Partial analytic integration of Cosserat PDE system describing dynamics of slender structures

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Contents

Research Object and Applications

Object

Slender (nearly one-dimensional) structures (e.g., fibers, rods, cables, ...).

Applications

- biophysics
- visual computing
- civil and mechanical engineering
- microelectronics and robotics
- ....

Model

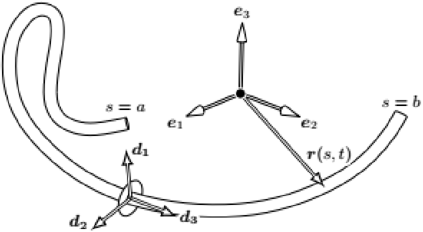
Model

Description of the dynamical behavior of flexible one-dimensional structures is provided by special Cosserat theory of elastic rods (Antmann'1995).
This is a **general and geometrically exact dynamical model** that takes bending, extension, shear, and torsion into account as well as rod deformations under external forces and torques.

Main Obstacle

Stiffness: the deformation modes of a rod evolve on different time scales that renders the problem inherently stiff and demands for appropriate methods in numerical simulation.

Cosserat Model of Rod



The vector set $\{\boldsymbol{d}_k\}$ forms a right-handed orthonormal basis at each point of the centerline. The directors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ span the local material cross-section, whereas $\boldsymbol{d}_3$ is perpendicular to the material cross-section.

In the (special) Cosserat theory of rods, the motion of a rod is defined by

$$[a,b]\times\mathbb{R}\ni(s,t)\mapsto(\boldsymbol{r}(s,t),\boldsymbol{d}_1(s,t),\boldsymbol{d}_2(s,t))\in\mathbb{E}^3\,.$$

Compatibility Equations

The directors evolve according to the kinematic relations:

$$\partial_s \boldsymbol{d}_k = \boldsymbol{\kappa} \times \boldsymbol{d}_k \,, \qquad \partial_t \boldsymbol{d}_k = \boldsymbol{\omega} \times \boldsymbol{d}_k \,,$$

where $\boldsymbol{\kappa}$ is the Darboux and $\boldsymbol{\omega}$ the twist vector function, respectively. Their coordinates with respect of the orthonormal basis are:

$$\boldsymbol{\kappa} = \sum_{k=1}^{3} \kappa_k \boldsymbol{d}_k \,, \qquad \boldsymbol{\omega} = \sum_{k=1}^{3} \omega_k \boldsymbol{d}_k \,.$$

The linear strains of the rod read are given by $\boldsymbol{\nu} = \sum_{k=1}^{3} \nu_k \boldsymbol{d}_k = \partial_s \boldsymbol{r}$, and the velocity of a cross-section material plane by $\boldsymbol{\upsilon} = \partial_t \boldsymbol{r}$.

The equality $\partial_t \partial_s \boldsymbol{d}_k = \partial_s \partial_t \boldsymbol{d}_k$ implies the compatibility equation

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\kappa} \,.$$

Similarly, $\partial_t \boldsymbol{\nu} = \partial_s \boldsymbol{\upsilon}$ implies another compatibility equation

$$\partial_t \boldsymbol{\nu} = \partial_s \boldsymbol{\upsilon} + \boldsymbol{\kappa} \times \boldsymbol{\upsilon} - \boldsymbol{\omega} \times \boldsymbol{\nu} \,.$$

Governing PDE System I

The full system of PDEs governing the deformation of an elastic rod reads:

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\kappa}\,, \tag{1}$$
$$\partial_t \boldsymbol{\nu} = \partial_s \boldsymbol{\upsilon} + \boldsymbol{\kappa} \times \boldsymbol{\upsilon} - \boldsymbol{\omega} \times \boldsymbol{\nu}\,, \tag{2}$$
$$\rho\, A\, \partial_t \boldsymbol{\upsilon} = \partial_s \boldsymbol{n} + \boldsymbol{F}\,, \tag{3}$$
$$\partial_t \boldsymbol{h} = \partial_s \boldsymbol{m} + \boldsymbol{\nu} \times \boldsymbol{n} + \boldsymbol{T}\,. \tag{4}$$

