Functional integral approach to system of stochastic differential equations

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System of stochastic differential equations

$$\begin{cases} dx_1(t) = a_1(\vec{x}, t)dt + \sum_{j=1}^{d} g_{1j}(\vec{x}, t)dw_j(t) \\ \dots \\ dx_d(t) = a_d(\vec{x}, t)dt + \sum_{j=1}^{d} g_{dj}(\vec{x}, t)dw_j(t) \end{cases} \tag{1}$$

$$\vec{x}(t_0) = \vec{x}_0.$$

Problem

solution

$$\vec{x}(\tau)$$

probability density function

$$p(\vec{x}, t, \vec{x}_0, t_0),$$

expectation

$$E[f(\vec{x})]$$

Functional integral

$$p(\vec{x}, t, \vec{x}_0, t_0) =$$
$$\int D[\vec{x}] \exp\{-\int_{t_0}^{t} L_0(\vec{x}(\tau), \dot{\vec{x}}(\tau)) d\tau\}, \qquad (2)$$

$$D[\vec{x}] = \lim_{N\to\infty} \prod_{j=1}^{N-1} dx_j \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi\Delta t}^d \sqrt{\det G(\vec{x}_{j-1}, t_{j-1})}},$$

$$L_0(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \frac{1}{2} \sum_{k,j=1}^{d} G^{-1}_{kj}(\vec{x}(\tau), \tau) \times$$
$$[\dot{x}_k - A_k(\vec{x}(\tau), \tau)][\dot{x}_j - A_j(\vec{x}(\tau), \tau)].$$

$$A_k(\vec{x}(\tau), \tau) = a_k(\vec{x}(\tau), \tau) - \frac{1}{2} \sum_{i,j=1}^{d} g_{ij}(\vec{x}(\tau), \tau) \frac{\partial}{\partial x_i} g_{kj}(\vec{x}(\tau), \tau)$$

G – matrix with elements

$$G_{ij}(\vec{x}(\tau), \tau) = \sum_{k=1}^{d} g_{ik}(\vec{x}(\tau), \tau) g_{jk}(\vec{x}(\tau), \tau)$$

Case of system of SDEs is more complicated than case of SDEs. Therefore we consider the Onsager-Machlup functional technique only for the flat space when the diffusion matrix for system of SDEs defines a Riemannian space with vanishing curvature.

Evaluation of functional integral

Following this method we distinguish among all trajectories the classical trajectory for which the action takes the extreme value. The classical trajectory is found as the solution of the multidimensional Euler-Lagrange equation. Further, to compute the integral, we use the decomposition of action with respect to the classical trajectory.

Euler-Lagrange equations

$$\begin{cases} \frac{d}{dt}(\frac{\partial L_0}{\partial \dot{x}_1}) - \frac{\partial L_0}{\partial x_1} = 0 \\ \dots \\ \frac{d}{dt}(\frac{\partial L_0}{\partial \dot{x}_d}) - \frac{\partial L_0}{\partial x_d} = 0 \end{cases} \tag{3}$$

$$S[y(\tau)] \approx S[y_{\text{cl}}(\tau)] + \frac{1}{2}\delta^2 S[y_{\text{cl}}(\tau)].$$

$$\delta^2 S[y_{\text{cl}}(\tau)] = \int_{t_0}^{t} \sum_{i,j=1}^{d} \delta y_i \Lambda_{ij} \delta y_j d\tau,$$

$$y = y_{\text{cl}} + \delta y,$$

$$\Lambda_{ij} = (\frac{\partial^2 L}{\partial y_i \partial y_j})_{y_{\text{cl}}} + (\frac{\partial^2 L}{\partial y_i \partial \dot{y}_j})_{y_{\text{cl}}} \frac{d}{dt} - \frac{d}{dt}(\frac{\partial^2 L}{\partial \dot{y}_i \partial y_j})_{y_{\text{cl}}} - \frac{d}{dt}(\frac{\partial^2 L}{\partial \dot{y}_i \partial \dot{y}_j})_{y_{\text{cl}}} \frac{d}{dt}.$$

Table 1: Approximate and exact values for expectations for $a = 2$, $c = -2$, $\sigma_1 = \sigma_2 = 1$, $b_1 = b_2 = \frac{1}{4}$, $t_0 = 0$, $t = 1$, $x_{10} = x_{20} = \frac{1}{4}$

	$E[\sqrt{x_1}]$	$E[\sqrt{x_2}]$	$E[(x_1 - x_2)]$
Approximate values	0.4651	0.4651	0.0001
Exact values	0.5	0.5	0

Table 2: Approximate and exact values for expectations for $a = 2$, $c = -2$, $\sigma_1 = 9$, $\sigma_2 = 10$, $b_1 = \frac{81}{4}$, $b_2 = \frac{100}{4}$, $t_0 = 0$, $t = 1$, $x_{10} = x_{20} = 4$

	$E[\sqrt{x_1}]$	$E[\sqrt{x_2}]$	$E[(x_1 - x_2)]$
Approximate values	6.037	10.72	-11.17
Exact values	5.806	13.19	-15.17

Example

$$\begin{cases} dx_1(t) = (a_1 x_1 + c_1\sqrt{x_1 x_2} + b_1)dt + \sigma_1\sqrt{x_1}dw_1(t) \\ \dots \\ dx_2(t) = (a_2 x_2 + c_2\sqrt{x_1 x_2} + b_2)dt + \sigma_2\sqrt{x_2}dw_2(t) \end{cases} \tag{4}$$

$$x_1(t_0) = x_{10}, x_2(t_0) = x_{20}.$$

Grid method for the solution of nonlinear boundary problems

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