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Higher – order partial differential equations for description of the Fermi-Pasta-Ulam an the Kontorova-Frenkel models

MMCP, Dubna, July 7, 2017

Outline of the Talk

1. A little bit of the history (KF model, FPU model, FPU paradox, Concept of Soliton is the explanation of the FPU paradox by ZK, 50 years of IST)
2. Properties of the higher-order nonlinear partial differential equation corresponding to FPU model.
3. Properties of higher-order nonlinear partial differential equation corresponding to the generalized of the KP model and the FPU model.

The Frenkel-Kontorova-Fermi-Pasta-Ulam model

$$m\,\frac{d^2 y_i}{dt} = (y_{i+1} - 2\,y_i + y_{i-1})\ [\mathbf{k} + \alpha\,(y_{i+1} - y_{i-1}) +$$

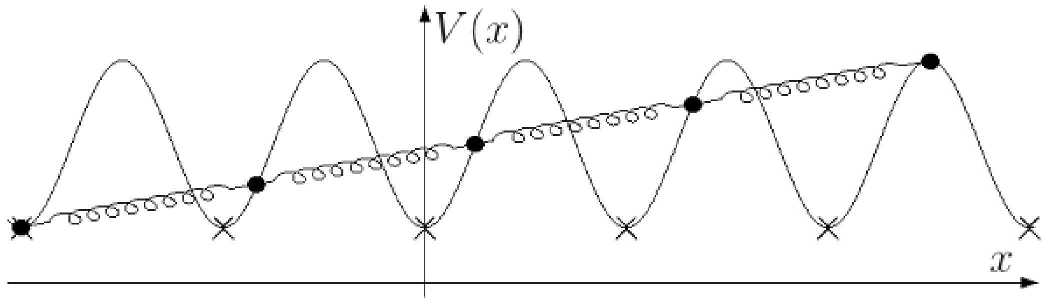
$$+\beta\,(y_{i+1}^2 + y_i^2 + y_{i-1}^2 - y_{i+1}y_i - y_{i+1}y_{i-1} - y_i y_{i-1})\Big] - f_0\,\sin\left(\frac{2\,\pi\,y_i}{a}\right),$$

$$(i = 1, \ldots, N),$$

where y_i measures the displacement of the i-th mass from equilibrium in time t, the force $F_{i+1,i}$ describes the nonlinear interaction between atoms in the crystal lattice in case of dislocations

$$F_{i+1,i} = \gamma\,(y_{i+1} - y_i) + \alpha\,(y_{i+1} - y_i)^2 + \beta\,(y_{i+1} - y_i)^3, \qquad (2)$$

and f_0, a, γ, α, β are constant parameters of system (1).



The Frenkel-Kontorova model

$$m\,\frac{d^2 y_i}{dt} = k\,(y_{i+1} - 2\,y_i + y_{i-1}) - f_0\,\sin\left(\frac{2\,\pi\,y_i}{a}\right), \quad (i = 1, \dots, N),$$

The system of equations (1) is the generalization of some well-known dynamical systems. At $\alpha = 0$ and $\beta = 0$ the system of equations (1) is the mathematical model introduced by Frenkel and Kontorova for the description of dislocations in the rigid body. It was suggested that the influence of atoms in the crystal is described by term $f_0\,\sin\frac{2\,\pi\,y_i}{a}$ but the atoms in case of dislocations interact by means of linear low. Assuming that $N \to \infty$ and $h \to 0$ where h is the distance between atoms, we can get the Sine-Gordon equation.

$$u_{x\tau} = \sin u,$$

T.A. Kontorova, Ya. I. Frenkel, On theory of plastic deformation, 8 (1938) JETP, 89, 1340, 1349 (in Russian)

The Fermi-Pasta-Ulam model

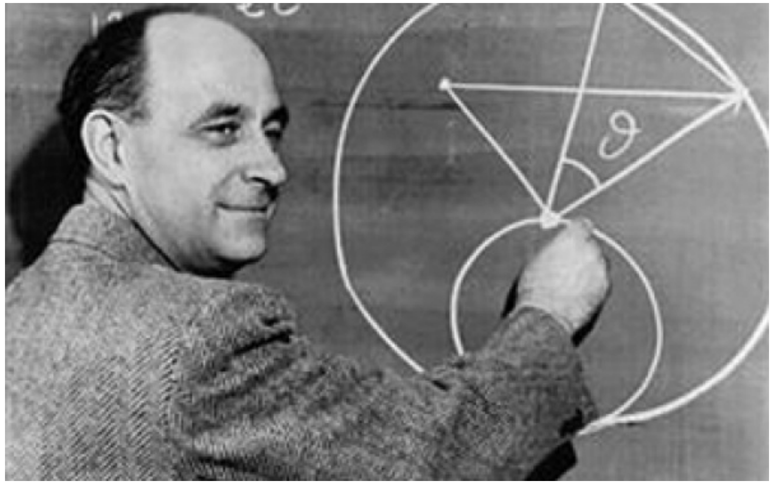
In case of $f = 0$ and $\beta = 0$ system of equations (1) is the well-known Fermi-Pasta-Ulam model which was studied many times.

$$m\,\frac{d^2 y_i}{dt} = (y_{i+1} - 2\,y_i + y_{i-1})\,\left[\mathbf{k} + \alpha\,(y_{i+1} - y_{i-1})\right], \quad (i = 1, \ldots, N),$$

E. Fermi, J.R. Pasta, S.Ulam, Studies of nonlinear problems, Report LA-1940, 1955. Los Alamos: Los Alamos Scientific Laboratory

Result of numerical modeling is the FPU paradox.

The Fermi – Pasta – Ulam model (1)

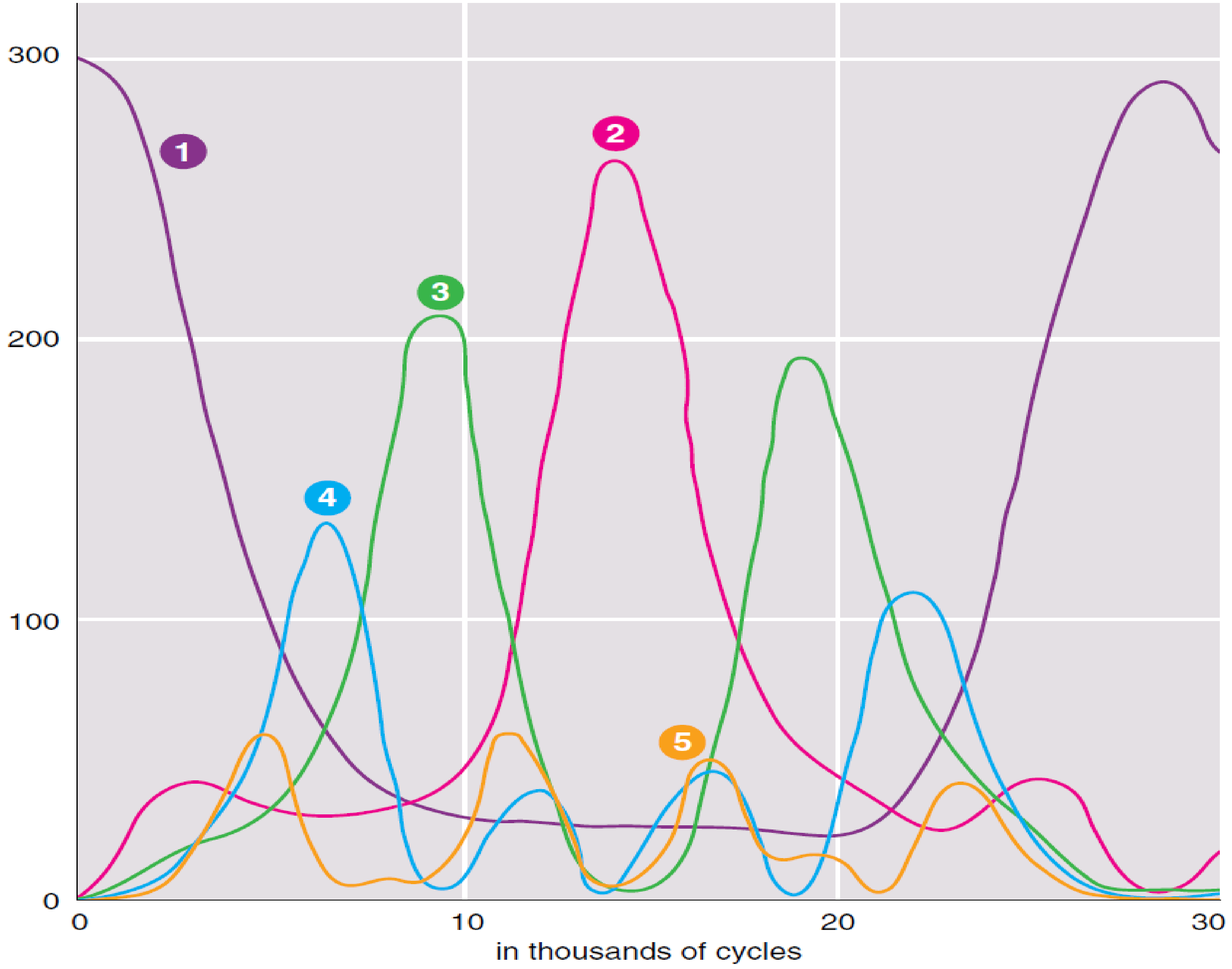


Enriko Fermi (1901-1954)

