))^2)^{-3/2}$$

Example: trigonometric nonlinearity

- The general solution can be expressed as follows

$$y = \left[\frac{6\lambda\sqrt{2\epsilon}\cosh^3\{\sqrt{\epsilon}(\zeta-\zeta_0)\}}{\sinh\{\sqrt{\epsilon}(\zeta-\zeta_0)\}(\cosh^2\{\sqrt{\epsilon}(\zeta-\zeta_0)\}+2)+C_1} \right]^{1/3}, z = \frac{5}{6\sqrt{2}\lambda}\int \cosh^{-1}(y)d\zeta.$$



Example: rational nonlinearity

- We assume that $f(y)=1/(2y),\ g(y)=-a^3/y^3,\ \kappa=0,\ \lambda=2^{-2/3}3^{-1/3}$

$$y_{zz}+\frac{1}{2y}y_z^2-\frac{a^3}{y^3}y_z+\frac{a^6}{2y^5}-\frac{5832H}{a^2y}=0$$

- The corresponding Lagrangian, first integral and Jacobi last multiplier

$$L=-\frac{\left(a^3+y^2y_z\right)^4}{2187y^6}+\frac{32H\left(a^3+y^2y_z\right)^2}{a^2y^2}+\frac{186624H^2y^2}{a^4}$$

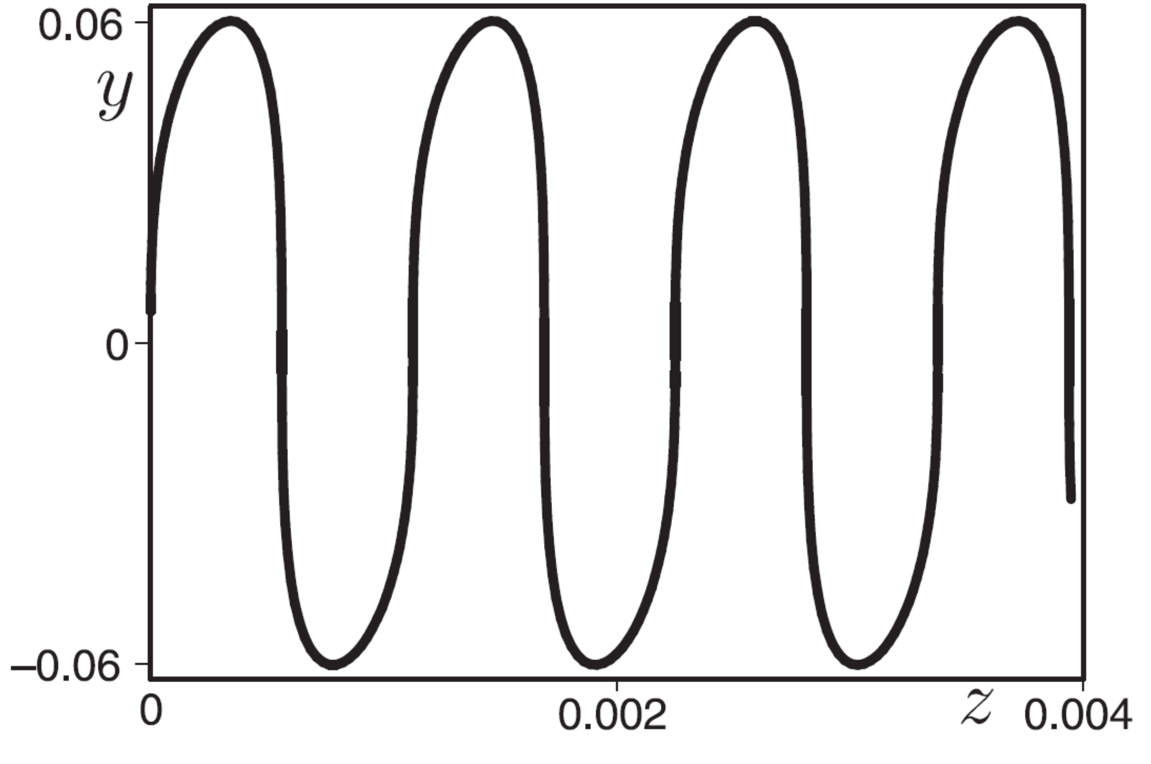
$$I=-\frac{\left(3\,y^2y_z-a^3\right)\left(y^2y_z+a^3\right)^3}{2187y^6}+\frac{32H\left(y^2y_z-a^3\right)\left(y^2y_z+a^3\right)}{y^2a^2}-\frac{186624H^2y^2}{a^4}$$

$$M=\frac{64Hy^2}{a^2}-\frac{4(y^2y_z+a^3)^2}{729y^2}$$

Example: rational nonlinearity

- The general solution has the form

$$y = \frac{a^2 \wp_z(\zeta - \zeta_0, 12\beta, g_3)}{18(\wp^2\{\zeta - \zeta_0, 12\beta, g_3\} - \beta)}, \qquad z = -\frac{1}{3a}\int y^2 d\zeta$$



Conclusion

- We have demonstrated that some of previously known analytical results, e.g. integrability criteria and Lagrangian structures, for the Liénard equation follows from linearizability by GST
- We have found new criteria for the integrability of the Liénard-type equations and constructed new expamples of integrable Liénard-type equations
- We have demonstrated that our approach can be applicable for finding Lagrangians, first integrals and Jacobi multipliers for the Lienard-type equations

Thank you for your attention!