The Numerical Evaluation of Universal Quantities of Directed Bond Percolation: Three-loop approximation

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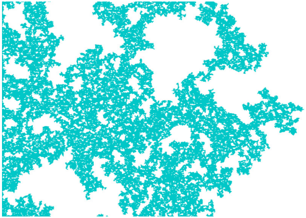
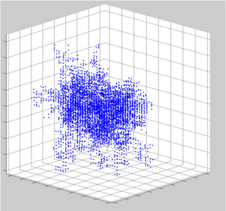
Percolation

- Percolation processes – agent and medium
- Percolation – passage of an agent through an irregularly structured medium





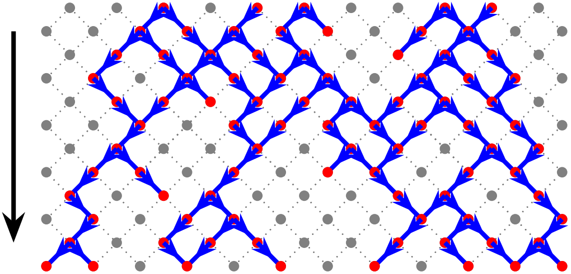
Examples

Replace magnetic atoms by nonmagnetic atoms, exploring path in labyrinth, movement of fluids through porous materials, ...

Directed Percolation

Directed percolation (DP)

The open bonds can be passaged of an agent only from one of the two connecting sites, whence the allowed passage direction globally defines a preferred direction in space.



Reaction scheme

$$A + A \xrightarrow{\kappa} A \qquad A \xrightarrow{\sigma} A + A \qquad A \xrightarrow{\mu} \varnothing$$

Directed Percolation

- Absorbing and active phase
- Non-equlibrium second order phase transition
- DP universal class
- The mean square radius $R(t)$ and dynamical exponent z

$$R^2(t) \sim t^{2/z}$$

- The numbers of active site $N(t)$ and exponent $\theta = -\frac{\eta}{2z}$

$$N(t) \sim t^{\theta}$$

Examples

Reggeon field theory, spreading fire or insect in forest, propagation illness in population, . . .

Directed Percolation

- Mean field equation

$$\partial_t n(t) = (\sigma - \mu) n(t) - \kappa n(t)^2$$

- Absorbing state: $(\sigma < \mu) \qquad n(\infty) = 0$
- Active state: $(\sigma > \mu) \qquad n(\infty) = (\sigma - \mu)/\kappa$
- Critical exponet

$$z = 2 \qquad\qquad \theta = \eta = 0$$

Reaction scheme

$$A + A \xrightarrow{\kappa} A \qquad A \xrightarrow{\sigma} A + A \qquad A \xrightarrow{\mu} \varnothing$$

Action functional

- Standard approach – constructing a field theory representation[1] for a stochastic system from the Langevin equation

$$\begin{aligned} \partial_t \psi(t, \boldsymbol{x}) &= D_0 \left(\nabla^2 - \tau_0 \right) \psi(t, \boldsymbol{x}) - \frac{\lambda_0 D_0}{2} \psi^2(t, \boldsymbol{x}) + \xi(t, \boldsymbol{x}) \\ \langle \xi(t, \boldsymbol{x}) \xi(t', \boldsymbol{x}') \rangle &= \lambda_0 D_0 \rho(t, \boldsymbol{x}) \delta(t - t') \delta^d(\boldsymbol{x} - \boldsymbol{x}') \end{aligned}$$

- Introducing Martin-Siggia-Rose response field $\tilde{\psi}(t; x)$ and integrating out the Gaussian noise to obtain action functional[2] for the directed percolation problem

$$S_0(\tilde{\psi}, \psi) = \tilde{\psi}[-\partial_t + D_0 \nabla^2 - D_0 \tau_0] \psi + \frac{D_o \lambda_0}{2} \left[\tilde{\psi}^2 \psi - \tilde{\psi} \psi^2 \right]$$

[1] J.L. Cardy and R.L. Sugar. In: *J. Phys. A: Math. Gen.* 13 (1980).
[2] P.C. Martin, E.D. Siggia, and H.A. Rose. In: *Phys. Rev A* 8.423 (1973).