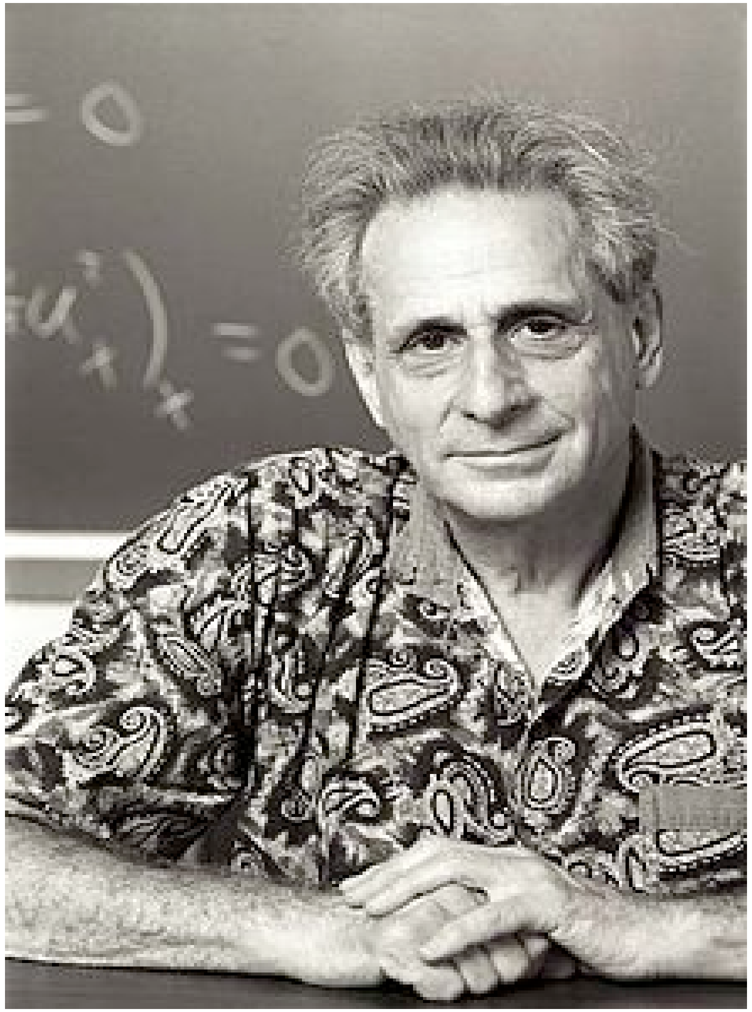
Jhon Pasta (1918-1981)

Stiv Ulam (1909 - 1984)

Fermi-Pasta-Ulam paradox



Martin D Kruskal (September 28, 1925 – December 26, 2006)



The concept of Soliton

The main result of work by M. Kruskal and N. Zabusky was the introduction of solitons as solutions of the Koryeweg-de Vries equation.

$$u_t + u\,u_x + \delta^2\,u_{xxx} = 0,$$

where δ is the parameter of equation.

The paradox was solved in the work by N.J. Zasbusky and M.D. Kruskal in 1965.

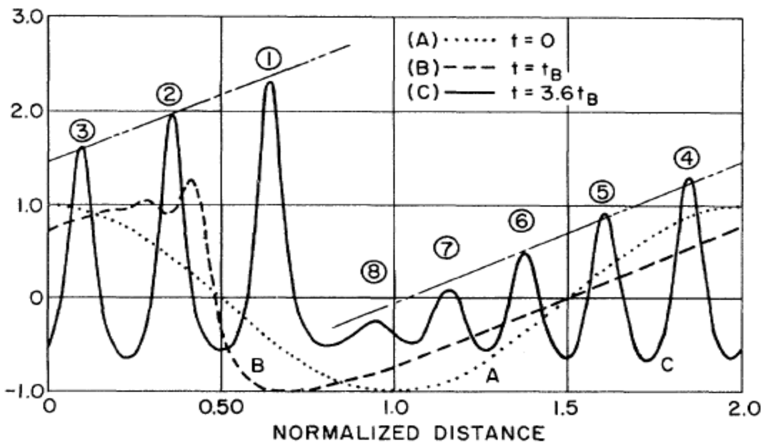


FIG. 1. The temporal development of the wave form $u(x)$.

The Lax pair for the KdV equation

$$\Psi_{xx} + (u + \lambda)\,\Psi = 0,$$

$$\Psi_t = (C + u_x)\,\Psi - 2\,(u - 2\,\lambda)\,\Psi_x.$$

$$(\Psi_{xx})_t = (\Psi_t)_{xx}.$$

$$u_t + 6\,u\,u_x + u_{xxx} = 0$$

C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. (ISSN: 0031-9007) 19 (19) (1967) 1095–1097. http://dx.doi.org/10.1103/PhysRevLett.19.1095.

Two cases of this talk

$$m\,\frac{d^2 y_i}{dt} = (y_{i+1} - 2\,y_i + y_{i-1})\;[\mathbf{k} + \alpha\,(y_{i+1} - y_{i-1}) +$$
$$+\beta\,(y_{i+1}^2 + y_i^2 + y_{i-1}^2 - y_{i+1}y_i - y_{i+1}y_{i-1} - y_i y_{i-1})] - f_0\,\sin\left(\frac{2\,\pi\,y_i}{a}\right),$$
$$(i = 1, \ldots, N),$$

First of all we want to study more exact model than M. Kruskal and N. Zabusky taking into account

$$f_0 = 0, \quad \beta = 0 \quad \alpha \neq 0, \quad k \neq 0, \quad h \to 0, \quad N \to \infty.$$

Secondly we consider

$$f_0 \neq 0, \quad \beta \neq 0, \quad \alpha \neq 0, \quad k \neq 0, \quad h \to 0, \quad N \to \infty.$$

Questions for the Fermi-Pasta-Ulam model:

What higher-order nonlinear equations can be used for description of the Fermi-Pasta-Ulam model?

Properties (integrable or not) of these equations?

Are there the recurrence of initial condition for the higher-order nonlinear partial differential equations?

Are there exact solutions the higher-order nonlinear partial differential equations?

What results of numerical modeling of nonlinear waves in the chain by Fermi-Pasta-Ulam?

Derivation of more exact nonlinear differential equations

Taking into account the expansion in the Teylor series, we have

$$y_{k\pm1} = y_k \pm h\, y_x + \frac{h^2}{2} y_{2\,x} \pm \frac{h^3}{6} y_{3\,x} + \frac{h^4}{24} y_{4\,x} \pm$$

$$\pm \frac{h^5}{120} y_{5\,x} + \frac{h^6}{720} y_{6\,x} + \frac{h^7}{5040} y_{7\,x} + \frac{h^8}{40320} y_{8\,x} + \ldots,$$

where

$$y_x = \frac{dy}{dx}, \qquad y_{m\,x} = \frac{d^m y}{dx^m}, \qquad (m = 2, \ldots, 8).$$

Why we took into account nine terms in expansion in Teylor series?

Derivation of more exact nonlinear differential equations

Using the parameter in the form

$$h=\frac{c\sqrt{m}}{\sqrt{\gamma}}$$

We have

$$y_{tt}-c^2\,y_{xx}=\frac{2\,\alpha\,c^3\,\sqrt{m}}{\gamma\,\sqrt{\gamma}}\,y_x\,y_{2\,x}+\frac{m\,c^2}{12\,\gamma}\,y_{4\,x}+\frac{\alpha\,c^5\,m\sqrt{m}}{6\,\gamma^2\,\sqrt{\gamma}}\,y_x\,y_{4\,x}+$$
$$+\frac{\alpha\,c^5\,m\sqrt{m}}{3\,\gamma^2\,\sqrt{\gamma}}\,y_{2\,x}\,y_{3\,x}+\frac{m^2\,c^6}{360\,\gamma^2}\,y_{6\,x}+\frac{\alpha\,c^7\,m^2\,\sqrt{m}}{60\,\gamma^3\,\sqrt{\gamma}}\,y_{2\,x}\,y_{5,x}+$$
$$+\frac{\alpha\,c^7\,m^2\,\sqrt{m}}{36\,\gamma^3\,\sqrt{\gamma}}\,y_{3\,x}\,y_{4\,x}+\frac{\alpha\,c^7\,m^2\,\sqrt{m}}{180\,\gamma^3\,\sqrt{\gamma}}\,y_x\,y_{6\,x}+\frac{m^3\,c^8}{20160\,\gamma^3}\,y_{8\,x}.$$

