Derivation of the system of N-order Riccati equations

Robert M. Yamaleev
Joint Institute for Nuclear Research, LIT, Dubna, Russia.
Email: yamaleev@jinr.ru

July 5, 2017

1 Formulation of the problem

Evolution generated by the finite dimensional operator H:

$$\frac{d}{dt}\Psi(t) = H\Psi(t), \quad \Psi(0) = \Psi_0,$$

$$\Psi(t) = \exp(tH)\Psi_0.$$

The finite dimensional operator H obeys its characteristic polynomial equation

$$f(H) = 0.$$

Besides of the evolution equation generated by operator H, one may define an evolution equation governed by the n-order Riccati equation of the form

$$\frac{d}{dt}U = f(U).$$

Polynomial $f(X)$ we present in the form

$$f(X) = X^n + \sum_{k=1}^{n}(-)^k a_k X^{n-k}, \quad a_k \in C.$$

Let E be a *companion matrix* of the operator H. The companion matrix satisfies the same characteristic equation , so that,

$$f(E) = 0.$$

The aim is to transform a linear system of evolution equations generated by finite dimensional matrix to the system of Riccati equations.

$$\frac{d}{dt}\Psi(t) = H\Psi(t) \quad \Leftarrow \quad \Rightarrow \quad \frac{d}{dt}U = f(U).$$

General complex algebra

$$f(\mathbf{e}) = \mathbf{0}.$$

Elements of the general complex algebra GC_n are defined by the series

$$Z = \sum_{k=0}^{n-1} \mathbf{e}^k q_k, \quad \mathbf{e}^0 = I, \quad Z \in GC_n.$$

In a matrix representation the generator $\mathbf{e} \rightarrow E$, correspondingly, elements of the general complex algebra are presented by $n \times n$ matrix of the form

$$Z = \sum_{k=0}^{n-1} E^k q_k, \quad E^0 = I.$$

It is supposed that the n-order polynomial $f(X)$ possesses with n distinct roots $x_k, k = 1, ..., n \in \mathcal{C}$. The *companion matrix* E of polynomial $f(X)$:

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & (-1)^{n+1} a_n \\ 1 & 0 & 0 & 0 & 0 & (-1)^n a_{n-1} \\ . & . & ... & . & . & . \\ 0 & 0 & ... & 1 & 0 & -a_2 \\ 0 & 0 & ... & 0 & 1 & a_1 \end{pmatrix}.$$

Thus, $Z \in GC_n$ is $(n-1)$ degree polynomial of the form

$$Q(U) = \sum_{k=0}^{n-1} U^k q_k, \;\; q_{n-1} \neq 0.$$

The modulus of $Z \in GC_n$ conventionally is defined by the determinant function

$$|Z|^n = Det(\sum_{k=0}^{n-1} E^k q_k).$$

Let $u_k, k = 1, 2, ..., n-1$ be roots of the polynomial $Q(U)$. Then the modulus of the GC_n-number admits another form of representation via the basic polynomial $f(X)$:

$$|Z|^n = Det(\sum_{k=0}^{n-1} E^k q_k) = q_{n-1}^n \prod_{k=1}^{n-1} f(u_k).$$

Examples

$$f(X) = X^2 - a_1 X + a_2.$$

$$f(\mathbf{e}) = \mathbf{0}.$$

$$Z = q_0 + \mathbf{e}\ \mathbf{q_1}.$$

$$E^2 - a_1 E + a_2 I = 0.$$

$$Z = q_0 + E q_1, \quad Z = x + iy, \quad i^2 + 1 = 0.$$

$$|Z|^2 = q_1^2 f(u), \quad u = -\frac{q_0}{q_1}$$

Companion matrix

$$E = \begin{pmatrix} 1 & -a_2 \\ 0 & a_1 \end{pmatrix}$$

The case $N = 3$.
Companion matrix

$$E = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}, \quad E^2 = \begin{pmatrix} 0 & a_3 & a_3 a - 1 \\ 0 & -a_2 & a_3 - a_1 a_2 \\ 0 & a_1 & a_1^2 - a_2 \end{pmatrix}$$

$$f(X) = X^3 - a_1 X^2 + a_2 X + a_3$$

$$E^3 - a_1 E^2 + a_2 E - a_3 I = 0.$$

$$Z = q_0 + E q_1 + E^2 q_2,$$

$$|Z|^3 = q_2^3 f(u_1) f(u_2), \quad q_2 u^2 + q_1 u + q_0 = 0.$$

$$|Z|^3 = Det(Z).$$

Trigonometry. Examples.

