High-Accuracy Finite Element Method For Elliptic Boundary-Value Problems

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The statement of the problem

A self-adjoint elliptic PDE in the region $z = (z_1, ..., z_d) \in \Omega \subset \mathcal{R}^d$ (Ω is polyhedra)

$$\left(-\frac{1}{g_0(z)} \sum_{ij=1}^{d} \frac{\partial}{\partial z_i} g_{ij}(z) \frac{\partial}{\partial z_j} + V(z) - E \right) \Phi(z) = 0,$$

$g_0(z) > 0$, $g_{ji}(z) = g_{ij}(z)$ and $V(z)$ are the real-valued functions, continuous together with their generalized derivatives to a given order.

Boundary conditions

$$(I): \ \Phi(z)|_S = 0, \ \ (II): \ \left.\frac{\partial \Phi(z)}{\partial n_D}\right|_S = 0, \ \ (III): \ \left.\frac{\partial \Phi(z)}{\partial n_D}\right|_S + \sigma(s)\Phi(z)|_S = 0,$$

$$\frac{\partial \Phi(z)}{\partial n_D} = \sum_{ij=1}^{d} (\hat{n}, \hat{e}_i) g_{ij}(z) \frac{\partial \Phi(z)}{\partial z_j},$$

$\frac{\partial \Phi_m(z)}{\partial n_D}$ is the derivative along the conormal direction
$\hat{n}$ is the outer normal to the boundary of the domain $\partial\Omega$.

О.А. Ладыженская, Краевые задачи математической физики (М., Наука, 1973)
В.В. Шайдуров, Многосеточные методы конечных элементов. (М., Наука, 1989).

The statement of the problem

For a discrete spectrum problem the functions $\Phi_m(z)$ from the Sobolev space $H_2^{s\geq 1}(\Omega)$, $\Phi_m(z) \in H_2^{s\geq 1}(\Omega)$, corresponding to the real eigenvalues E: $E_1 \leq E_2 \leq \ldots \leq E_m \leq \ldots$ satisfy the conditions of normalization and orthogonality

$$\langle \Phi_m(z) | \Phi_{m'}(z) \rangle = \int_\Omega dz g_0(z) \Phi_m(z) \Phi_{m'}(z) = \delta_{mm'}, \quad dz = dz_1 \ldots dz_d.$$

The FEM solution of the BVP is reduced to the determination of stationary points of the variational functional

$$\Xi(\Phi_m, E_m, z) \equiv \int_\Omega dz g_0(z) \Phi_m(z) \left(D - E_m\right) \Phi(z) = \Pi(\Phi_m, E_m, z) - \oint_S \Phi_m(z) \frac{\partial \Phi_m(z)}{\partial n_D},$$

$$\Pi(\Phi_m, E_m, z) = \int_\Omega dz \left[\sum_{ij=1}^{d} g_{ij}(z) \frac{\partial \Phi_m(z)}{\partial z_i} \frac{\partial \Phi_m(z)}{\partial z_j} + g_0(z) \Phi_m(z) (V(z) - E_m) \Phi_m(z) \right].$$

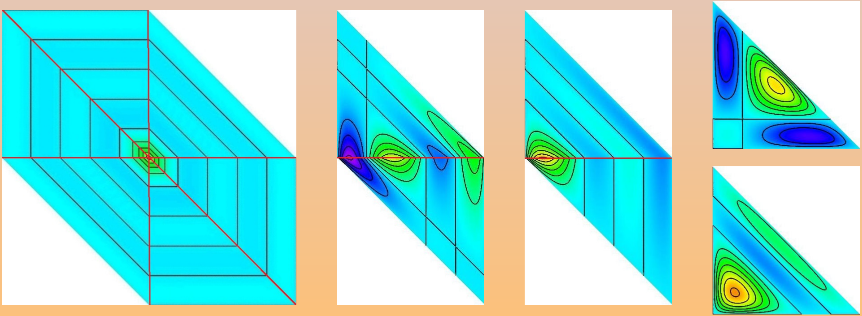
Strang, G., Fix, G.J.: An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, New York (1973)

Lagrange Finite Elements

The piecewise polynomial functions $N_i(z)$ are constructed by joining the shape functions $\varphi_l(z)$ in the triangle Δ_q:

$$N_i(z) = \left\{ \varphi_l(z), A_l \in \Delta_q; 0, A_l \notin \Delta_q \right\}$$

and possess the following properties:
functions $N_i(z)$ are continuous in the domain Ω;
the functions $N_i(z)$ equal 1 in one of the points A_l and zero in the rest points.