where (1)-(2) are kinematic equations, (3)-(4) are dynamic equations and

- $\boldsymbol{n} = \sum_{k=1}^{3} n_k \boldsymbol{d}_k$ and $\boldsymbol{m} = \sum_{k=1}^{3} m_k \boldsymbol{d}_k$ are the internal stresses ,
- $\boldsymbol{h} = \sum_{k=1}^{3} h_k\, \boldsymbol{d}_k$ are the angular momenta ,
- $\boldsymbol{F}$ and $\boldsymbol{T}$ are the external forces and torques acting on the rod ,
- $\rho = \rho(s)$ is the linear density and $A = A(s)$ is the cross-section area .

Governing PDE System II

The internal stresses $\boldsymbol{m}(s,t)$ and $\boldsymbol{n}(s,t)$ are related to the extension and shear strains $\boldsymbol{\nu}(s,t)$ as well as to the flexure and torsion strains $\boldsymbol{\kappa}(s,t)$ by the constitutive relations

$$\boldsymbol{m}(s,t) = \hat{\boldsymbol{m}}\left(\boldsymbol{\kappa}(s,t), \boldsymbol{\nu}(s,t), s\right)\,, \quad \boldsymbol{n}(s,t) = \hat{\boldsymbol{n}}\left(\boldsymbol{\kappa}(s,t), \boldsymbol{\nu}(s,t), s\right)\,.$$

Under certain reasonable assumptions (Antman'1995) on the structure of the right-hand sides in (3)-(4), they take the form

$$\begin{aligned} \rho \boldsymbol{J} \cdot \partial_t \boldsymbol{\omega} &= \partial_s \hat{\boldsymbol{m}} + \boldsymbol{\kappa} \times \hat{\boldsymbol{m}} + \boldsymbol{\nu} \times \hat{\boldsymbol{n}} - \boldsymbol{\omega} \times (\rho \boldsymbol{J} \cdot \boldsymbol{\omega}) + \boldsymbol{\upsilon}\,, \\ \rho\, A\, \partial_t \boldsymbol{\upsilon} &= \partial_s \hat{\boldsymbol{n}} + \boldsymbol{\kappa} \times \hat{\boldsymbol{n}} - \boldsymbol{\omega} \times (\rho\, A\, \boldsymbol{\upsilon}) + \boldsymbol{F}\,. \end{aligned}$$

where $\boldsymbol{J}$ is the inertia tensor of the cross-section per unit length.

Infinitesimal Symmetry

System of differential equations of order q with $n-$independent and $m-$dependent variables:

$$F_k(\boldsymbol{x}, \boldsymbol{u}^{(q)}) = 0\,, \qquad k = 1, 2, \ldots, p\,.$$

A symmetry is a transformation that maps solutions of differential system to solutions.

Sophus Lie: To find the symmetry group work infinitesimally.

The vector field

$$\mathcal{X} = \sum_{i=1}^{n} \xi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \theta_\alpha(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry or symmetry generator if its flow $\exp(a\,\mathcal{X})$ is a one-parameter symmetry group of the differential system.

Determining equations

To find the infinitesimal symmetry, the vector field $\mathcal{X}$ is prolonged to the jet space whose coordinates are the derivatives occurring in the differential system:

$$\mathcal{X}^{(q)} \equiv \mathcal{X}^{(pr)} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{\#J=0}^{q} \theta_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\theta_\alpha^J = D_J \left(\theta^\alpha - \sum_{i=1}^{n} u_i^\alpha \xi^i \right) + \sum_{i=1}^{n} u_{J,i}^\alpha \xi^i \,.$$

and D_J is the total derivative operator.

Infinitesimal invariance criterion:

$$\mathcal{X}^{(pr)}(F_k) = 0 \qquad \text{whenever} \qquad F_k = 0\,, \qquad k = 1, 2, \ldots, p\,.$$

It generates the overdetermined system

$$\mathcal{L}\left(\boldsymbol{x}, \boldsymbol{u}\,;\, \xi^{(q)}, \theta^{(q)}\right) = 0$$

of linear PDEs in ξ^i, θ^α called infinitesimal determining equations.