Renormalization group

- Renormalized action

$$\mathsf{S_R} = \tilde{\psi}(-\mathsf{Z_1}\partial_\mathsf{t} + \mathsf{Z_2 D}\partial^2 - \mathsf{Z_3 D}\tau)\psi + \frac{Z_4 D\lambda\mu^\varepsilon}{2}[\tilde{\psi}^2\psi - \tilde{\psi}\psi^2]$$

- Multiplicative renormalization of fields an parameters

$$\psi_0 = \psi Z_\psi, \quad \tilde{\psi}_0 = \tilde{\psi} Z_{\tilde{\psi}}, \quad D_0 = D Z_D, \quad \lambda_0 = \lambda Z_\lambda, \quad \tau_0 = \tau Z_\tau$$

- The basic RG differential equation for the renormalized Greens function Γ_R

$$\left(\mu\partial_\mu + \beta_\lambda\partial_\lambda - \tau\gamma_\tau\partial_\tau - D\gamma_D\partial_D - n_\psi\gamma_\psi - n_{\tilde{\psi}}\gamma_{\tilde{\psi}}\right)\Gamma_R = 0$$

- β and γ functions

$$\gamma_x = \mu\partial_\mu\Big|_0 \ln Z_x, \qquad \beta_\lambda = \mu\partial_\mu\Big|_0 \lambda$$

Renormalization group

- Analytical calculation using the renormalization group method and ε – expansion encountered considerable problems.
- Renormalization procedure in terms of the $\mathcal{R}$ operation

$$\Gamma_R = \mathcal{R}\Gamma = (1-K)\mathcal{R}'\Gamma$$

- The choice of K is ambiguous - Null-momentum substraction scheme
- $\mathcal{R}$-operation on graphs χ[3]

$$\mathcal{R}\chi = \left[\prod_{\mathrm{i}} \left(\frac{1}{\mathrm{n_i}!}\int_0^1 \mathrm{da_i}(1-\mathrm{a_i})^{\mathrm{n_i}}\partial_{\mathrm{a_i}}^{\mathrm{n_i}+1}\right)\right]\chi(\{\mathrm{a}\})$$

where the product is taken over all relevant subgraphs χ_{i} (including the diagram χ as a whole) with the canonical dimension $\mathrm{n_i} \geq 0$ and $\mathrm{a_i}$ is the parameter of stretching of momenta flowing into the *i*-th subgraph inside this graph.

[3] L. Ts. Adzhemyan and M. V. Kompaniets. In: *Theor. Math. Phys.* 169.1 (2011).

Renormalization group

- Using $\mathcal{R}$ operation let us define the following functions[4]

$$f_i = \mathcal{R}[-\tilde{\tau}\partial_{\tilde{\tau}}\bar{\Gamma}_i(\tilde{\tau})]\Big|_{\tilde{\tau}=1}, \qquad \tilde{\tau} = \frac{\tau}{\mu^2}$$

- RG functions using diagrams of one ireducible functions reduce to convergent integrals
- Normalized Green function

$$\bar{\Gamma}_1 = \partial_{i\omega}\Gamma_{\psi\tilde{\psi}}\big|_{p=0,\omega=0}, \qquad \bar{\Gamma}_3 = -\frac{\Gamma_{\psi\tilde{\psi}} - \Gamma_{\psi\tilde{\psi}}\big|_{\tau=0}}{D\tau}\Big|_{p=0,\omega=0},$$

$$\bar{\Gamma}_2 = -\frac{1}{2D}\partial_p^2\Gamma_{\psi\tilde{\psi}}\big|_{p=0,\omega=0}, \qquad \bar{\Gamma}_4 = \frac{\Gamma_{\psi\tilde{\psi}\tilde{\psi}} - \Gamma_{\psi\psi\tilde{\psi}}}{D\lambda\mu^{\varepsilon}}\Big|_{p=0,\omega=0},$$

satisfying the conditions

$$\bar{\Gamma}_i\big|_{\tau=\mu^2} = 1, \quad i = 1, 2, 3, 4$$

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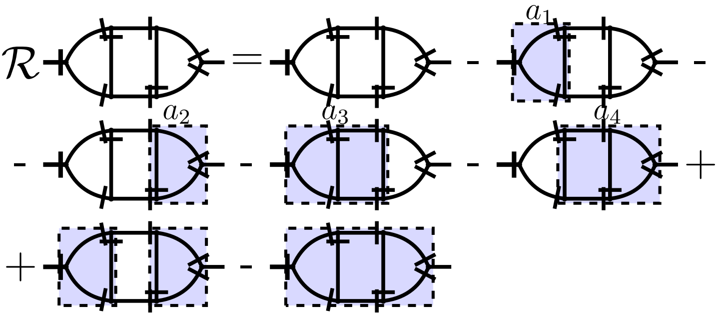
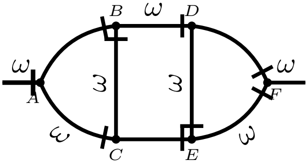
[4] L. Ts. Adzhemyan and M. V. Kompaniets. In: *Theor. Math. Phys.* 169.1 (2011), L. Ts. Adzhemyan et al. In: *Theor. Math. Phys.* 175.3 (2013).

Using $\mathcal{R}$-operation



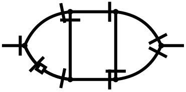
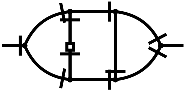
The diagram has four relevant subgraphs with the dimension $n_{a_i} = 0$.