Finite Element Method

Solutions $\hat{\Phi}(z)$ are sought in the form of a finite sum over the basis of local functions $N_\mu^g(z)$ in each nodal point $z = z_k$ of the grid $\Omega_h(z)$:

$$\hat{\Phi}(z) = \sum_{\mu=0}^{L-1} \Phi_\mu^h N_\mu^g(z),$$

where L is number of local functions, and Φ_μ^h are nodal values of function $\hat{\Phi}(z)$ at nodal points z_l.

After substituting the expansion into the variational functional and minimizing it, we obtain the generalized eigenvalue problem

$$\mathbf{A}^p \boldsymbol{\xi}^h = \varepsilon^h \mathbf{B}^p \boldsymbol{\xi}^h.$$

Here $\mathbf{A}^p$ is the stiffness matrix; $\mathbf{B}^p$ is the positive definite mass matrix; $\boldsymbol{\xi}^h$ is the vector approximating the solution on the finite-element grid; and ε^h is the corresponding eigenvalue.

FEM calculation scheme

The polyhedron $\Omega = \bigcup_{q=1}^{Q} \Delta_q$ is covered with simplexes Δ_q with $d+1$ vertices:

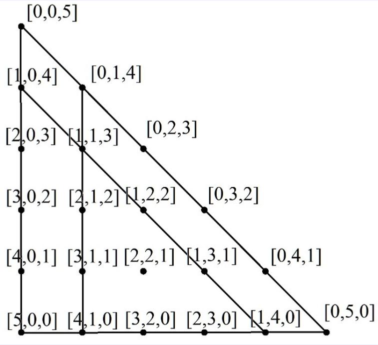
$$\hat{z}_i = (\hat{z}_{i1}, \hat{z}_{i2}, ..., \hat{z}_{id}), \quad i = 0, ..., d.$$

Each edge of the simplex Δ_q is divided into p equal parts and the families of parallel hyperplanes $H(i,k)$, $k = 0, ..., p$ are drawn. The equation of the hyperplane $H(i,k)$: $H(i;z) - k/p = 0$, $H(i;z)$ is a linear on z.

The points A_r of hyperplanes crossingare enumerated with sets of hyperplane numbers: $[n_0, ..., n_d]$, $n_i \geq 0$, $n_0 + ... + n_d = p$.

The coordinates $\xi_r = (\xi_{r1}, ..., \xi_{rd})$ of $A_r \in \Delta_q$:

$$\xi_r = \hat{z}_0 n_0/p + \hat{z}_1 n_1/p + ... + \hat{z}_d n_d/p.$$



Lagrange Interpolation Polynomials

$$\varphi_r(z) = \left(\prod_{i=0}^{d} \prod_{n_i'=0}^{n_i-1} \frac{H(i;z) - n_i'/p}{H(i;\xi_r) - n_i'/p} \right),$$

$$\varphi_r(\xi_{r'}) = \delta_{rr'}, \quad \xi_r \leftrightarrow [n_0, n_1, ..., n_d].$$

The economical implementation, accepted in FEM:

1. The calculations are performed in the local coordinates z', in which the coordinates of the simplex vertices are the following: $\hat{z}'_j = (\hat{z}'_{j1}, ..., \hat{z}'_{jd})$, $\hat{z}'_{jk} = \delta_{jk}$

$$z_i = \hat{z}_{0i} + \sum_{j=1}^{d} J_{ij} z'_j, \quad z'_i = \sum_{j=1}^{d} (J^{-1})_{ij} (z_j - \hat{z}_{0j}), \quad J_{ij} = \hat{z}_{ji} - \hat{z}_{0i}, \quad i = 1, ..., d.$$

$$\frac{\partial}{\partial z'_i} = \sum_{j=1}^{d} J_{ji} \frac{\partial}{\partial z_j}, \quad \frac{\partial}{\partial z_i} = \sum_{j=1}^{d} (J^{-1})_{ji} \frac{\partial}{\partial z'_j}.$$

2. The calculation of FEM integrals is executed in the local coordinates.

$$\int_{\Delta_q} dz g_0(z) \varphi_r^{\kappa}(z) \varphi_{r'}^{\kappa''}(z) U(z) = \hat{J} \int_{\Delta} dz' g_0(z(z')) \varphi_r^{\kappa}(z') \varphi_{r'}^{\kappa''}(z') U(z(z')), \quad \hat{J} = \det(J_{ij}) > 0$$