Derivation of nonlinear evolution differential equations

$$y(x,t) = f(x', T) + \varepsilon\, y_1(x,t),$$

where

$$x' = x - c\,t, \qquad T = \varepsilon\, t$$

(primes are omitted) we obtain the equation in the form

$$2\, f_{xT} + \frac{m^2 c^3}{12\,\gamma}\, f_{xxxx} + \frac{2\,\alpha\, c^2 \sqrt{m}}{\gamma\,\sqrt{\gamma}}\, f_x\, f_{xx} + \frac{c^5 m^3}{360\,\gamma^2}\, f_{xxxxxx}$$
$$+ \frac{\alpha\, c^4 m^2 \sqrt{m}}{3\,\gamma^2 \sqrt{\gamma}}\, f_{xx}\, f_{xxx} + \frac{\alpha\, c^4 m^2 \sqrt{m}}{6\,\gamma\,\sqrt{\gamma}}\, f_x\, f_{xxxx} = 0.$$

Assuming

$$\tau = \frac{c\,T}{2}, \qquad u = f_x,$$

Nonlinear evolution equation of fifth order

$$u_\tau + \frac{m\,c^2}{12\,\gamma}\left(u_{xxx} + \frac{24\,\alpha}{c\,\sqrt{m\,\gamma}}\,u\,u_x\right) + \frac{m^2\,c^4}{360\,\gamma^2}\left(u_{xxxxx} + \frac{120\,\alpha}{c\,\sqrt{m\,\gamma}}\,u_x\,u_{xx} + \frac{60\,\alpha}{c\,\sqrt{m\,\gamma}}\,u\,u_{xxx}\right) = 0.$$

Taking into account variables in the form:

$$\delta^2 = \frac{mc^2}{12\,\gamma}, \quad u' = \frac{c\,\sqrt{m\,\gamma}}{24\,\alpha\,\delta^2}, \quad t = \tau \tag{13}$$

(primes are omitted too) we write the fifth-order nonlinear evolution equation (10) can be written in the form

Nonlinear higher-order differential equations:

$$u_t + u\,u_x + \delta^2\,u_{xxx} + 2\,\delta^2\,u_x\,u_{xx} + \delta^2\,u\,u_{xxx} + \frac{2}{5}\,\delta^4\,u_{xxxxx} = 0. \qquad (16)$$

$$\begin{gathered} u_t + u\,u_x + \delta^2\,u_{xxx} + 2\,\delta^2\,u_x\,u_{xx} + \delta^2\,u\,u_{xxx} + \frac{2}{5}\,\delta^4\,u_{xxxxx} + \\ + \frac{2}{5}\,\delta^4\,u\,u_{xxxxx} + 2\,\delta^4\,u_{xx}\,u_{xxx} + \frac{6}{5}\,\delta^4\,u_x\,u_{xxxx} + \frac{3}{35}\,\delta^6\,u_{xxxxxxx} = 0. \end{gathered} \qquad (17)$$

N.A. Kudryashov, Refinment of the Korteweg-de Vries equation from the Fermi-Pasta-Ulam. Phys Lett A 2015; 379 (40-41):2610-14

Painleve test for fifth-order differential equation

$$u_t + uu_x + \delta^2 u_{xxx} + 2\delta^2 u_x u_{xx} + \delta^2 u u_{xxx} + \frac{2}{5}\delta^4 u_{xxxxx} = 0.$$

Using the variables

$$u(x,t) = w(z), \qquad z = x - C_0\, t$$

we have

$$\frac{2}{5}\delta^4\, w_{zzzz} + \delta^2\, w\, w_{zz} + \frac{1}{2}\delta^2\, w_z^2 + \frac{1}{2}\, w^2 + \delta^2\, w_{zz} - C_0\, w + C_1 = 0$$

$$(a_0, p) = (-1, 2).$$

$$j_1 = -1, \qquad j_2 = 8, \qquad j_{3,4} = \frac{7}{2} \pm \frac{i\sqrt{11}}{2}.$$

First integral for the fifth-order differential equation

There is the first integral for the fifth-order differential equation

$$I_2 = C_2 + C_1\, w - \frac{C_0}{2}\, w^2 + \frac{1}{6}\, w^3 + \frac{1}{2}\, \delta^2\, w_z^2 + \frac{1}{2}\, \delta^2\, w\, w_z^2 +$$

$$+\frac{2}{5}\, \delta^4\, w_z\, w_{zzz} - \frac{1}{5}\, \delta^4\, w_{zz}^2 = 0$$

$$j_1 = -1, \qquad j_{2,3} = \frac{7}{2} \pm \frac{i\sqrt{11}}{2}.$$

Results of the Painleve test of higher-order equations

- The fifth and seventh-order nonlinear evolution equations do not pass the Painleve test

- The Cauchy problems for these equations cannot be solved by means of the inverse scattering transform.

- There are expansions for solutions of these equations in the Laurent series.

- Some exact solutions of these equations can be found.

Exact solution of higher order differential equation

$$C_0 = -\frac{1}{2\beta} - \frac{\beta k^4}{10}, \qquad C_1 = \frac{\beta k^6}{120} + \frac{1}{48\beta^2} - \frac{k^4}{80}.$$

$$\begin{aligned} w_1(z) &= -\frac{1+\beta k^2}{12\beta} + \frac{k^2}{4}\left[1 - \tanh^2\left(\frac{k(z-z_0')}{2}\right)\right] \\ &= -\frac{1+\beta k^2}{12\beta} + \frac{k^2}{4}\cosh^{-2}\left(\frac{k(z-z_0')}{2}\right) \end{aligned}$$

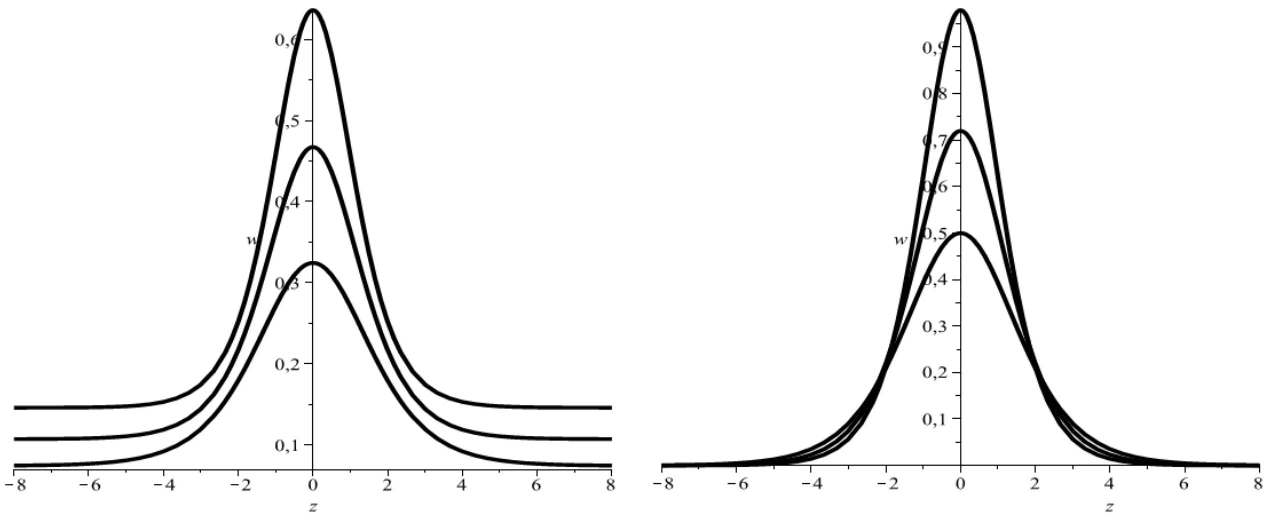
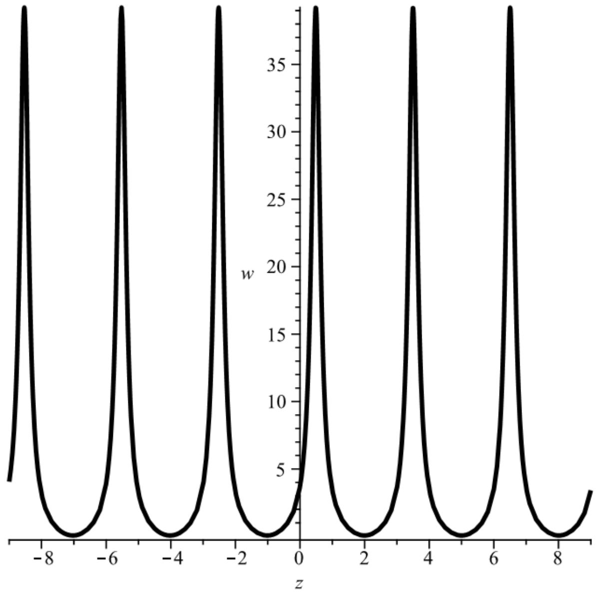


Fig. 1. Solitary wave solutions $w_{(1)}(z)$ (left) and $w_{KdV}(z)$ (right), of equations (17) and (15) at $k_1 = 1.0$, $k_2 = 1.2$, $k_3 = 1.4$ and at $C_0^{(1)} = 1$, $C_0^{(2)} = 1.44$, $C_0^{(3)} = 1.96$.