Infinitesimal Symmetry for Compatibility Equation

Consider now the subsystem of the Cosserat governing PDEs which is the vector form of compatibility equations:

$$\boldsymbol{F} = \mathbf{0}, \quad \boldsymbol{F} := \partial_s \boldsymbol{\omega} - \partial_t \boldsymbol{\kappa} + \boldsymbol{\omega} \times \boldsymbol{\kappa},$$

Its Lie symmetry transformation group

$$\begin{aligned} s' &= s'(s, t, \boldsymbol{\omega}(s,t), \boldsymbol{\kappa}(s,t)), \quad & \boldsymbol{\omega}' &= \boldsymbol{\omega}'(s, t, \boldsymbol{\omega}(s,t), \boldsymbol{\kappa}(s,t)), \\ t' &= t'(s, t, \boldsymbol{\omega}(s,t), \boldsymbol{\kappa}(s,t)), \quad & \boldsymbol{\kappa}' &= \boldsymbol{\kappa}'(s, t, \boldsymbol{\omega}(s,t), \boldsymbol{\kappa}(s,t)), \end{aligned}$$

has infinitesimal generator:

$$\mathcal{X} := \xi^1 \partial_s + \xi^2 \partial_t + \boldsymbol{\theta} \cdot \partial_{\boldsymbol{\omega}} + \boldsymbol{\vartheta} \cdot \partial_{\boldsymbol{\kappa}}\,.$$

Here we use the vector notation:

$$\boldsymbol{\theta} := \{\theta^1, \theta^2, \theta^3\}\,, \qquad \boldsymbol{\vartheta} := \{\vartheta^1, \vartheta^2, \vartheta^3\}\,.$$

Determining Equations and Janet Basis

To compute the determining equations that follow from the infinitesimal criterion of invariance

$$\mathcal{X}^{(pr)}\boldsymbol{F}\mid_{\boldsymbol{F}=\boldsymbol{0}}=\boldsymbol{0}\,,$$

we used the Maple package Desolv (Carminati,Vu'2000) and its routine gendef. It outputs 42 first-order equations.

The most universal algorithmic way to solve the determining system is its completion to a canonical involutive (or to a Gröbner basis) form and solving afterwards. For completion to involution we applied the Maple package Janet Janet (Blinkov,Cid,Gerdt,Plesken,Robertz'2003).

The output Janet basis contains 86 linear PDEs. Unlike the input set of determining equations, in the Janet basis form it is solvable by both pdsolve, the built-in Maple routine, and by the subroutine pdsolv of Desolv.

We prefer the output form of the latter solver, since one of the former solver contains non-local expressions (integrals).

Arbitrariness in Infinitesimal Symmetry

Although the outputs of solvers pdsolv in Desolv and pdsolve are dissimilar, both of them contain five arbitrary functions in independent variables (s, t).

Question: Is there a solution with a larger number of arbitrary functions in the two variables?

Answer: No. This can be detected from the structure of differential dimensional polynomial (Lange-Hegermann'2014). It is easily computed by the routine DifferentialSystemDimensionPolynomial of the Maple package DifferentialThomas (Bächler,Gerdt,Lange-Hegermann,Robertz'2012).

It takes Janet basis of the determining system as an input and outputs the differential dimensional polynomial:

$$\frac{5}{2}\sigma^2 + \frac{21}{2}\sigma + 11 = 5\binom{\sigma+2}{\sigma} + 3\binom{\sigma+1}{\sigma} + 3.$$

The first term shows that the general analytic solution depends on five arbitrary functions in two variables.

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Solution to Determining System

Two of arbitrary function in the solution of determining system are superfluous for us and can be omitted.

These functions appear in the solution as shifts in the independent variables (PDE system under consideration is autonomous).