Evolution equations

the case N=2.

$$E^2 - a_1 E + a_2 = 0.$$

$$\exp(E\phi) = g_0(\phi, a_1, a_2) + x_1\ g_1(\phi, a_1, a_2).$$

$$\frac{d}{d\phi}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 1 & -a_2 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}.$$

the case $N = 3$.

$$\exp(E\phi_1 + E^2\phi_2) = g_0(\phi_1, \phi_2) + Eg_1(\phi_1, \phi_2) + E^2 g_2(\phi_1, \phi_2).$$

$$\frac{\partial}{\partial\phi_1}\exp(E\phi_1 + E^2\phi_2) = E\ \exp(E\phi_1 + E^2\phi_2),$$

$$\frac{\partial}{\partial\phi_2}\exp(E\phi_1 + E^2\phi_2) = E^2\ \exp(E\phi_1 + E^2\phi_2).$$

$$\frac{d}{d\phi_1}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}.$$

$$\frac{d}{d\phi_2}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & a_3 & a_3 a - 1 \\ 0 & -a_2 & a_3 - a_1 a_2 \\ 0 & a_1 & a_1^2 - a_2 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}.$$

Trigonometry. The general case.

Euler formula for the exponential matrix is defined by the series

$$\exp(\sum_{k=1}^{n-1} E^k \phi_k) = g_0(\phi) + Eg_1(\phi) + E^2 g_2(\phi) + ... + E^{n-1} g_{n-1}(\phi).$$

Here ϕ means the set of $(n-1)$ parameters $\phi := (\phi_1, \phi_2, \phi_3, ...\phi_{n-1})$.

Define $(n-1)$-order polynomial

$$Q(U) = g_0(\phi) + Ug_1(\phi) + U^2 g_2(\phi) + ... + U^{n-1} g_{n-1}(\phi).$$

Evolution equation

$$\frac{\partial}{\partial \phi_k} \mathbf{v}^{\mathbf{g}}(\phi) = E^k \mathbf{v}^{\mathbf{g}}(\phi), \ \phi = (\phi_1, \phi_2, ..., \phi_{n-1}), \ k = 1, ..., n-1.$$

where $\mathbf{v}^{\mathbf{g}}(\phi)$ is a vector with components

$$\mathbf{v}^{\mathbf{g}} = [g_0, g_1, g_2, ..., g_{n-1}]^T.$$

Theorem.

The system of evolution equations are reduced to n-order Riccati equation of the form

$$\frac{d}{d\phi_{n-1}} U = f(U),$$

under the set of constraints

$$g_k(\phi) = 0, k = 2, 3, ..., n-1,$$

the solution of the n-order Riccati equation is defined as a fraction of two trigonometric functions by

$$U(\phi_{n-1}) = -\frac{g_0(\phi_{n-1})}{g_1(\phi_{n-1})},$$

where ϕ_{n-1} depends of $(n-2)$ other parameters $\phi_{n-1} = \phi_{n-1}(\phi_1, \phi_2, ..., \phi_{n-2})$, this dependence is implicitly defined by the constraints.

Thus, transformation of the linear system of evolution equations into canonical form of n-order Riccati equation requires $(n-2)$ constraints. Under these constraints the $(n-1)$ order derivative polynomial $Q(U)$ is reduced into the linear function of the form

$$Q(U) = g_0 + U g_1.$$

Then the solution of equation $Q(U) = 0$ it turns out to be the solution to n-order Riccati equation. This observation prompts us an idea to seek differential equations for the roots of the $(n-1)$ order polynomial $Q(U)$. In the result we expect to obtain a system of Riccati-type equations for functions

$$u_k = u_k(\phi),\ k = 1, 2, 3, ..., n-1,\ \phi = (\phi_1, \phi_2, ..., \phi_{n-1}),$$

where u_k are roots of the polynomial $Q(u_k)$:

$$Q(U) = 0 \quad \rightarrow \quad g_0(\phi) + U g_1(\phi) + U^2 g_2(\phi) + ... + U^{n-1} g_{n-1}(\phi) = 0.$$

2 Theorem

The system of generalized Riccati equations.