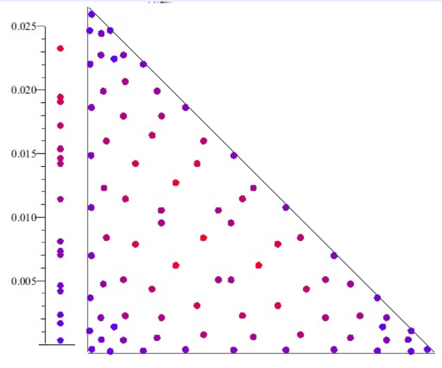
$$\int_{\Delta_q} dz g_{s_1 s_2}(z) \frac{\partial \varphi_r^{\kappa}(z)}{\partial z_{s_1}} \frac{\partial \varphi_{r'}^{\kappa''}(z)}{\partial z_{s_2}} = \hat{J} \sum_{t_1, t_2 = 1}^{d} (J^{-1})_{t_1 s_1} (J^{-1})_{t_2 s_2} \int_{\Delta} dz' g_{s_1 s_2}(z(z')) \frac{\partial \varphi_r^{\kappa}(z')}{\partial z'_{t_1}} \frac{\partial \varphi_{r'}^{\kappa''}(z')}{\partial z'_{t_2}},$$

Calculations of FEM integrals

Fully symmetric high-order Gaussian quadratures

In the paper [a] was presented weights and coordinates of the fully symmetric rules up to order $p = 20$ with minimal number of points using the moment equations. Calculation was preformed with double precision accuracy. However, the some rules has points outside the triangle and/or negative weights. We need to use Gaussian quadrature rules with positive weights, and no points are outside the triangle (so-called PI type).
A new high ordered PI type rules was calculated by the elaborated algorithm implemented in Maple-Fortran.
Alternative results in [b]

The quadrature rule for $p = 20$, $n_p = 85$, type $[n_0, n_1, n_2] = [1, 8, 10]$



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[a] Dunavant, D. A.: High Degree Efficient Symmetrical Gaussian Quadrature Rules for the Triangle, International journal for numerical methods and engineering, 21, 1129–1148, 1985

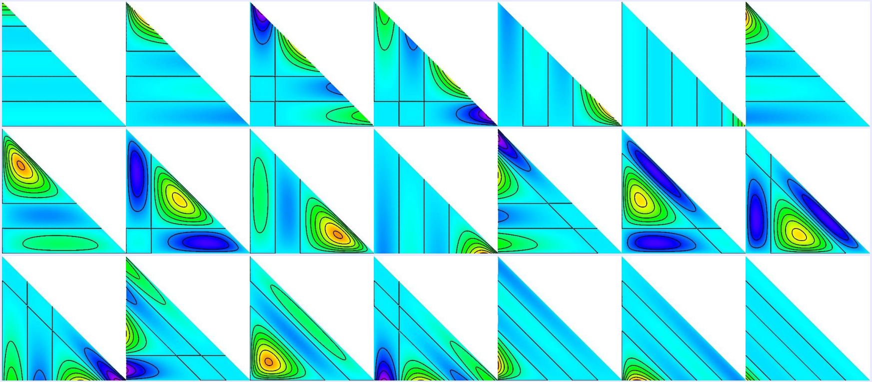
[b] L. Zhang, T. Cui, and H. Liu, A set of symmetric quadrature rules on triangles and tetrahedra. Journal of Computational Mathematics, Vol.27, No.1, 2009, 89-96

Lagrange Finite Elements

Lagrange Interpolation Polynomials (in the local coordinates)

$$\varphi_r(z') = \left(\prod_{i=1}^{d} \prod_{n_i'=0}^{n_i-1} \frac{z_i' - n_i'/p}{n_i/p - n_i'/p} \right) \left(\prod_{n_0'=0}^{n_0-1} \frac{1 - z_1' - \ldots - z_d' - n_0'/p}{n_0/p - n_0'/p} \right).$$

2D ILP at $p = 5$, $\hat{z}_0' = (\hat{z}_{01}', \hat{z}_{02}') = (0, 0)$, $\hat{z}_1' = (\hat{z}_{11}', \hat{z}_{12}') = (1, 0)$, $\hat{z}_2' = (\hat{z}_{21}', \hat{z}_{22}') = (0, 1)$



Algorithm for calculating the basis of Hermite interpolating polynomials

The problem

Constructions of the HIP of the order p', joining which the piecewise polynomial functions can be obtained that possess continuous derivatives to the given order κ'.