Taking this into account, the obtained solution is analytic and can be presented as

$$\xi^1 = \xi^2 = 0, \quad \boldsymbol{\theta} = \hat{\boldsymbol{A}}\,\boldsymbol{\omega} + \partial_t \boldsymbol{p}, \quad \boldsymbol{\vartheta} = \hat{\boldsymbol{A}}\,\boldsymbol{\kappa} + \partial_s \boldsymbol{p}\,,$$

where

$$\hat{\boldsymbol{A}}\,(s,t) = \begin{bmatrix} 0 & -c(s,t) & b(s,t) \\ c(s,t) & 0 & -a(s,t) \\ -b(s,t) & a(s,t) & 0 \end{bmatrix}, \quad \boldsymbol{p}(s,t) := \begin{bmatrix} a(s,t) \\ b(s,t) \\ c(s,t) \end{bmatrix}.$$

Here $a(s,t), b(s,t), c(s,t)$ are arbitrary analytic functions.

Lie Equations

Having obtained the infinitesimal symmetry generator, to find the symmetry group one has to solve the system of two scalar Lie equations:

$$\mathrm{d}_a s' = 0, \quad s'(0) = s \quad \Longrightarrow \quad s' = s,$$
$$\mathrm{d}_a t' = 0, \quad t'(0) = t \quad \Longrightarrow \quad t' = t,$$

and two vector ones:

$$\mathrm{d}_a \boldsymbol{\omega}' = \hat{\boldsymbol{A}}\, \boldsymbol{\omega}' + \partial_t \boldsymbol{p}\,, \quad \boldsymbol{\omega}'(0) = \boldsymbol{\omega},$$
$$\mathrm{d}_a \boldsymbol{\kappa}' = \hat{\boldsymbol{A}}\, \boldsymbol{\kappa}' + \partial_s \boldsymbol{p}\,, \quad \boldsymbol{\kappa}'(0) = \boldsymbol{\kappa}.$$

The last equations can be solved with the Maple command dsolve. However, its output is awkward, and it is not easy to convert it into a compact form.

A compact form of the solution can be obtained "by hand".

Solution to Lie Equations

It is easy to see that expression

$$\omega' = \exp(\, a\hat{\boldsymbol{A}}\,)\,\omega + \exp(\, a\,\hat{\boldsymbol{A}}\,)\left(\int_0^a \exp(-y\hat{\boldsymbol{A}}\,)\,dy\right)\,\partial_t \boldsymbol{p}$$

satisfies the vector Lie equations for, and it is the solution by the classical existence and uniqueness theorem for systems of ODEs.

Furthermore, application of the Cayley-Hamilton theorem gives:

$$\hat{\boldsymbol{A}}^{\,3} = -p^2\hat{\boldsymbol{A}}\,, \qquad p := |\boldsymbol{p}| = \sqrt{a^2(s,t)+b^2(s,t)+c^2(s,t)}\,.$$

Thereby,

$$\omega' = \left(\hat{\boldsymbol{I}} + \frac{\sin(p\,a)\,\hat{\boldsymbol{A}}}{p} + \frac{(1-\cos(p\,a))\,\hat{\boldsymbol{A}}^{\,2}}{p^2}\right)\omega +$$
$$+\left(\frac{p\,a\hat{\boldsymbol{A}}^{\,2} + p^3\,a\hat{\boldsymbol{I}} - \sin(p\,a)\,\hat{\boldsymbol{A}}^{\,2} - \cos(p\,a)\,p\hat{\boldsymbol{A}} + p\hat{\boldsymbol{A}}}{p^3}\right)\partial_t \boldsymbol{p}\,,$$

Lie Symmetry Transformations

Without loss of generality the arbitrary vector $\boldsymbol{p}$ and matrix $\hat{\boldsymbol{A}}$ for $a \neq 0$ can be rescaled to absorb the group parameter a. It is equivalent to putting $a := 1$. In so doing, the transformation for ω can be rewritten in terms of arbitrary vector-function $\boldsymbol{p}$:

$$\boldsymbol{\omega}' = \boldsymbol{\omega} - \frac{\sin(p)}{p}\,\boldsymbol{p}\times\boldsymbol{\omega} + \frac{1-\cos(p)}{p^2}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\boldsymbol{\omega}) - p^2\,\boldsymbol{\omega}\right) + \\ + \partial_t\boldsymbol{p} + \frac{p-\sin(p)}{p^3}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\partial_t\boldsymbol{p}) - p^2\,\partial_t\boldsymbol{p}\right) - \frac{1-\cos(p)}{p^2}\,\boldsymbol{p}\times\partial_t\boldsymbol{p}\,.$$