Denote by $u_k(\phi),\ k = 1, 2, 3, ..., n-1;\ \phi = (\phi_1, \phi_2, ..., \phi_{n-1})$, the set of roots of the polynomial

$$Q(U) = \sum_{j=0}^{n-1} U^j g_j(\phi),$$

where coefficients $g_j(\phi), j = 0, 1, 2, ..., n-1$ are solutions of the linear system of evolution equations:

$$\partial_i\ g_j = \sum_{m=1}^{n} (H^i)_j^m g_{m-1},\ i = 1, ..., n-1.$$

Theorem

The functions $u_k(\phi), k = 1, ..., n-1$ obey the following system of nonlinear equations

$$F(u_m) \sum_{k=1}^{n-p} a_{n-k-p}\partial_k\ u_m = A_p\ f(u_m), \quad m = 1, ..., n-1.$$

where $F(u_m)$ is $(n-2)$-degree truncated polynomial of the form

$$F(u_m) = \frac{dQ(U)}{dU}|_{U=u_m} = u_m^{n-2} + \sum_{k=0}^{n-3} u_m^k A_k(m) = \prod_{k=1,k\neq m}^{n-1} (u_m - u_k),$$

and $A_p(m)$ is p-th coefficient of the polynomial $F(u_m)$.

Proof.

Example, $N = 6$**.**

Let us illustrate the method by taking in a quality of a basic polynomial the six-order polynomial of the form

$$f(X) = X^6 - a_1 X^5 + a_2 X^4 - a_3 X^3 + a_4 X^2 - a_5 X + a_6.$$

Then, the derived polynomial has the form

$$Q(U) = g_5 U^5 + g_4 U^4 + g_3 U^3 + g_2 U^2 + g_1 U + g_0.$$

Let U be one of the roots of the polynomial $Q(U)$. Then, according to Riccati-type equations the function U obeys the following system of equations

$$F(U)\ \partial_k\ U = \sum_{j=1}^{5} M_{kj}\ A_j\ f(U),\ \ k = 1,2,3,4,5;$$

where the explicit form of the matrix M_{ij} is

$$M_{kj} = \begin{pmatrix} 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1 a_2 + a_1^3 & -a_4 + 2a_1 a_3 - 3a_1^2 a_2 + a_2^2 + a_1^4 \\ 0 & 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1 a_2 + a_1^3 \\ 0 & 0 & 1 & a_1 & -a_2 + a_1^2 \\ 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The polynomial $F(U)$ has the form

$$F(U) = (U-V)(U-W)(U-Y)(U-Z) = A_1 U^4 + A_2 U^3 + A_3 U^2 + A_4 U + A_5,$$

where coefficients $A_p, p = 1,2,3,4,5$ are defined by Vieta's formulae

$$A_1 = 1,\ \ -A_2 = V + W + Y + Z,\ A_3 = VW + VY + VZ + WY + WZ + YZ,$$

$$-A_4 = WVY + WVZ + WYZ + VYZ,\ A_5 = WVYZ,$$

and V, W, Y, Z are the other roots of $Q(U)$. By using the inverse matrix

$$M_{ij}^{-1} = \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

these equations are transformed to the following Riccati-type equations

$$F(U)\ (\partial_5 - a_1\partial_4 + a_2\partial_3 - a_3\partial_2 + a_4\partial_1)U = A_5\ f(U),$$

$$F(U)\ (\partial_4 - a_1\partial_3 + a_2\partial_2 - a_3\partial_1)U = A_4\ f(U),$$
$$F(U)\ (\partial_3 - a_1\partial_2 + a_2\partial_1)U = A_3\ f(U),$$
$$F(U)(\partial_2 - a_1\partial_1)U = A_2\ f(U),$$
$$F(U)\partial_1 U = A_1\ f(U).$$

Now, we collect these equations into unique equation for function U:

$$\begin{pmatrix} 1 & U & U^2 & U^3 & U^4 \end{pmatrix} \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_5 \\ \partial_4 \\ \partial_3 \\ \partial_2 \\ \partial_1 \end{pmatrix} U = f(U).$$