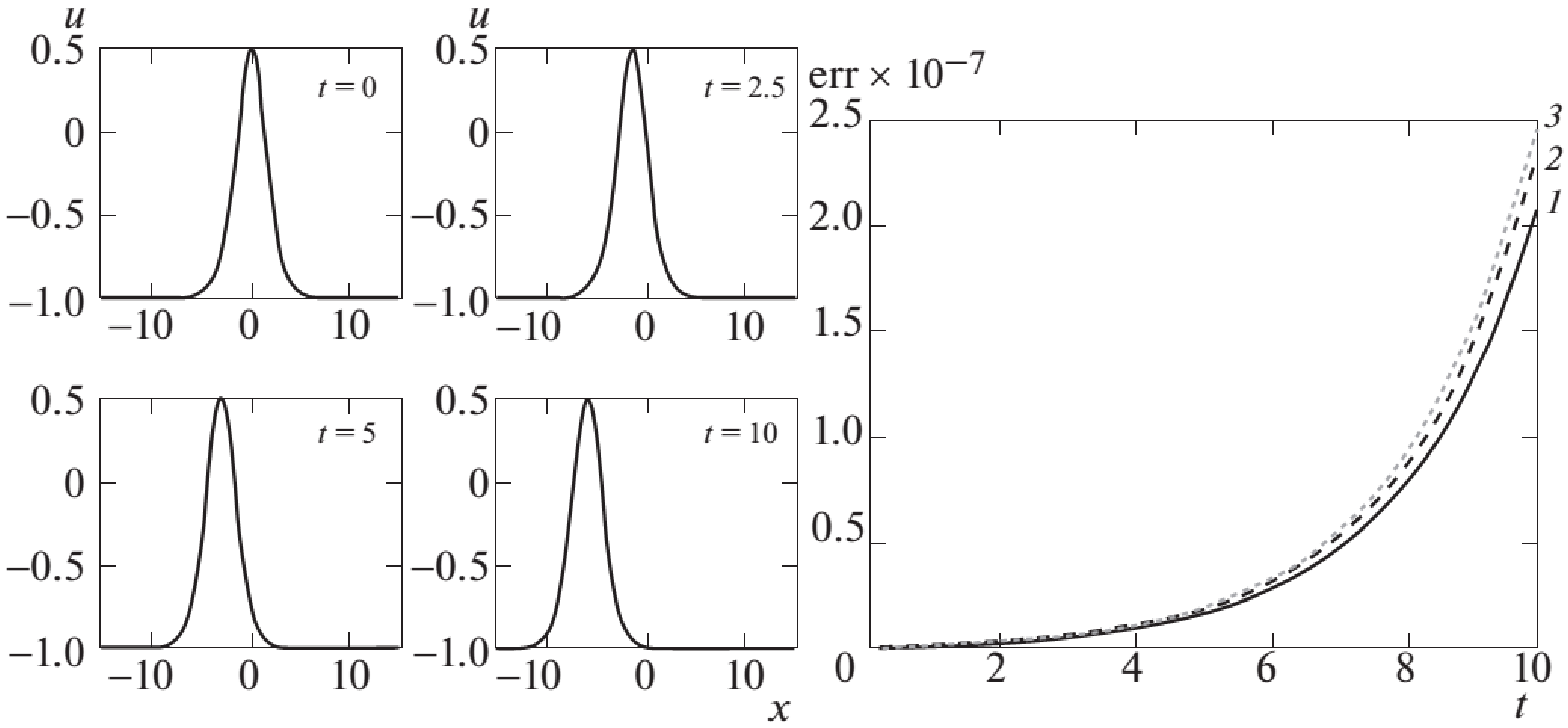
Elliptic solution of higher order differential equation

$$w(z) = A + B\,\wp(z - z_0, g_2, g_3),$$

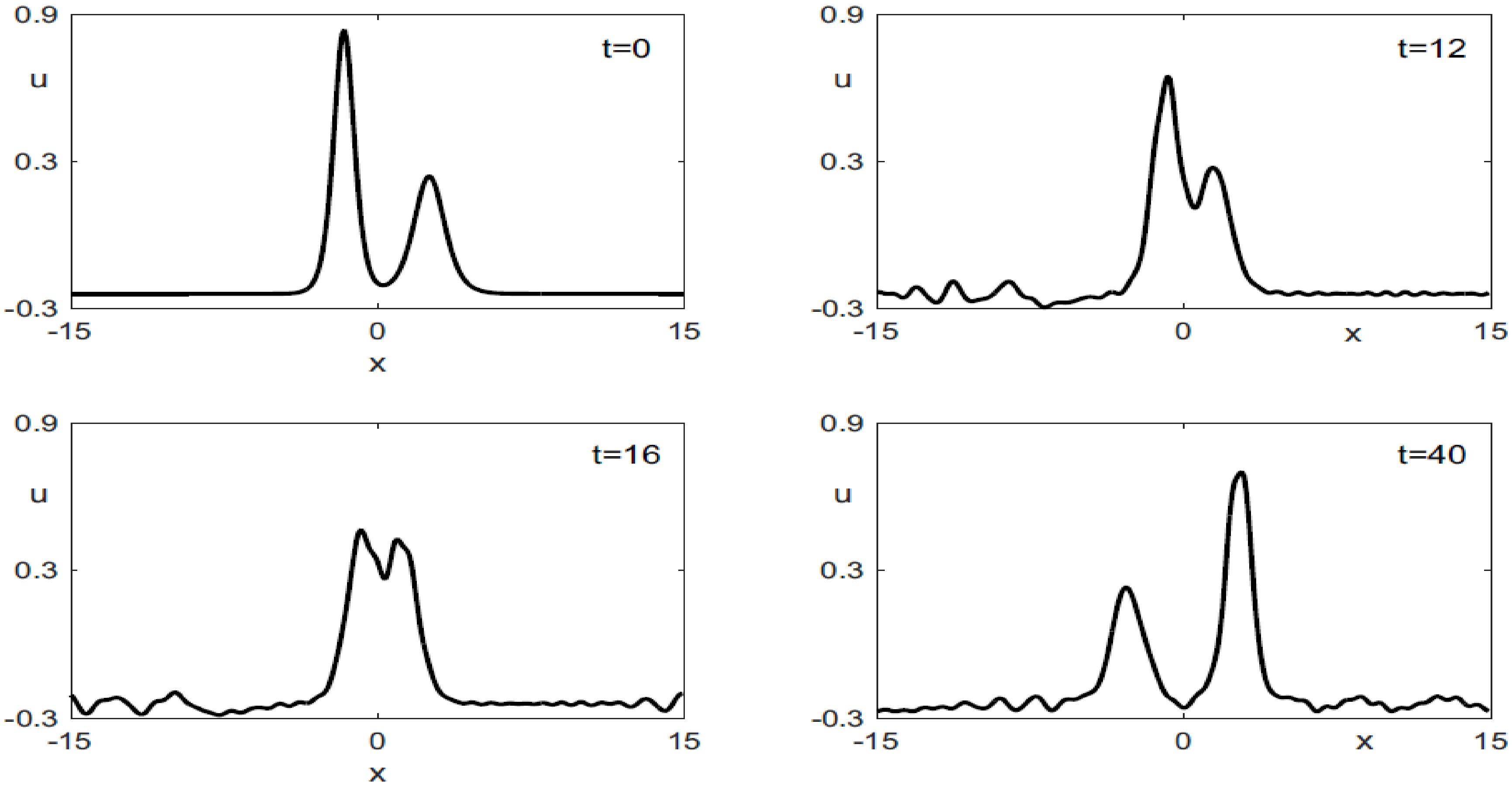
$$w_4(z) = -\frac{1}{2} - 6\,\delta^2 \wp\left(z - z_0,\ \frac{5}{12}\,\frac{2\,C_0 - 1}{\delta^4},\ -\frac{5}{216}\,\frac{3\,C_0 + 4\,C_1 - 2}{\delta^6}\right).$$