Step 1. Auxiliary polynomials (AP1)

$$\varphi_r^{\kappa_1\cdots\kappa_d}(\xi'_r)=\delta_{rr'}\delta_{\kappa_1 0}\ldots\delta_{\kappa_d 0}, \quad \left.\frac{\partial^{\mu_1\cdots\mu_d}\varphi_r^{\kappa_1\cdots\kappa_d}(z')}{\partial {z'_1}^{\mu_1}\ldots\partial {z'_d}^{\mu_d}}\right|_{z'=\xi'_{r'}}=\delta_{rr'}\delta_{\kappa_1\mu_1}\ldots\delta_{\kappa_d\mu_d},$$

$$0\leq\kappa_1+\kappa_2+\ldots+\kappa_d\leq\kappa_{\max}-1, \quad 0\leq\mu_1+\mu_2+\ldots+\mu_d\leq\kappa_{\max}-1.$$

Here in the node points ξ'_r, in contrast to LIP, the values of not only the functions themselves, but of their derivatives to the order $\kappa_{\max}-1$ are specified.

Algorithm for calculating the basis of Hermite interpolating polynomials

AP1 are given by the expressions

$$\varphi_r^{\kappa_1\kappa_2\ldots\kappa_d}(z') = w_r(z') \sum_{\mu\in\Delta_\kappa} a_r^{\kappa_1\ldots\kappa_d,\mu_1\ldots\mu_d}(z'_1-\xi'_{r1})^{\mu_1}\times\ldots\times(z'_d-\xi'_{rd})^{\mu_d},$$

$$w_r(z') = \left(\prod_{i=1}^{d}\prod_{n'_i=0}^{n_i-1}\frac{(z'_i-n'_i/p)^{\kappa^{\max}}}{(n_i/p-n'_i/p)^{\kappa^{\max}}}\right)\left(\prod_{n'_0=0}^{n_0-1}\frac{(1-z'_1-\ldots-z'_d-n'_0/p)^{\kappa^{\max}}}{(n_0/p-n'_0/p)^{\kappa^{\max}}}\right),\quad w_r(\xi'_r)=1,$$

where the coefficients $a_r^{\kappa_1\ldots\kappa_d,\mu_1\ldots\mu_d}$ are calculated from recurrence relations

$$a_r^{\kappa_1\ldots\kappa_d,\mu_1\ldots\mu_d} = \begin{cases} 0, \quad \mu_1+\ldots+\mu_d \le \kappa_1+\ldots+\kappa_d, (\mu_1,\ldots,\mu_d)\neq(\kappa_1,\ldots,\kappa_d),\\ \prod_{i=1}^{d}\frac{1}{\mu_i!}, \qquad (\mu_1,\ldots,\mu_d)=(\kappa_1,\ldots,\kappa_d);\\ -\sum_{\nu\in\Delta_\nu}\left(\prod_{i=1}^{d}\frac{1}{(\mu_i-\nu_i)!}\right) g_r^{\mu_1-\nu_1,\ldots,\mu_d-\nu_d}(\xi'_r)a_r^{\kappa_1\ldots\kappa_d,\nu_1\ldots\nu_d},\\ \qquad \mu_1+\ldots+\mu_d > \kappa_1+\ldots+\kappa_d; \end{cases}$$

$$g^{\kappa_1\kappa_2\ldots\kappa_d}(z') = \frac{1}{w_r(z')}\frac{\partial^{\kappa_1\kappa_2\ldots\kappa_d}w_r(z')}{\partial z_1'^{\kappa_1}\partial z_2'^{\kappa_2}\ldots\partial z_d'^{\kappa_d}}.$$

Algorithm for calculating the basis of Hermite interpolating polynomials

For $d > 1$ and $\kappa_{\max} > 1$, the number $N_{\kappa_{\max} p'}$ of HIP of the order p' and the multiplicity of nodes $\kappa_{\max}$ are smaller than the number $N_{1p'}$ of the polynomials that form the basis in the space of polynomials of the order p', i.e., these polynomials, are determined ambiguously.