Three-loop Feynman diagrams $\Gamma_{\psi\tilde{\psi}}$ with external frequency

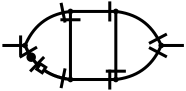
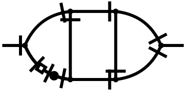


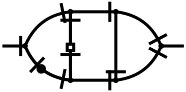
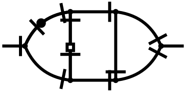
Using $\mathcal{R}$-operation

Contributions Feynman diagrams to f_1

$\partial_{i\omega}\chi =$  $+$  $+ \dots$

The action of the operation $-\tilde{\tau}\partial_{\tilde{\tau}}$ on the line $G(k) = 1/(k^2+\tilde{\tau})$ gives $1/(k^2+\tilde{\tau})$ and graphically, insert unit vertex into the line.

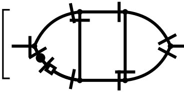
$\underbrace{-\tilde{\tau}\partial_{\tilde{\tau}}\left(\partial_{i\omega}\chi\right)\Big|_{p=\omega=0}}_{\bar{\Gamma}_1} =$  $+$ 

$+$  $+$  $+ \dots$

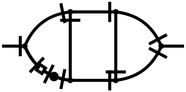
Using $\mathcal{R}$-operation

The next step is including $\mathcal{R}$ operation.

$$\underbrace{\mathcal{R}\left(-\tilde{\tau}\partial_{\tilde{\tau}}\bar{\Gamma}_1\right)\Big|_{\tilde{\tau}=1}}_{f_1} = \int_0^1 da_2 da_4 \partial_{a_2}\partial_{a_4}\Bigg[$$

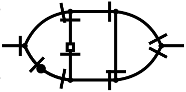


$$+$$

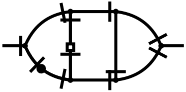


$$\Bigg]$$

$$+\int_0^1 da_2\partial_{a_2}\Bigg[$$



$$+$$



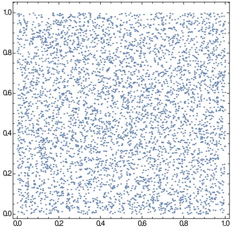
$$\Bigg] + \dots$$

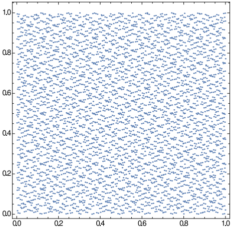
The number of Feynman diagrams

	1 - loop	2 - loop	3 - loop
$\Gamma_{\tilde{\psi}\psi}$	1	2	17
$\Gamma_{\tilde{\psi}\psi\psi}$	1	12	150

Numerical Calculation

- Program for numerical calculation
- GiNaC : from Graphine, GraphState (Python 2.7) to GiNaC archive file (.gar)
- Numcal : is interface between Cuba and GiNaC
- Vegas[5] - quasi Monte Carlo algorithm that use s importance sampling as a variance-reduction technique and Sobol quasi-random sample are used as basic integration mathod.



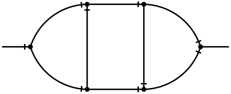


[5] T. Hahn. In: *Comput. Phys. Commun* 168 (2005).

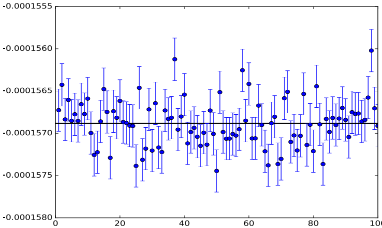
Numeric Evaluation

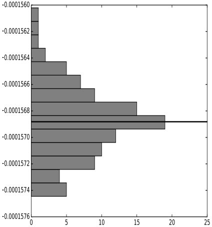
Numeric integration - quasi Monte Carlo method

$$\int\limits_{[0,1]^d} du \chi(u) \approx \frac{1}{N}\sum_{n=1}^{N} \chi(x_n)$$

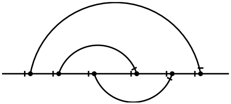


Result: $-1.568818E-04 \pm 2.503367E-08$

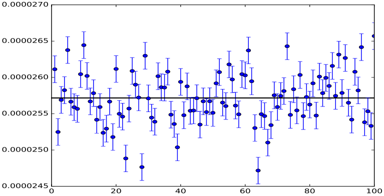


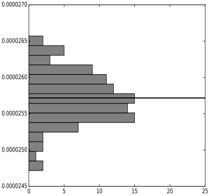


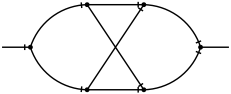
Numeric Evaluation



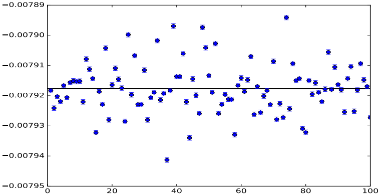
Result: $2.571713E - 05 \pm 1.839028E - 08$