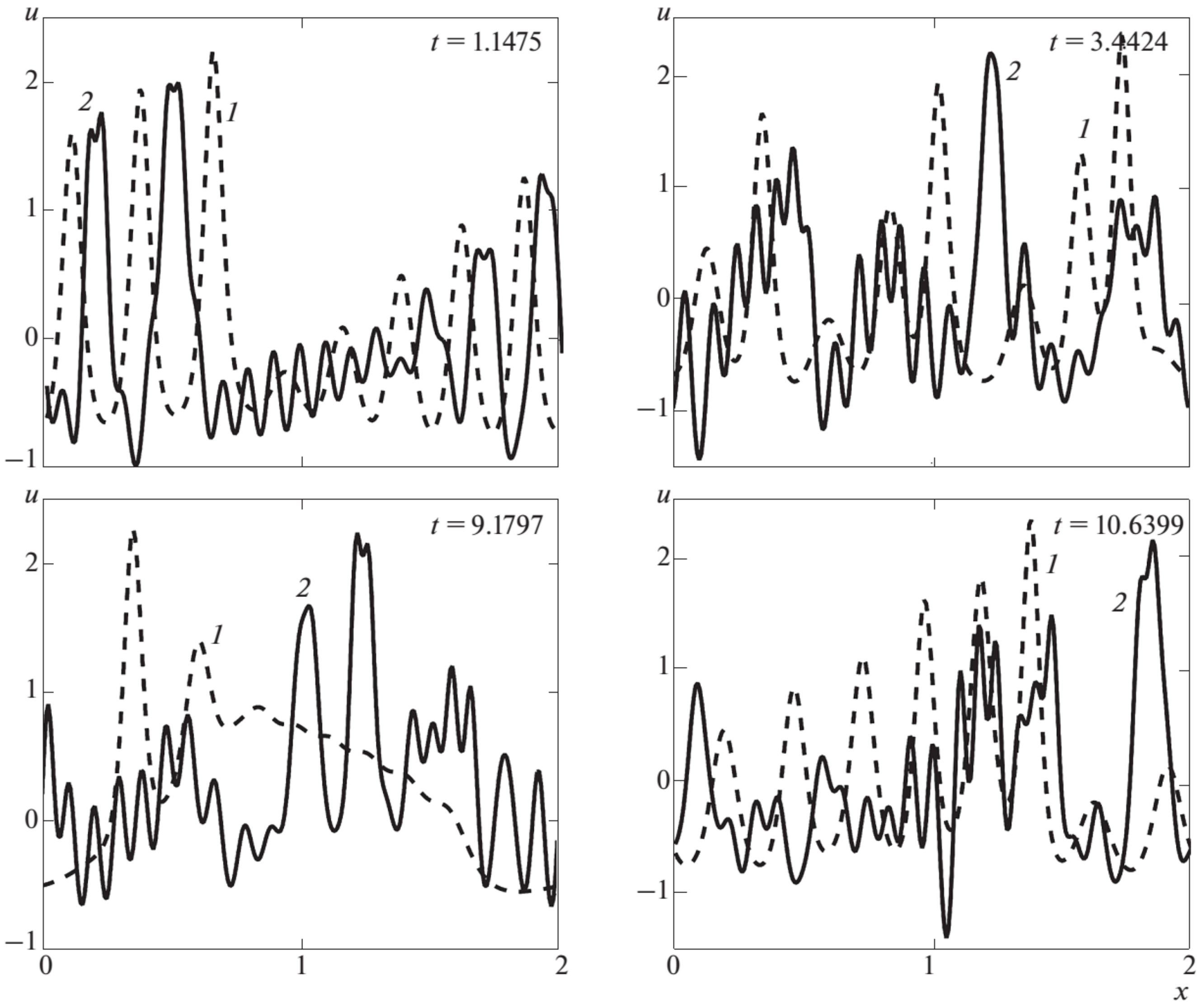
Test for mathematical modeling



Interactions of solitary waves for the fifth-order differential equations



Comparison with numerical solution of the KdV model



The Frenkel-Kontorova-Fermi-Pasta-Ulam model

$$m\,\frac{d^2 y_i}{dt} = (y_{i+1} - 2\,y_i + y_{i-1})\;[\mathbf{k} + \alpha\,(y_{i+1} - y_{i-1}) +$$

$$+\beta\,(y_{i+1}^2 + y_i^2 + y_{i-1}^2 - y_{i+1}y_i - y_{i+1}y_{i-1} - y_i y_{i-1})] - f_0\,\sin\left(\frac{2\,\pi\,y_i}{a}\right),$$

$$(i = 1, \ldots, N),$$

$$f_0 \neq 0, \quad \beta \neq 0 \quad \alpha \neq 0, \quad k \neq 0, \quad h \to 0, \quad N \to \infty.$$

Questions for the Frenkel-Kontorova-Fermi-Pasta-Ulam model

What higher-order nonlinear partial differential equations can be used for description of the Kontorova-Frenkel-Fermi-Pasta-Ulam model?

Properties (integrable or not) of these equations?

Are there exact solutions the higher-order nonlinear partial differential equations?

What results of numerical modeling of nonlinear waves in the chain by Kontorova-Frenkel-Fermi-Pasta-Ulam?

Equation for description of nonlinear dislocations

$$y_{i\pm1} = y_i \pm h\, y_{i,x} + \frac{h^2}{2} y_{i,xx} \pm \frac{h^3}{6} y_{i,xxx} + \frac{h^4}{24} y_{i,xxxx} + \dots .$$

$$m\, y_{tt} = k\, h^2\, y_{xx} + 2\alpha h^3 y_x\, y_{xx} + 3\beta h^4 y_x^2\, y_{xx} + \frac{kh^4}{12} y_{xxxx} - f_0 \sin\left(\frac{2\pi y}{a}\right).$$

$$c = \frac{h\sqrt{k}}{\sqrt{m}}, \quad \varepsilon = \frac{4\,h\,\pi}{a\,k}, \quad \beta' = \frac{\beta\, h}{2},$$

$$v(x,\tau) = \frac{2\,\pi}{a}\, y(x,\tau), \quad \gamma' = \frac{a\,k\,h}{48\,pi}, \quad \delta = \frac{a\, f_0}{4\,\pi\, h^3}.$$

$$v_{tt} = c^2\, v_{xx} + \alpha\, c^2\, \varepsilon\, v_x\, v_{xx} + 3\,\beta'\, c^2\, \varepsilon\, v_x^2\, v_{xx} + \gamma'\, c^2\, \varepsilon\, v_{xxxx} - \delta\, c^2\, \varepsilon\, \sin v.$$

Equation for description of nonlinear dislocations

Using the nonlinear wave with a motion in the right hand side

$$v(x,t) = u(x^{'},t^{'}) + \varepsilon\, v_1(x,t),$$

where

$$x^{'} = x - c\,t^{'}, \qquad t^{'} = \frac{c}{2}\,\varepsilon\, t$$

(primes are omitted) we obtain the equation in the form

$$u_{xt} + \alpha\, u_x\, u_{xx} + 3\,\beta\, u_x^2\, u_{xx} + \gamma\, u_{xxxx} = \delta\, \sin u.$$

Painleve test of equation for nonlinear dislocations

$$u(x,t) = y(z), \qquad z = x - C_0\, t.$$

$$\gamma\, y_{zzzz} + 3\,\beta\, y_z^2\, y_{zz} + \alpha\, y_z\, y_{zz} - C_0\, y_{zz} - \delta\, \sin y = 0.$$

$y(z) = \frac{i}{2}\, \ln(v(z))$

$$\gamma\, v^3\, v_{zzzz} - 4\,\gamma\, v^2\, v_z\, v_{zzz} + 12\,\gamma\, v\, v_z^2\, v_{zz} - 6\,\gamma\, v_z^4 - i\,\alpha\, v^2\, v_z\, v_{zz} +$$

$$+ i\,\alpha\, v\, v_z^3 - -3\,\beta\, v\, v_z^2\, v_{zz} + 3\,\beta\, v_z^4 - \frac{1}{2}\,\delta\, v^5 + \frac{1}{2}\,\delta\, v^3 -$$

$$-C_0\, v^3\, v_{zz} + C_0\, v^2\, v_z^2 = 0.$$

Painleve test of equation for description of dislocations

$$v = \frac{48(\gamma - 8\,\delta)}{\delta\, z^4} + \ldots.$$

$$j_1 = -1, \quad j_2 = 4, \quad j_{3,4} = \frac{3\,\gamma \pm \sqrt{192\,\beta\,\gamma - 15\,\gamma^2}}{2\,\gamma}.$$

$$\gamma = \frac{48\,\beta}{N^2 - 3\,N + 6},$$

$$j_1 = -1, \quad j_2 = 4, \quad j_3 = N, \quad j_4 = 3 - N.$$

$$\alpha = 0,\ \beta \neq 0,\ \gamma = 2\,\beta.$$

Painleve test of equation for description of dislocations

Table 1
Some results of the Painlevé test: the parameters of Eq. (11) when the Fuchs indices are integers.