Respectively, the transformation for κ reads

$$\boldsymbol{\kappa}' = \boldsymbol{\kappa} - \frac{\sin(p)}{p}\,\boldsymbol{p}\times\boldsymbol{\kappa} + \frac{1-\cos(p)}{p^2}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\boldsymbol{\kappa}) - p^2\,\boldsymbol{\kappa}\right) + \\ + \partial_s\boldsymbol{p} + \frac{p-\sin(p)}{p^3}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\partial_s\boldsymbol{p}) - p^2\,\partial_s\boldsymbol{p}\right) - \frac{1-\cos(p)}{p^2}\,\boldsymbol{p}\times\partial_s\boldsymbol{p}\,.$$

General Solution to the Kinematic subsystem I

Proposition 1

The general analytic solution to

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\kappa}$$

depends on the arbitrary vector function $\boldsymbol{p}(s,t)$ and reads

$$\begin{aligned}
\boldsymbol{\omega} =& \partial_t \boldsymbol{p} + \frac{p - \sin(p)}{p^3}\, \left(\boldsymbol{p}\, (\boldsymbol{p} \cdot \partial_t \boldsymbol{p}) - p^2\, \partial_t \boldsymbol{p}\right) - \frac{1 - \cos(p)}{p^2}\, \boldsymbol{p} \times \partial_t \boldsymbol{p}\,, \\
\boldsymbol{\kappa} =& \partial_s \boldsymbol{p} + \frac{p - \sin(p)}{p^3}\, \left(\boldsymbol{p}\, (\boldsymbol{p} \cdot \partial_s \boldsymbol{p}) - p^2\, \partial_s \boldsymbol{p}\right) - \frac{1 - \cos(p)}{p^2}\, \boldsymbol{p} \times \partial_s \boldsymbol{p}\,.
\end{aligned}$$

Corrolary 1

The Lie symmetry group acts transitively.

General Solution to the Kinematic subsystem II

The determining equation system for kinematic subsystem of Cosserat rod

$$\partial_t \boldsymbol{\kappa} = \boldsymbol{\partial_s \omega} + \boldsymbol{\omega} \times \boldsymbol{\kappa}\,, \quad \partial_t \boldsymbol{\nu} = \partial_s \boldsymbol{\upsilon} + \boldsymbol{\kappa} \times \boldsymbol{\upsilon} - \boldsymbol{\omega} \times \boldsymbol{\nu}$$

contains 138 PDEs and takes about an hour computational time to complete it to involution. Integration of this system yields the symmetry generator

$$\begin{aligned} \mathcal{X}_0 := \quad & (-\partial_s q_1 + q_2\kappa_3 - q_3\kappa_2)\,\partial_{\nu_1} + (-\partial_s q_2 + q_3\kappa_1 - q_1\kappa_2)\,\partial_{\nu_2} + \\ & (-\partial_s q_3 + q_1\kappa_3 - q_3\kappa_1)\,\partial_{\nu_3} + (-\partial_t q_1 + q_2\omega_3 - q_3\omega_2)\,\partial_{v_1} + \\ & (-\partial_t q_2 + q_3\omega_1 - q_1\omega_2)\,\partial_{v_2} + (-\partial_t q_3 + q_1\omega_3 - q_3\omega_1)\,\partial_{v_3}\,. \end{aligned}$$

depending on arbitrary vector function $\boldsymbol{q}(s,t)$. Integration the Lie equations leads to

$$\boldsymbol{\nu}' = (\boldsymbol{q} \times \boldsymbol{\kappa} - \partial_s \boldsymbol{q})a + \boldsymbol{\nu}, \quad \boldsymbol{\upsilon}' = (\boldsymbol{q} \times \boldsymbol{\omega} - \partial_t \boldsymbol{q})a + \boldsymbol{\upsilon}, \quad \boldsymbol{\omega}' = \boldsymbol{\omega}, \quad \boldsymbol{\kappa}' = \boldsymbol{\kappa}.$$