Step 2. Auxiliary polynomials (AP2 and AP3)

For unambiguous determination of the polynomial basis let us introduce $K = N_{1p'} - N_{\kappa_{\max} p'}$ auxiliary polynomials $Q_s(z)$ of two types: AP2 and AP3, linear independent of AP1 and satisfying the conditions in the node points $\xi'_{r'}$ of AP1:

$$Q_s(\xi'_{r'})=0, \quad \left.\frac{\partial^{\kappa'_1 \kappa'_2 \ldots \kappa'_d} Q_s(z')}{\partial {z'_1}^{\mu_1} \partial {z'_2}^{\mu_2} \ldots \partial {z'_d}^{\mu_d}}\right|_{z'=\xi'_{r'}}=0, \quad s = 1, \ldots, K,$$

$$0 \le \kappa_1 + \kappa_2 + \ldots + \kappa_d \le \kappa_{\max}-1, \quad 0 \le \mu_1 + \mu_2 + \ldots + \mu_d \le \kappa_{\max}-1.$$

AP2 for cont. of derivs. ($\eta'_{s'}$ on bounds of Δ):

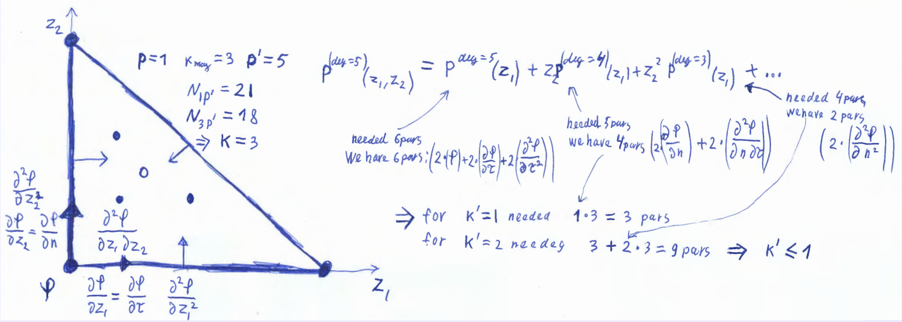
$$\left.\frac{\partial^k Q_s(z')}{\partial n^k_{i(s)}}\right|_{z'=\eta'_{s'}}=\delta_{ss'}, \quad s, s'=1, \ldots, T_1(\kappa').$$

AP3 ($\zeta'_{s'}$ inside Δ):

$$Q_s(\zeta'_{s'})=\delta_{ss'}, \quad s, s'=T_1(\kappa')+1, \ldots, K.$$

Construction of AP2 and AP3 at $d = 2$

Example: $p = 1$, $\kappa_{\max} = 3$, $p' = 5$, $\Rightarrow \kappa' = 1$ (the Argyris triangle)



Argyris triangle ($\kappa' = 1$): AP1 (18 elements) + AP2 (3 elements: $\left.\frac{\partial^k Q_s(z')}{\partial n^k_{i(s)}}\right|_{z'=\eta'_{s'}} = \delta_{ss'}$ at $\eta'_{s'} \in \{(0, 1/2), (1/2, 0), (1/2, 1/2)\}$).

Alt. variant (Bell triangle, $\kappa' = 1$): $z_2 P^{\mathrm{deg}=4}(z_1) \to z_2 P^{\mathrm{deg}=3}(z_1), \Leftrightarrow \left.\frac{\partial^5 \varphi(z')}{\partial n \partial \tau^4}\right|_{\delta\Delta} = 0$.

Alt. variant ($\kappa' = 0$): AP1 (18 elements) + AP3 (3 elements: $Q_s(\zeta'_{s'}) = \delta_{ss'}$ at $\eta'_{s'} \in \{(1/2, 1/4), (1/4, 1/2), (1/4, 1/4)\}$ or (Q_s or $\frac{\partial Q_s}{\partial z_1}$ or $\frac{\partial Q_s}{\partial z_2}$) $= \delta_{ss'}$ at $\eta'_{s'} \in (1/3, 1/3)$).

Characteristics of the HIP bases at $d = 2$

$[p\kappa_{\max}\kappa']$	[120]	[131]	[141]	[231]	[152]	[162]	[241]	[173]
p'	3	5	7	8	9	11	11	13
$N_{\kappa_{\max}p'}$	9	18	30	36	45	63	60	84
$N_{1p'}$	10	21	36	45	55	78	78	105
K	1	3	6	9	10	15	9	21
$T_1(1) = 3p$	3	3	3	6	3	3	6	3
$T_1(2) = 9p$	9	9	9	18	9	9	18	9
$N(\mathrm{AP1}) = N_{\kappa_{\max}p'}$	9	18	30	36	45	63	60	84
$N(\mathrm{AP2}) = T_1(\kappa')$	0	3	3	6	9	9	6	18
$N(\mathrm{AP3}) = K - T_1(\kappa')$	1	0	3	3	1	6	12	3