N	g	b	j_1	j_2	j_3	a
5	1	1	-1	5	-2	0
6	1	$\frac{3}{2}$	-1	6	-3	0
7	1	$\frac{17}{8}$	-1	7	-4	0
9	1	$\frac{15}{4}$	-1	9	-6	0
11	1	$\frac{47}{24}$	-1	11	-8	0

where N is one integer Fuchs index. In the case we obtain the following Fuchs indices

$$j_1 = -1, \quad j_2 = N, \quad j_3 = 3 - N.$$

Lax pair corresponding to special case **at** $\alpha = 0$ **and** $\gamma = 2\beta$

$$\psi_{1,x} = -i\,\lambda\,\psi_1 - \frac{1}{2}\,u_x\,\psi_2,$$
$$\psi_{2,x} = \frac{1}{2}\,u_x\,\psi_1 + i\,\lambda\,\psi_2,$$

$$\psi_{1,t} = h\,\psi_1 + e\,\psi_2,$$
$$\psi_{2,t} = f\,\psi_1 - h\,\psi_2,$$

$$h = -\frac{i\,\delta}{4\,\lambda}\,\cos u - i\,\beta\,u_x^2\,\lambda + 8\,i\,\beta\,\lambda^3,$$

$$f = -\frac{i\,\delta}{4\,\lambda}\,\sin u - \beta\,u_{xxx} - \frac{\beta}{2}\,u_x^3 + 2\,i\,\beta\,u_{xx}\,\lambda + 4\,\beta\,u_x\,\lambda^2,$$

$$e = -\frac{i\,\delta}{4\,\lambda}\,\sin u + \beta\,u_{xxx} + \frac{\beta}{2}\,u_x^3 + 2\,i\,\beta\,u_{xx}\,\lambda - 4\,\beta\,u_x\,\lambda^2.$$

The solution of the Cauchy problem

$$u_{xt} + 3\,\beta\,u_x^2\,u_{xx} + 2\,\beta\,u_{xxxx} = \delta\,\sin u$$

$$\int_{-\infty}^{\infty} |u_x|\,dx < \infty, \qquad \lim_{x\to\pm\infty} u = \pi\,k.$$

$$K_{11}(x,z,t) + \int_x^{\infty} K_{12}(x,y,t)\,f(y+z;t)\,dy,$$

$$K_{12}(x,z,t) - f(x+z;t) - \int_x^{\infty} K_{11}(x,y;t)\,f(y+z;t)dy = 0,$$

$$f(X;t) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \frac{b(k)}{a(k)}\,\exp\left(ik\,X + \frac{i\,\delta\,t}{2\,k} - i\,\beta\,k^3\,t\right)dk -$$

$$-i\sum_{n=1}^{N} c_n\,\exp\left\{i\,\zeta_n\,X + \frac{i\,\delta\,t}{2\,\zeta_n} - i\,\beta\,\zeta_n^3\,t\right\}.$$

$$u(x,t) = -2\,K_{12}(x,x,t).$$

Multi-soliton solutions of the forth-order equration

$$u(x,t) = -\frac{i}{2}\,\ln\left\{\frac{\det(I + A(x,t))}{\det(I - A(x,t)}\right\},$$

$$A_{k\,j}(x,t) = \frac{\zeta_j}{\lambda_k + \lambda_j}\,\exp\left(i\,\lambda_j\,x + \frac{i\,\delta\,t}{\lambda_j} - 2\,\beta\,\lambda_j^3\,t\right).$$

$$u(x,t) = \pm\,4\,\arctan\left(e^{\theta}\right), \quad \theta = \mu + \lambda\,x + \frac{\delta\,t}{\lambda} - 2\,\beta\,\lambda^3\,t$$

$$u(x,t) = \pm\,4\,\arctan\left(\left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right)\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1 + \theta_2}}\right),$$

$$\theta_i = \mu_i + \lambda_i\,x + \frac{\delta\,t}{\lambda_i} - 2\,\beta\,\lambda_i^3\,t, \qquad (i = 1,\ 2).$$

The evolution of the one -soliton solutions

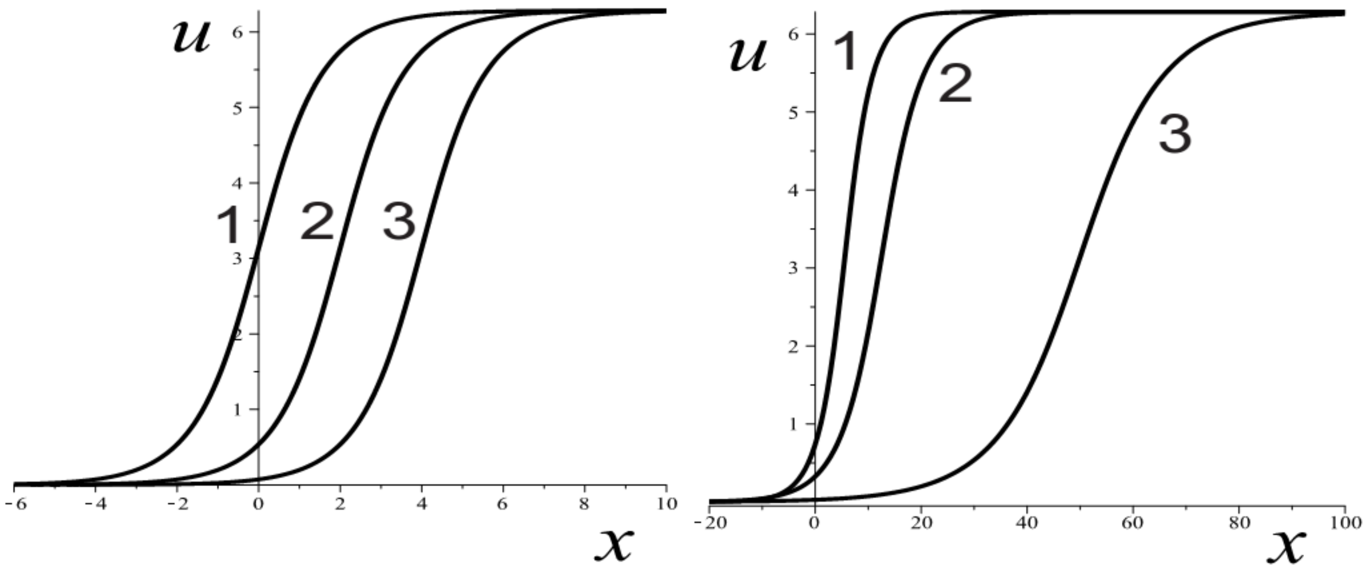
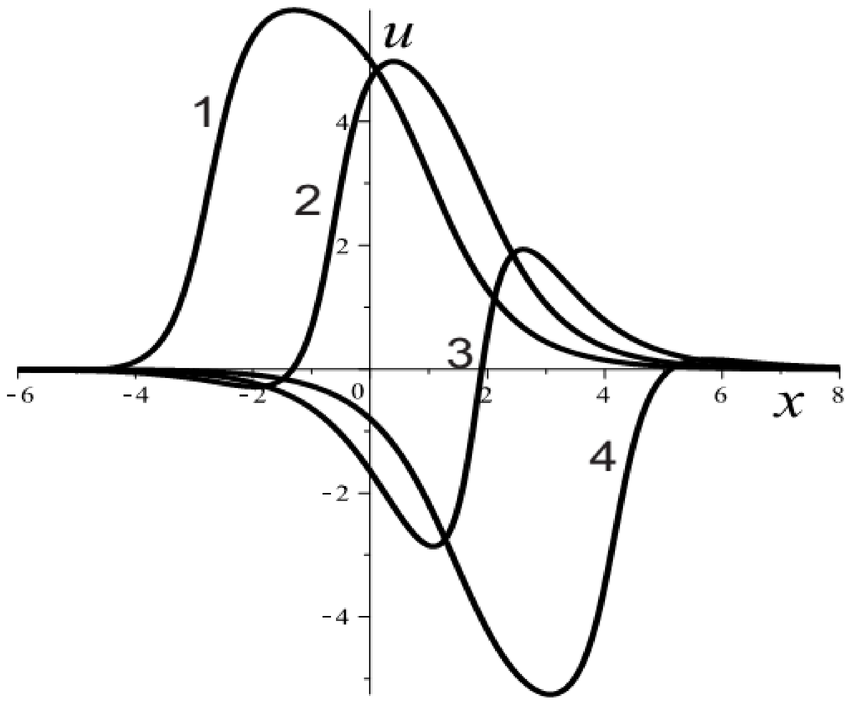


Fig. 1. The evolution of the one-soliton solution for equation (25) at $\delta = -1$, $\beta = 0.5$ and $\lambda = 1.0$ in time $t = 0.0,\ 1.0,\ 2.0$ (left); The one-soliton solution of equation (25) in time $t = 0.5$ at $\beta = 0.5$ for different values of λ: $\lambda = 0.1,\ 0.2,\ 0.3$ (right).