General Solution to the Kinematic subsystem III

Proposition 2

The general analytic solution to the kinematic part of Cosserat PDE system is

$$\boldsymbol{\omega} = \partial_t \boldsymbol{p} + \frac{p - \sin(p)}{p^3}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\partial_t\boldsymbol{p}) - p^2\,\partial_t\boldsymbol{p}\right) - \frac{1-\cos(p)}{p^2}\,\boldsymbol{p}\times\partial_t\boldsymbol{p}\,,$$

$$\boldsymbol{\kappa} = \partial_s \boldsymbol{p} + \frac{p - \sin(p)}{p^3}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\partial_s\boldsymbol{p}) - p^2\,\partial_s\boldsymbol{p}\right) - \frac{1-\cos(p)}{p^2}\,\boldsymbol{p}\times\partial_s\boldsymbol{p}\,,$$

$$\boldsymbol{\nu} = \boldsymbol{q}\times\boldsymbol{\kappa} - \partial_s\boldsymbol{q}\,,\quad \boldsymbol{\upsilon} = \boldsymbol{q}\times\boldsymbol{\omega} - \partial_t\boldsymbol{q}\,.$$

with two arbitrary analytical functions $\boldsymbol{p}(s,t)$ and $\boldsymbol{q}(s,t)$.

Corrolary 2

The symmetry group of kinematic equations acts transitively.

Equivalent Form of the Governing System I

Proposition 3

Solving the equation

$$\omega = \partial_t \boldsymbol{p} + \frac{p - \sin(p)}{p^3} \left(\boldsymbol{p} \left(\boldsymbol{p} \cdot \partial_t \boldsymbol{p} \right) - p^2 \, \partial_t \boldsymbol{p} \right) - \frac{1 - \cos(p)}{p^2} \, \boldsymbol{p} \times \partial_t \boldsymbol{p} \qquad (5)$$

with respect to $\partial_t \boldsymbol{p}$ gives

$$\partial_t \boldsymbol{p} = \frac{\boldsymbol{p} \cdot \omega}{p^2} \, \boldsymbol{p} + \frac{1}{2} \, \boldsymbol{p} \times \omega - \frac{p}{2} \cot\left(\frac{p}{2}\right) \cdot \frac{\boldsymbol{p} \times (\boldsymbol{p} \times \omega)}{p^2} \, . \qquad (6)$$

Remark

This can be verified either by hand computation or by using the routines of the Maple package VectorCalculus after the substitution of (6) into the right-hand side of (5) and simplification of the obtained expression to ω.

Equivalent Form of the Governing System II

Corrolary 3

The governing Cosserat system, after explicit integration of its kinematic part, can be rewritten as PDE system

$$\begin{aligned}
\partial_t \boldsymbol{p} &= \frac{\boldsymbol{p}\cdot\omega}{p^2}\,\boldsymbol{p} + \frac{1}{2}\,\boldsymbol{p}\times\omega - \frac{p}{2}\cot\left(\frac{p}{2}\right)\cdot\frac{\boldsymbol{p}\times(\boldsymbol{p}\times\omega)}{p^2}\,,\\
\partial_t \boldsymbol{q} &= \boldsymbol{q}\times\boldsymbol{\omega} - \boldsymbol{\upsilon}\,,\\
\rho\,\boldsymbol{J}\cdot\partial_t\boldsymbol{\omega} &= \partial_s\hat{\boldsymbol{m}} + \boldsymbol{\kappa}\times\hat{\boldsymbol{m}} + \boldsymbol{\nu}\times\hat{\boldsymbol{n}} - \boldsymbol{\omega}\times(\rho\,\boldsymbol{J}\cdot\boldsymbol{\omega}) + \boldsymbol{L}\,,\\
\rho\,A\,\partial_t\boldsymbol{\upsilon} &= \partial_s\hat{\boldsymbol{n}} + \boldsymbol{\kappa}\times\hat{\boldsymbol{n}} - \boldsymbol{\omega}\times(\rho\,A\,\boldsymbol{\upsilon}) + \boldsymbol{F}
\end{aligned}$$