The matrix equation can be written of the following forms either

$$(U^4\partial_1 + U^3(\partial_2 - a_1\partial_1) + U^2(\partial_3 - a_1\partial_2 + a_2\partial_1)$$

$$+U(\partial_4 - a_1\partial_3 + a_2\partial_2 - a_3\partial_1)$$
$$+(\partial_5 - a_1\partial_4 + a_2\partial_3 - a_3\partial_2 + a_4\partial_1))\ U = f(U);$$

or,

$$(\ (U^4 - a_1U^3 + a_2U^2 - a_3U + a_4)\partial_1 +$$
$$+(U^3 - a_1U^2 + a_2U - a_3)\partial_2 +$$
$$+(U^2 - a_1U + a_2)\partial_3 +$$
$$+(U - a_1)\partial_4 + \partial_5)\ U = f(U).$$

Inverse system of generalized Riccati equations.

The n-order Riccati equation

$$\frac{dU}{d\phi} = f(U),$$

with constant coefficient directly is integrated with respect to inverse function $\phi = \phi(U)$ by

$$d\phi = \frac{dU}{f(U)}.$$

Thus, in order to integrate the Riccati equation one has to revert this equation. The system of n-order Riccati equations also admit an inverse system of equations where the set of variables $\psi_k, k = 1, 2, 3, ..., n-1$ are functions of the roots $u_k, k = 1, 2, 3, ..., n-1$. The differential of u_i is

$$du_i = \sum_{k=1}^{n-1} \frac{\partial u_i}{\partial \psi_k}\ d\psi_k,\ i = 1, 2, ..., n-1.$$

In general case the elements of Jacobian matrix are defined as follows

$$J(\frac{Du}{D\psi}) = \begin{pmatrix} A_{1,1} & A_{1,2} & ... & A_{1,n-1} \\ A_{2,1} & A_{2,2} & ... & A_{2,n-1} \\ ... & ... & ... & ... \\ A_{n-1,1} & A_{n-1,2} & ... & A_{n-1,3} \end{pmatrix} \begin{pmatrix} 1 & a_1 & ... & M_{1,n-1} \\ 0 & 1 & a_1... & M_{2,n-1} \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_1 \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

Now let us define the inverse Jacobian matrix

$$J^{-1}(\frac{Du}{D\psi}) = J(\frac{D\psi}{Du}) = \begin{pmatrix} \partial_{u_1}\phi_1 & \partial_{u_2}\phi_1 & ... & \partial_{u_{n-1}}\phi_1 \\ ... & ... & ... & ... \\ \partial_{u_1}\phi_{n-2} & \partial_{u_2}\phi_{n-2} & ... & \partial_{u_{n-1}}\phi_{n-2} \\ \partial_{u_1}\phi_{n-1} & \partial_{u_2}\phi_{n-1} & ... & \partial_{u_{n-1}}\phi_{n-1} \end{pmatrix}.$$

The inverse Jacobian matrix as a product of two matrices:

$$J(\frac{D\psi}{Du}) = \begin{pmatrix} 1 & -a_1 & ... & a_{n-2}(-1)^n \\ 0 & 1 & ... & a_{n-3}(-1)^{n-1} \\ ... & ... & ... & ... \\ 0 & 0 & ... & -a_1 \\ 0 & 0 & ... & 1 \end{pmatrix} \begin{pmatrix} u_1^{n-2} & u_2^{n-2} & ... & u_{n-1}^{n-2} \\ u_1^{n-1} & u_2^{n-1} & ... & u_{n-1}^{n-1} \\ ... & ... & ... & ... \\ u_1 & u_2 & ... & u_{n-1} \\ 1 & 1 & ... & 1 \end{pmatrix} = \frac{\partial}{\partial u_i} \begin{pmatrix} \phi_1 \\ \phi_2 \\ ... \\ \phi_{n-2} \\ \phi_{n-1} \end{pmatrix}$$