The evolution of the two-soliton solutions



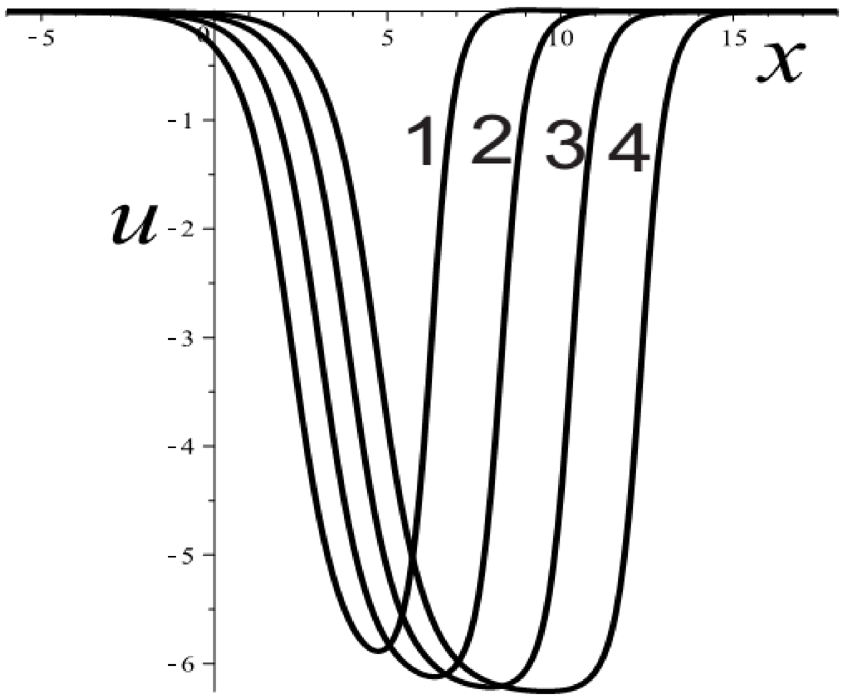


Fig. 4. The two-soliton solution of equation (25) at $\delta = -1$, $\mu_1 = 0.0$, $\mu_2 = 5.0$, $\lambda_1 = 1.1$, $\lambda_2 = 2.2$ and $\beta = 0.5$ for different values of time: $t = 0.0,\ 0.4,\ 0.8,\ 1,2$ (curve line 1 - 4 in left); $t = 1.6,\ 2.0,\ 2.4,\ 2.8$ (curve lines 1-4 in right).

The breather solution for nonlinear dislocations

$$u(z) = \pm 4 \arctan\left[\frac{\mu \sin\left(\lambda x - 2\beta\lambda^3 t + 6\beta\lambda\mu^2 t + \frac{\delta\lambda t}{\lambda^2+\mu^2} + b\right)}{\lambda \cosh\left(\mu x + 2\beta\mu^3 t - 6\beta\lambda^2 t - \frac{\delta\mu t}{\lambda^2+\mu^2} + a\right)}\right],$$

where μ, λ, a and b are arbitrary constants; β and δ are parameters.

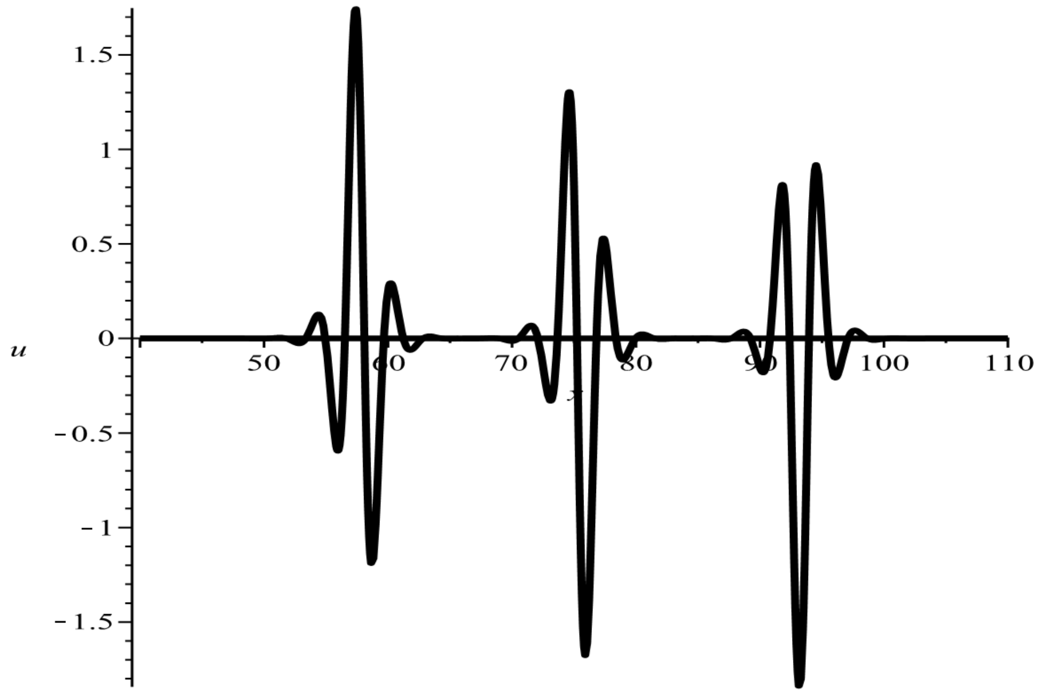


Fig. 7. The dependence of the breather solution in moments $t = 2.4,\ 3.2,\ 4.0$ at $\beta = 1.0,\ \delta = 1.0,\ \lambda = 2.0,\ \mu = 1.0$.

The Hierarchy of nonlinear integrable equations

Equation is the first member of the following integrable hierarchy

$$u_{xt} + 2\,\beta\,\frac{\partial}{\partial x}\left(u_x - i\,\frac{\partial}{\partial x}\right) L_n\left[\frac{i\,u_{xx}}{2} + \frac{u_x^2}{4}\right] = \delta\,\sin u, \qquad (68)$$

where $L_n[v]$ is the Lenard recursion operator which is determined by means of formula [18]

$$\frac{L_{n+1}[v]}{\partial x} = \left(\frac{\partial^3}{\partial x^3} + 4\,v\frac{\partial}{\partial x} + 2\,\frac{\partial v}{\partial x}\right) L_n[v], \qquad L_1[v] = v. \qquad (69)$$

$$u_{xt} + \frac{5}{2}\,u_x^2\,u_{xxxx} + \frac{5}{2}\,u_{xx}^3 + \frac{15}{8}\,u_x^4\,u_{xx} +$$

$$+10\,u_x\,u_{xx}\,u_{xxx} + u_{xxxxxx} = \delta\,\sin u$$

Exact solutions for nonintegrable cases

$$v_{\tau\tau} = v_{xx} + a\,v_x\,v_{xx} + b\,v_x^2\,v_{xx} + g\,v_{xxxx} - d\,\sin v,$$

$$u_{xt} + \alpha\,u_x\,u_{xx} + \beta\,u_x^2\,u_{xx} + \gamma\,u_{xxxx} = \delta\,\sin u.$$

$$w_{zzzz} + b\,w_z^2\,w_{zz} + a\,w_z\,w_{zz} - c_0\,w_{zz} - d\,\sin w = 0, \qquad c_0 = C_0^2 - 1.$$

$$w_{zzz}\,w_z - \frac{1}{2}\,w_{zz}^2 + \frac{a}{3}\,w_z^3 + \frac{b}{4}\,w_z^4 - \frac{c_0}{2}\,w_z^2 + d\,\cos w + C_1 = 0,$$

$$w(z) = -2\,i\,\ln(\varphi(z))$$

$$C_1\,\varphi^4 + 2\,c_0\,\varphi^2\,\varphi_z^2 + \frac{8}{3}\,i\,a\,\varphi\,\varphi_z^3 + 4\,b\,\varphi_z^4 - 4\,\varphi^2\,\varphi_z\,\varphi_{zzz} + 8\,\varphi\,\varphi_z^2\,\varphi_{zz}$$
$$-6\,\varphi_z^4 - 2\,\varphi_z^4 + 2\,\varphi^2\,\varphi_{zz}^2 + \frac{1}{2}\,d\,\varphi^6 + \frac{1}{2}\,d\,\varphi^2 = 0.$$

Kudryashov N. A. Analytical properties of nonlinear dislocation equation // Applied Mathematics Letters. — 2017. — jul. — Vol. 69. — P. 29–34. — URL: `http://linkinghub.elsevier.com/retrieve/pii/S0893965917300320`.