in unknown vector functions $(\boldsymbol{p},\boldsymbol{q},\omega,\boldsymbol{\upsilon})$, where $\boldsymbol{\kappa}$ and $\boldsymbol{\nu}$ read

$$\begin{aligned}
\boldsymbol{\kappa} &= \partial_s\boldsymbol{p} + \frac{p-\sin(p)}{p^3}\,\left(\boldsymbol{p}\,(\boldsymbol{p}\cdot\partial_s\boldsymbol{p}) - p^2\,\partial_s\boldsymbol{p}\right) - \frac{1-\cos(p)}{p^2}\,\boldsymbol{p}\times\partial_s\boldsymbol{p}\,,\\
\boldsymbol{\nu} &= \boldsymbol{q}\times\boldsymbol{\kappa} - \partial_s\boldsymbol{q}\,.
\end{aligned}$$

Symbolic-Numeric (Exponential) Integration I

Let $\boldsymbol{p} = p\boldsymbol{e}$. Then the first equation in Cosserat system reads

$$p_t = \boldsymbol{e} \cdot \omega, \quad 2\,\boldsymbol{e}_t = \boldsymbol{e} \times \boldsymbol{\omega} - \cot\left(\frac{p}{2}\right) \boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{\omega})$$

Assume that $\boldsymbol{\omega}$ is independent on t on a time interval Δt and choose the Cartesian coordinate system $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ such that $\boldsymbol{e}_3 || \boldsymbol{\omega}$

$$\boldsymbol{e} = A_1\,\boldsymbol{e}_1 + A_2\,\boldsymbol{e}_2 + A_3\,\boldsymbol{e}_3\,, \quad \boldsymbol{\omega} = \omega\,\boldsymbol{e}_3\,, \quad \omega := \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\,.$$

It follows

$$\begin{aligned}
2\,(A_1)_t &= A_2\,\omega - \cot\left(\frac{p}{2}\right) A_1 A_3\,\omega\,, \\
2\,(A_2)_t &= -A_1\,\omega - \cot\left(\frac{p}{2}\right) A_2 A_3\,\omega\,, \\
2\,(A_3)_t &= -\cot\left(\frac{p}{2}\right) (A_3^2 - 1)\,\omega\,, \\
p_t &= A_3\,\omega\,.
\end{aligned}$$

Symbolic-Numeric (Exponential) Integration II

With a help of Maple the last system can be analytically integrated

$$A_1(s,t) = -\frac{\sqrt{C}\cdot sin(\frac{1}{2}\omega(C_2-t))}{\sqrt{\omega^2\cos^2(\frac{1}{2}\omega(C_1-t)) + C\sin^2(\frac{1}{2}\omega(C_1-t))}},$$

$$A_2(s,t) = \frac{\sqrt{C}\cdot cos(\frac{1}{2}\omega(C_2-t))}{\sqrt{\omega^2\cos^2(\frac{1}{2}\omega(C_1-t)) + C\sin^2(\frac{1}{2}\omega(C_1-t))}},$$

$$A_3(s,t) = \frac{\sqrt{\omega^2-C}\cdot cos(\frac{1}{2}\omega(C_1-t))}{\sqrt{\omega^2\cos^2(\frac{1}{2}\omega(C_1-t)) + C\sin^2(\frac{1}{2}\omega(C_1-t))}},$$

$$p(s,t) = 2\arccos\left(\frac{\sqrt{\omega^2-C}\sin(\frac{1}{2}\omega(C_1-t))}{\omega}\right),$$

Symbolic-Numeric (Exponential) Integration III

where C, C_1, C_2 are functions of s determined by the initial data

$$C(s) := \omega^2 \left(1 - A_3^2(s,t_0)\right) \sin^2\left(\frac{p(s,t_0)}{2}\right) \,,$$

$$C_1(s) := t_0 + \frac{A_3(s,t_0)|\sin(p(s,t_0))|}{\sqrt{\omega^2 - C(s)}} \,,$$

$$C_2(s) := t_0 + \frac{2}{\omega} \arctan\left(\frac{A_1(s,t_0)}{A_2(s,t_0)}\right) \,.$$