$$p' = \kappa_{\max}(p+1) - 1$$

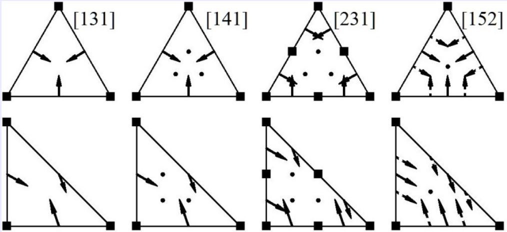
$$N_{\kappa_{\max}p'} = (p+1)(p+2)\kappa_{\max}(\kappa_{\max}+1)/4$$

$$N_{1p'} = (p'+1)(p'+2)/2$$

$$K = p(p+1)\kappa_{\max}(\kappa_{\max}-1)/4$$

Restriction of derivative order κ' : $3p\kappa'(\kappa'+1)/2 \leq K$.

Characteristics of the HIP bases at $d = 2$



The squares: the functions with their derivatives are fixed;
The solid (dashed) arrows: first (second) normal derivative are fixed;
The circles: the functions are fixed.

The normal derivatives:

$$\frac{\partial}{\partial n_i}=f_{i1}\frac{\partial}{\partial z_1'}+f_{i2}\frac{\partial}{\partial z_2'}, \quad i=1,2, \quad \frac{\partial}{\partial n_0}=(f_{01}+f_{02})\frac{\partial}{\partial z_1'}+(f_{01}-f_{02})\frac{\partial}{\partial z_2'},$$

$$f_{11}=J^{-1}R(\hat z_2,\hat z_0), \qquad f_{12}=-((\hat z_{12}-\hat z_{02})(\hat z_{22}-\hat z_{02})+(\hat z_{21}-\hat z_{01})(\hat z_{11}-\hat z_{01}))/(JR(\hat z_2,\hat z_0)),$$
$$f_{22}=J^{-1}R(\hat z_1,\hat z_0), \qquad f_{21}=-((\hat z_{12}-\hat z_{02})(\hat z_{22}-\hat z_{02})+(\hat z_{21}-\hat z_{01})(\hat z_{11}-\hat z_{01}))/(JR(\hat z_1,\hat z_0)),$$
$$f_{01}=-(2J)^{-1}R(\hat z_2,\hat z_1), \qquad f_{02}=((\hat z_{11}-\hat z_{01})^2+(\hat z_{12}-\hat z_{02})^2-(\hat z_{22}-\hat z_{02})^2-(\hat z_{21}-\hat z_{01})^2)/(2JR(\hat z_2,\hat z_1)),$$
$$R(\hat z_j,\hat z_{j'}) = ((\hat z_{1j}-\hat z_{1j'})^2+(\hat z_{2j}-\hat z_{2j'})^2)^{1/2}.$$

The auxiliary polynomials AP2 and AP3:

$$Q_s(z') = {z'_1}^{k_1} \ldots {z'_d}^{k_d} (1 - z'_1 - \ldots - z'_d)^{k_0} \sum_{j_1,\ldots,j_d} b_{j_1,\ldots,j_d;s} {z'_1}^{j_1} \ldots {z'_d}^{j_d},$$

where $k_t = 1$, if the point η_s, in which the additional conditions are specified, lies on the corresponding face of the simplex Δ and $k_t = \max(1, \kappa')$, if $H(t, \eta_s) \neq 0$.
The coefficients $b_{j_1,\ldots,j_d;s}$ are determined from the unambiguously solvable system of linear equations, obtained as a result of the substitution of this expression into the above conditions of Step 2.

Step 3: Recalculation of AP1

$$\check{\varphi}_r^{\kappa}(z') = \varphi_r^{\kappa}(z') - \sum_{s=1}^{K} c_{\kappa;r;s} Q_s(z'), \quad c_{\kappa;r;s} = \begin{cases} \left. \dfrac{\partial^k \varphi_r^{\kappa}(z')}{\partial n_{i(s)}^k} \right|_{z'=\eta'_s}, & Q_s(z') \in \mathrm{AP2}, \\ \varphi_r^{\kappa}(\zeta_s), & Q_s(z') \in \mathrm{AP3}. \end{cases}$$

Step 4.