Example, $N = 6$**.**
The inverse system of equations are given by

$$\frac{\partial \phi_1}{\partial u_i} = \frac{1}{f(u_i)}(u_i^4 - a_1 u_i^3 + a_2 u_i^2 - a_3 u_i + a_4),$$

$$\frac{\partial \phi_2}{\partial u_i} = \frac{1}{f(u_i)}(u_i^3 - a_1 u_i^2 + a_2 u_i + a_3),$$

$$\frac{\partial \phi_3}{\partial u_i} = \frac{1}{f(u_i)}(u_i^2 - a_1 u_i + a_2),$$

$$\frac{\partial \phi_4}{\partial u_i} = \frac{1}{f(u_i)}(u_i - a_1),$$

$$\frac{\partial \phi_5}{\partial u_i} = \frac{1}{f(u_i)},$$

where $u_1 = U, u_2 = V, u_3 = W, u_4 = Y, u_5 = Z$. These equations can be cast into the following matrix form

$$\frac{\partial}{\partial u_i}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \frac{1}{f(u_i)}\begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_i^4 \\ u_i^3 \\ u_i^2 \\ u_i \\ 1 \end{pmatrix}.$$

Applications

1. **Riccati and relativistic mechanics**.

$$f(X)=X^2-2p_0X+p^2,\quad x_1=p_0+mc,\ x_2=p_0-mc.$$

$$\frac{du}{d\phi}=f(u).$$

$$u(\phi,\phi_0)=p_0(\phi_0)-p_0(\phi).$$

2. **Riccati and cross-ratio**.

$$\exp(mc\phi)=\frac{p_0+mc}{p_0-mc}=\frac{x_1}{x_2}$$

$$\exp(mc(\phi_1-\phi_2))=\frac{p_0+mc-u_1}{p_0-mc-u_1}\frac{p_0-mc-u_2}{p_0+mc-u_2}=\frac{x_1-u_1}{x_2-u_1}\frac{x_2-u_2}{x_1-u_2}.$$

3. **Riccati and hyperbolic and elliptic geometries**.
Geodesic line — circle with $R=mc$. End points of geodesic line $x_1=p_0+mc,\quad x_2=p_0-mc$.
4. $N=3$ case.

$$f(X)=X^3-a_1X^2+a_2X-a_3,\quad x_1,x_2,x_3.$$

$$\exp(V\phi)=(u-x_1)^{m_{32}}(u-x_2)^{m_{13}}(u-x_3)^{m_21}$$

$$\frac{du}{d\phi}=u^3-a_1u^2+a_2u-a_3.$$

$N=3$.

$$(U-V)\partial_1U=f(U)$$

$$(U-V)\partial_2U=(a_1-V)\ f(U)$$

$N=4$.

$$(U-V)(U-W)\partial_1U=f(U)$$

$$(U-V)(U-W)\partial_2U=(a_1-V-W)\ f(U)$$

$$(U-V)(U-W)\partial_3U=((a_1^2-a_2)-a_1(V+W)+VW)f(U)$$

Inverse system of equations.

$$\frac{\partial \phi_1}{\partial u} = \frac{1}{f(u)}(u^2 - a_1 u + a_2),$$

$$\frac{\partial \phi_2}{\partial u_i} = \frac{1}{f(u)}(u - a_1),$$

$$\frac{\partial \phi_3}{\partial u} = \frac{1}{f(u)}.$$

References

[1] R.M.Yamaleev, Geometrical and physical interpretation of evolution governed by general complex algebra. *J.Math. Anal. Appl.* **340(1)** (2008),p.1046-1057.

[2] R.M.Yamaleev, Representation of solutions of n-order Riccati equation via generalized trigonometric functions. *J.Math. Anal. Appl.* **420(1)** 12 (2014),p.334-347. http://dx.doi.org/10.1016/j.jmaa.2014.05.066

[3] R.M.Yamaleev, Multicomplex algebras on polynomials and generalized Hamilton dynamics. *J.Math.Anal. Appl.* **322** (2006) 815-824.

[4] R.M.Yamaleev, Complex algebras on n-order polynomials and generalizations of trigonometry, oscillator model and Hamilton dynamics. *J. Adv. Appl. Clifford Al.* **15** No.2 (2005) 123-150.

[5] R.M.Yamaleev, Solutions of Riccati-Abel equation in terms of characteristics of general complex algebra. *Communications of Joint Institute for Nuclear Research* **E5-2012-129** (2012) 1-16.

[6] R.M.Yamaleev, Transformation of linear system of differential equations to the system of generalized Riccati equations. *Europian J. of Mathematics* **10(4)** (2017),p.139-160.