Symbolic-Numeric (Exponential) Integration IV

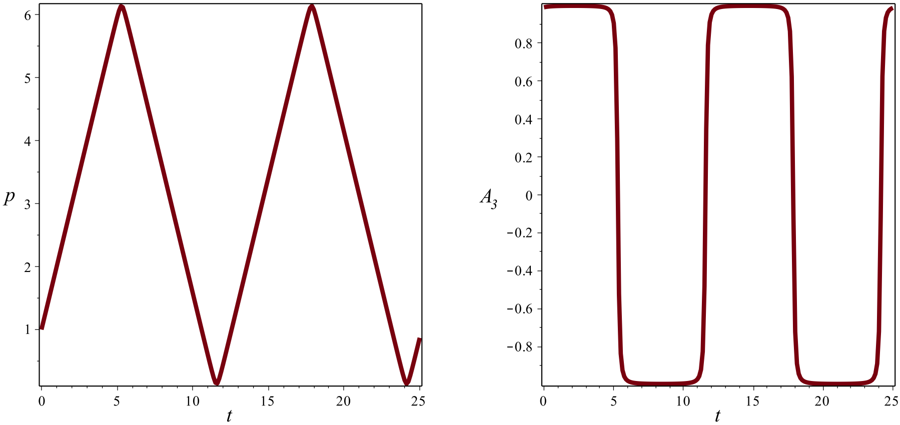


Figure: Illustration of the temporal evolution of $p(s_0, t)$ (left) and $A_3(s_0, t)$ (right).

Computational Experiments I

We compared our symbolic-numeric method with the so-called generalized α-method (Chung,Hulbert'1993) which is well established and commonly considered as the best numerical integration method in structural mechanics.

Test cases:

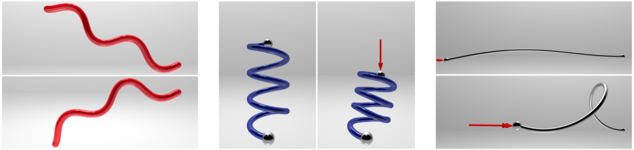
(i) a sinus-like shaped rod which is released under gravity from a horizontal position (no damping);

(ii) a highly damped helical rod subject to a time-varying end point load;

(iii) a straight rod (45 cm) subject to a time-varying torque;

(iv) a helical rod with low damping that is excited by a force parallel to the axis of the helix and released after 0.1 s showing the typical oscillating behavior of a steel-like coil spring.

Computational Experiments II



We measured the required computation time on a machine with an Intel(R) Xeon E5 with 3.5 GHz and 32 GB DDR-RAM without parallelization.

For all test cases, we obtained significant speedup of our symbolic-numeric method ("snm") compared to the generalized α-method ("α"):

(i) speedup of a factor more than 20$\times$ (α: 4.1 s; snm: 0.2 s);
(ii) speedup of over 21$\times$ (α: 4.3 s; snm: 0.2 s);
(iii) speedup of approx. 19$\times$ (α: 3.8 s; snm: 0.2 s);
(iv) speedup of approx. 34$\times$ (α: 6.8 s; snm: 0.2 s).

Conclusions

- The main barrier to numerical solving of the governing PDE system of equations in the Special Cosserat Theory of a Rod is their stiffness. One way to avoid this barrier is (partial) symbolic integration of the equations.
- Lie symmetry-based approach is the most universal one to symbolic integration of differential equations.
- We applied this approach to the parameter-free subsystem of the PDE system which contains the kinematic relations.
- With assistance of computer algebra-based software we were able to find the symmetry group of the subsystem and explicitly construct its general analytic solution.
- The obtained general analytic solution to the kinematic subsystem allows to rewrite the Cosserat system into equivalent form which is more appropriate for numerical solving, since it avoids stiffness.
- We developed a symbolic-numeric integration method based on ideas of exponential integration and demonstrated, by computational experiments, its superiority over the generalized $\alpha-$method, the best known pure numerical method of solving Cosserat PDE system.

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