The AP1 $\check{\varphi}_r^{\kappa}(z')$, where κ denotes the directional derivatives along the local coordinate axes, are recalculated into $\check{\varphi}_r^{\kappa;z}(z')$, in the local coordinates, but now κ denotes already the directional derivatives along the initial coordinate axes using the formulas

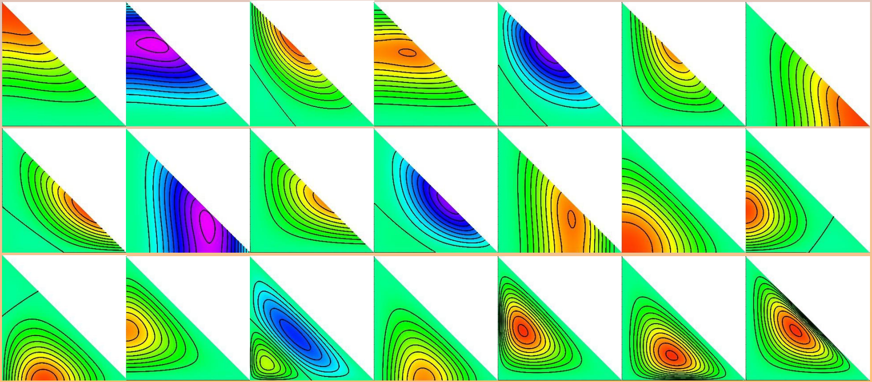
$$\frac{\partial}{\partial z_i}=\sum_{j=1}^{d}(\hat{J}^{-1})_{ji}\frac{\partial}{\partial z'_j}.$$

At $d=2$:

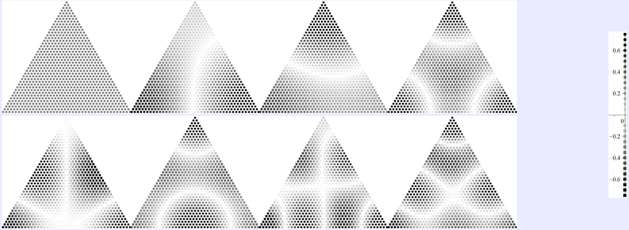
$$\begin{aligned}
&\check{\varphi}_r^{00;z}(z')=\check{\varphi}_r^{00}(z'),\\
&\check{\varphi}_r^{10;z}(z')=(J^{-1})_{11}\check{\varphi}_r^{10}(z')+(J^{-1})_{21}\check{\varphi}_r^{01}(z'),\\
&\check{\varphi}_r^{01;z}(z')=(J^{-1})_{12}\check{\varphi}_r^{10}(z')+(J^{-1})_{22}\check{\varphi}_r^{01}(z'),\\
&\check{\varphi}_r^{20;z}(z')=(J^{-1})_{11}(J^{-1})_{11}\check{\varphi}_r^{20}(z')+(J^{-1})_{11}(J^{-1})_{21}\check{\varphi}_r^{11}(z')+(J^{-1})_{21}(J^{-1})_{21}\check{\varphi}_r^{02}(z'),\\
&\check{\varphi}_r^{11;z}=2(J^{-1})_{11}(J^{-1})_{12}\check{\varphi}_r^{20}(z')+((J^{-1})_{11}(J^{-1})_{22}+(J^{-1})_{12}(J^{-1})_{21})\check{\varphi}_r^{11}(z')\\
&\qquad\qquad+2(J^{-1})_{21}(J^{-1})_{22}\check{\varphi}_r^{02}(z'),\\
&\check{\varphi}_r^{02;z}(z')=(J^{-1})_{12}(J^{-1})_{12}\check{\varphi}_r^{20}(z')+(J^{-1})_{12}(J^{-1})_{22}\check{\varphi}_r^{11}(z')+(J^{-1})_{22}(J^{-1})_{22}\check{\varphi}_r^{02}(z'),\\
&\dots
\end{aligned}$$

The HIP $p=1$, $\kappa_{\max}=3$, $\kappa'=1$, $p'=5$ (the Argyris element)