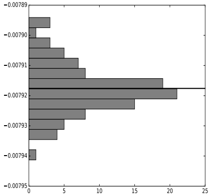






Result: $-7.917592E - 03 \pm 7.671648E - 08$





Renormalization group

- Rewrite γ functions[6]

$$\gamma_i = \frac{2f_i}{1+f_2}, \qquad f_i = \mathcal{R}[-\tilde{\tau}\partial_{\tilde{\tau}}\bar{\Gamma}_i(\tilde{\tau})]|_{\tilde{\tau}=1}$$

- Anomalous dimension fields and parameters

$$\begin{aligned}\gamma_\psi \equiv \gamma_{\tilde{\psi}} &= \frac{\gamma_1}{2}\\ \gamma_D &= \gamma_2 - \gamma_1\\ \gamma_\lambda &= \gamma_4 - \frac{\gamma_1}{2} - \gamma_2\\ \gamma_u = 2\gamma_\lambda &= 2\gamma_4 - \gamma_1 - 2\gamma_2\end{aligned}$$

- Beta function ($u = \lambda^2$)

$$\beta_u = u(-2\varepsilon - \gamma_u)$$

[6] L. Ts. Adzhemyan and M. V. Kompaniets. In: *Theor. Math. Phys.* 169.1 (2011), L. Ts. Adzhemyan et al. In: *Theor. Math. Phys.* 175.3 (2013).

Results

- Deviation from the critical space dimension $\varepsilon = 4 - d$
- Fixed point $u^* = \frac{4}{3}\varepsilon + 1.59024\varepsilon^2 - 0.40931\varepsilon^3 + O(\epsilon^4)$
- Critical exponents

$$z = 2 - \gamma_D^* = 2 - \frac{\varepsilon}{6} - 0.116824\varepsilon^2 + 0.034404\varepsilon^3 + O(\epsilon^4)$$
$$\eta = -\frac{\varepsilon}{3} - 0.27228\varepsilon^2 + 0.074165\varepsilon^3 + O(\epsilon^4)$$

- The calculation accurary for integrals was 10^{-5}
- Critical exponents with data from the analytic calculation[7]

$$z = 2 - \frac{\epsilon}{6}\left[1 + \left(\frac{67}{144} + \frac{59}{72}\ln\frac{4}{3}\right)\epsilon + O(\epsilon^2)\right] = 2 - \frac{\epsilon}{6} - 0.116836\epsilon^2 + O(\epsilon^3)$$
$$\eta = -\frac{\epsilon}{3}\left[1 + \left(\frac{25}{144} + \frac{161}{72}\ln\frac{4}{3}\right)\epsilon + O(\epsilon^2)\right] = -\frac{\epsilon}{3} - 0.272316\epsilon^2 + O(\epsilon^3)$$

[7] H. K. Janssen and U. C. Tauber. In: *Ann. Phys.* 315.147192 (2004).

Exponent

	z		$\delta = \frac{d+\eta}{2z}$		$\theta = -\frac{\eta}{2z}$	
	$d=3$	$d=2$	$d=3$	$d=2$	$d=3$	$d=2$
T_2	1.8874	1.7164	0.7371	0.4486	0.1208	0.3167
T_3	1.9218	1.9917	0.7387	0.4989	0.0835	0.0061
P_1^1	1.8716	1.4424	0.7364	0.4428	0.1515	1.6709
P_2^1	1.9048	1.8091	0.7339	0.4187	0.0715	0.0651
P_1^2	1.9032	1.7985	0.7335	0.4077	0.1029	0.2197
[8] T_2	1.8874	1.7165	0.7371	0.4486	0.1208	0.3167
P_1^1	1.8716	1.4425	0.7364	0.4428	0.1515	1.6705
[9] S	1.901(3)	1.765(2)	0.732(4)	0.451(3)	0.114(4)	0.229(3)

[8] H. K. Janssen. In: *Zeitschrift für Physik B Condensed Matter* 42.2 (1981), pp. 151–154, J. B. Bronzan and J. W. Dash. In: *Physics Letters B* 51.5 (1974), pp. 496–498.

[9] M. Henkel et al. *Non-equilibrium phase transitions*. Vol. 1. Springer, 2008.

Conclusion

- Our two-loop result are in agreement with the analytic calculations and our numerical method is suitable for the calculation of the Feynman graphs to the three loops order
- CUBA - multi-dimensional integration
- Critical exponent z, δ, θ – good agreement with numerical simulation
- DP universal class - numerical method for calculation all anomalous dimesion γ

Thank you for your attention.