Exact solutions for nonintegrable cases

$$\varphi(z) = A_0 + A_1\,Q(z) + A_2\,Q(z)^2, \qquad Q(z) = \frac{1}{1+e^{z-z_0}}.$$

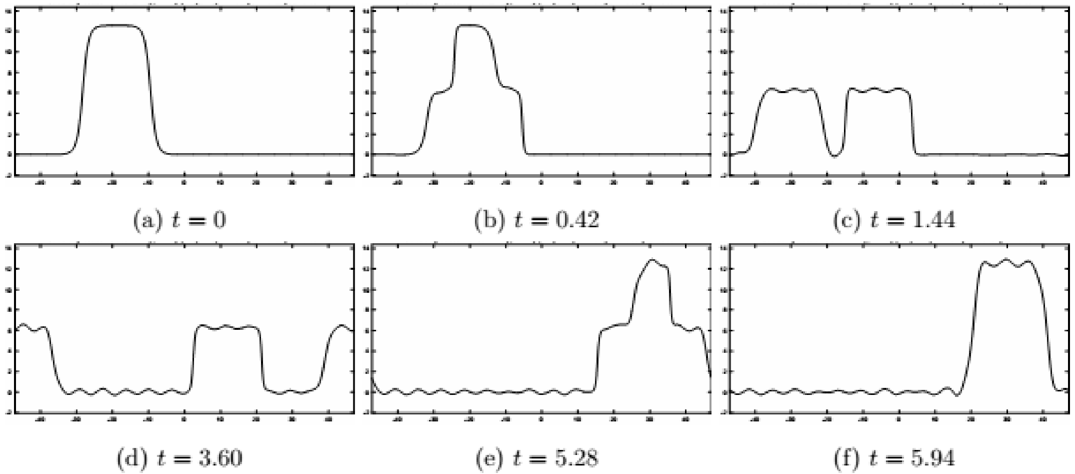
$$w_{zzzz} + b\,w_z^2\,w_{zz} - (8\,b-2)\,w_{zz} - (3-8\,b)\,\sin w = 0.$$

$$w(z) = \pm\,2\,i\,\ln\left[\pm\,i\left(\frac{1-\exp(z-z_0)}{1+\exp(z-z_0)}\right)^2\right].$$

$$u(x,t) = \pm\,4\,\arctan\left[\exp\left(\mu + \,k\,x\, - k^3\,t + \frac{\delta\,t}{k}\right)\right].$$

$$v(x,t) = \pm\,4\,\arctan\left[\exp\left(\mu + k\,x \mp\,k\,\tau\,\sqrt{1-\frac{d}{k^2}+k^2}\right)\right]$$

Results of numerical modeling

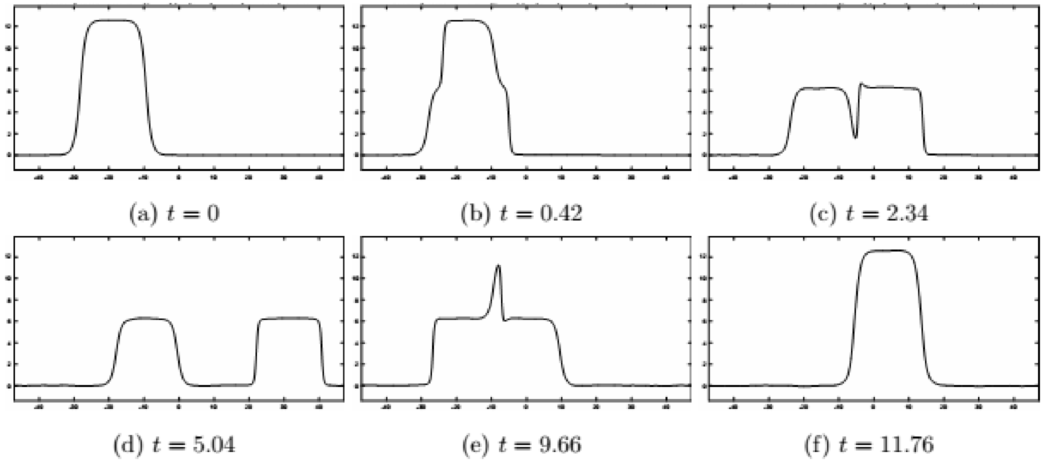


(a) $t = 0$ (b) $t = 0.42$ (c) $t = 1.44$

(d) $t = 3.60$ (e) $t = 5.28$ (f) $t = 5.94$

$$\beta = 1.6,\ \delta = -9,\ \gamma = 1,\ \alpha = 0$$

$$\beta = 1.6,\ \delta = 1,\ \gamma = 1,\ \alpha = 0$$

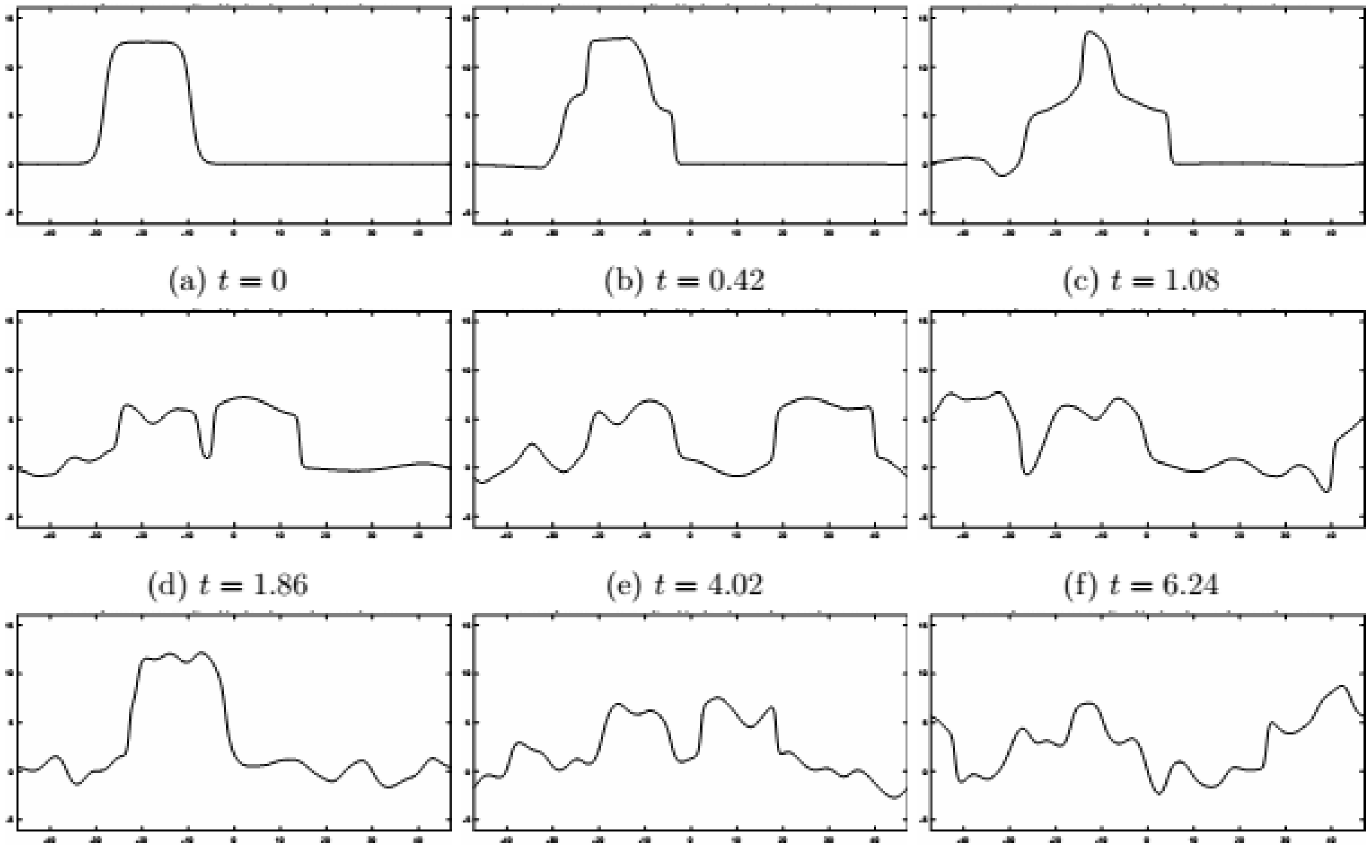


(a) $t = 0$ (b) $t = 0.42$ (c) $t = 2.34$

(d) $t = 5.04$ (e) $t = 9.66$ (f) $t = 11.76$

Results of numerical modeling

Parameters: $\beta = 2.0,\ \delta = 1,\ \gamma = 1,\ \alpha = 0$



(a) $t = 0$ (b) $t = 0.42$ (c) $t = 1.08$

(d) $t = 1.86$ (e) $t = 4.02$ (f) $t = 6.24$

(g) $t = 8.80$ (h) $t = 11.160$ (i) $t = 15$

Conclusion

1) Nonlinear differential obtained from the α-FPU model are nonintegraable except the KdV equation.

2) Nonlinear differential obtained from the β-FPU model are nonintegraable except the mKdV equation.

3) We can expect that FPU model leads to the chaotic behaviour.

4) There is the special case of nonlinear differential equation found from the FKFPU model.

5) This equation is the generalization of the mKdV and the Sine-Gordon equation.

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THNAK YOU FOR ATTENSION!