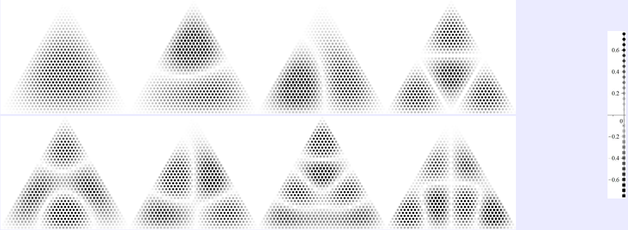
AP1 : $\xi_1=(0,1)$, $\xi_2=(1,0)$, $\xi_3=(0,0)$		
$\varphi_1^{0,0}=z_2^3(6z_2^2-15z_2+10)$	$\varphi_2^{0,0}=z_1^3(6z_1^2-15z_1+10)$	$\varphi_3^{0,0}=z_0^3(6z_0^2-15z_0+10)$
$\varphi_1^{0,1}=-z_2^3(z_2-1)(3z_2-4)$	$\varphi_2^{0,1}=-z_1^3 z_2(3z_1-4)$	$\varphi_3^{0,1}=-z_0^3 z_2(3z_0-4)$
$\varphi_1^{1,0}=-z_1 z_2^3(3z_2-4)$	$\varphi_2^{1,0}=-z_1^3(z_1-1)(3z_1-4)$	$\varphi_3^{1,0}=-z_0^3 z_1(3z_0-4)$
$\varphi_1^{0,2}=z_2^3(z_2-1)^2/2$	$\varphi_2^{0,2}=z_1^3 z_2^2/2$	$\varphi_3^{0,2}=z_0^3 z_2^2/2$
$\varphi_1^{1,1}=z_1 z_2^3(z_2-1)$	$\varphi_2^{1,1}=(z_1-1)z_1^3 z_2$	$\varphi_3^{1,1}=z_0^3 z_1 z_2$
$\varphi_1^{2,0}=z_1^2 z_2^3/2$	$\varphi_2^{2,0}=z_1^3(z_1-1)^2/2$	$\varphi_3^{2,0}=z_0^3 z_1^2/2$
AP2 : $\eta_1=(0,1/2)$, $\eta_2=(1/2,0)$, $\eta_3=(1/2,1/2)$		
$Q_1=16z_0^2 z_1 z_2^2/f_{11}$	$Q_2=16z_0^2 z_1^2 z_2/f_{22}$	$Q_3=-8z_0 z_1^2 z_2^2/f_{01}$



Example: the BVP for Helmholtz Eq. for the equilateral triangle with side equal to $4\pi/3$



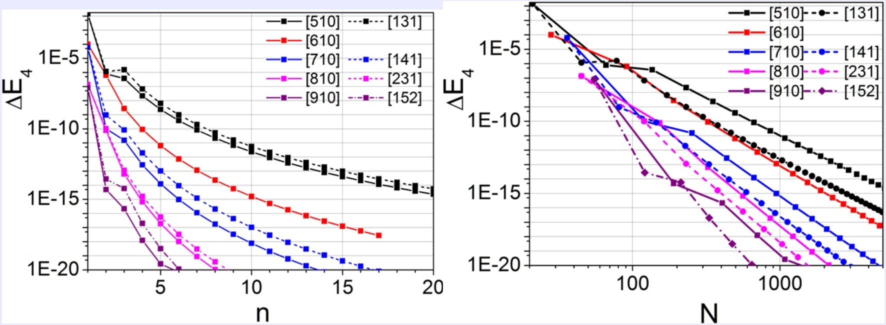
The BVP with the Neumann boundary conditions: the eigenvalues $\varepsilon_i = 0, 1, 1, 3, 4, 4, 7, 7, \ldots$.



The BVP with the Dirichlet boundary conditionshe eigenvalues $\varepsilon_i = 3, 7, 7, 12, 13, 13, 19, 19, \ldots$.

Pockels, F., Über die partielle Differential-Gleichung $\Delta u + k^2 u = 0$ und deren auftreten in der mathematischen physik (Leipzig, 1891)

Example



The absolute errors $\sigma_1^h(E_4)$ of fourth eigenvalue $E_4 = 3$ vs number n of divisions of the triangle side $4\pi/3$ (n^2 is a number of equilateral triangles with side $h = 4\pi/(3n)$) and length N of eigenvector of algebraic eigenvalue problem for the schemes with ILPs and IHPs since 5th till 9th order. $[p\kappa_{\max}\kappa']$: $[131], [141], [231], [152]$

Resume

- High-accuracy finite element method for elliptic boundary-value problems is presented.
- The basis functions of finite elements are high-order polynomials, determined from a specially constructed set of values of the polynomials themselves, their partial derivatives, and their derivatives along the directions of the normals to the boundaries of finite elements.
- Such a choice of the polynomials allows us to construct a piecewise polynomial basis continuous on the boundaries of elements together with the derivatives up to a given order. In present talk we show how this basis is applied to solve elliptic boundary value problems in the limited domain of multidimensional Euclidean space, specified as a polyhedron.
- The efficiency and the accuracy order of the finite element scheme, algorithm and program are demonstrated by the example of exactly solvable boundary-value problem for a triangular membrane, depending on the number of finite elements of the partition of the domain and the number of piecewise polynomial basis functions.

Thank you for your attention