Diffusion processes in A model of vector admixture: Turbulent Prandtl number

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Outline

Introduction

Passive vector advection

Formulation of the Model

Analysis of the Model

Results

Conclusion

Introduction - Turbulent Prandtl numbers

- Fully developed turbulence $\rightarrow$ Turbulence at very high Reynolds numbers
- Diffusion processes in fully developed turbulence are characterized by the turbulent Prandtl numbers (ratio of the turbulent viscosity to the corresponding coefficients of diffusivity): Pr_t, $Pr_{m,t}$ and $Pr_{v,t}$
- Interval of experimentally obtained values for Prandtl number: $Pr_t \in \langle 0.7, 0.9 \rangle$

  A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics: Mechanics of Trubulence, (1971)
- Pr_t for a passive scalar advection (temperature, concentration of an impurity) - one-loop RG result: $Pr_t = 0.7179$

  L. Ts. Adzhemyan, A. N. Vasilev, and M. Hnatich, Teor. Mat. Fiz. 58, 72 (1984)
- Two-loop RG calculations give: $Pr_t = 0.7051$

  L. Ts. Adzhemyan, etal., Phys. Rev E 71, 056311 (2005)

  E. Jurčišinová, etal., Phys. Rev. E 82, 028301 (2010)
- The two-loop corrections are less than 2% of the one-loop value

Turbulent Prandtl numbers

- Open question: influence of internal structure of the advected field on the diffusion processes
- Two-loop value of the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence: $Pr_{m,t} = 0.7051$

  E. Jurčišinová, etal., Phys. Rev. E 84, 046311 (2011)
- There is no difference between diffusion processes of a scalar quantity (e.g., temperature) and the weak magnetic field in the kinematic MHD turbulence!
- Two-loop value of the turbulent vector Prandtl number in the framework of the so called $\mathcal{A} = 0$ model: $Pr_{v,t} = 0.7307$

  E. Jurčišinová, etal., Phys. Rev. E 89, 043023 (2014)
- As we can see, the $\mathcal{A} = 0$ model feels the vector structure of the advected field so $Pr_{v,t} \neq Pr_{m,t}$ while $Pr_{m,t} = Pr_t$

Three models of passive vector advection

Three models of passive vector advection in fully developed turbulence

$$
\begin{aligned}
&& \mathcal{A} = 0 \quad \textit{passive admixture} && \\
\partial_t \mathbf{b} + (\mathbf{v}\cdot\partial)\mathbf{b} &=& u_0\nu_0 \Delta \mathbf{b} + \mathbf{f}^{\mathbf{b}}, && (1) \\
&& \mathcal{A} = 1 \quad \textit{kinematic MHD} && \\
\partial_t \mathbf{b} + (\mathbf{v}\cdot\partial)\mathbf{b} &=& u_0\nu_0 \Delta \mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, && (2) \\
&& \mathcal{A} = -1 \quad \textit{linearized } N - S && \\
\partial_t \mathbf{b} + (\mathbf{v}\cdot\partial)\mathbf{b} &=& u_0\nu_0 \Delta \mathbf{b} - (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, && (3)
\end{aligned}
$$

Stochastic formulation of the model

The passive vector advection is described by the following system of stochastic equations

$$\begin{aligned} \partial_t \mathbf{b} + (\mathbf{v}\cdot\partial)\mathbf{b} &= u_0\nu_0\Delta\mathbf{b} + \mathcal{A}(\mathbf{b}\cdot\partial)\mathbf{v} - \partial\mathcal{Q} + \mathbf{f}^{\mathbf{b}} && (4) \\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\partial)\mathbf{v} &= \nu_0\Delta\mathbf{v} - \partial\mathcal{P} + \mathbf{f}^{\mathbf{v}} && (5) \end{aligned}$$

u_0 is inverse Prandtl number, ν_0 is kinematical viscosity, f is a random force, $\mathbf{v}$ means incompressible velocity field (for this model) and $\mathcal{Q}$, $\mathcal{P}$ represent corresponding pressures.

In (4) we use standard Gaussian random noise with zero mean and the correlation function

$$\langle f_i^b(x) f_j^b(x')\rangle = \delta(t-t')C_{ij}(|x-x'|/L) \qquad (6)$$

Stochastic formulation of the model

The correlation function of the stochastic Navier-Stokes equation has the standard form

$$\langle f_i^v(x) f_j^v(x') \rangle = \delta(t-t')(2\pi)^{-d} \int d\mathbf{k} P_{ij}(\mathbf{k}) D_f(k) \times \exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}')] \quad (7)$$

with

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2. \quad (8)$$

in an isotropic incompressible flow

Field theoretic formulation of the model

The stochastic model is equivalent to the quantum field model with double set of fields

$$\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\} \tag{9}$$

and action functional of the model

$$\begin{aligned} S(\Phi) &= \mathbf{v}' D_{f^{\mathbf{v}}} \mathbf{v}'/2 + \mathbf{b}' D_{f^{\mathbf{b}}} \mathbf{b}'/2 + \mathbf{v}'[-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - (\mathbf{v} \cdot \partial)\mathbf{v}] \\ &+ \mathbf{b}'[-\partial_t \mathbf{b} + \nu_0 u_0 \Delta \mathbf{b} - (\mathbf{v} \cdot \partial)\mathbf{b} + \mathcal{A}(\mathbf{b} \cdot \partial)\mathbf{v}] \end{aligned} \tag{10}$$

where D_f are correlation functions of the random force. Necessary integrations over $\{t, \mathbf{x}\}$ and summations over vector indices are implied.

Field theoretic formulation of the model

Propagators

$$\langle b'_i b_j\rangle_{0(\mathbf{k})} = \langle b_i b'_j\rangle^*_{0(\mathbf{k})} = \frac{P_{ij}(\mathbf{k})}{i\omega_k + \nu_0 u_0 k^2}, \tag{11}$$

$$\langle v'_i v_j\rangle_{0(\mathbf{k})} = \langle v_i v'_j\rangle^*_{0(\mathbf{k})} = \frac{P_{ij}(\mathbf{k})}{i\omega_k + \nu_0 k^2}, \tag{12}$$

$$\langle b_i b_j\rangle_{0(\mathbf{k})} = \frac{C_{ij}(\mathbf{k})}{(i\omega_k + \nu_0 u_0 k^2)(-i\omega_k + \nu_0 u_0 k^2)}, \tag{13}$$

$$\langle v_i v_j\rangle_{0(\mathbf{k})} = \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k})}{(i\omega_k + \nu_0 k^2)(-i\omega_k + \nu_0 k^2)}, \tag{14}$$

Vertices

$$b'_i v_j V_{ijl} b_l \to V_{ijl} = i(k_j \delta_{il} - \mathcal{A} k_l \delta_{ij}), \tag{15}$$

$$v'_i v_j W_{ijl} v_l/2 \to W_{ijl} = i(k_l \delta_{ij} + k_j \delta_{il}), \tag{16}$$

Feynman diagrams, one-loop approximation

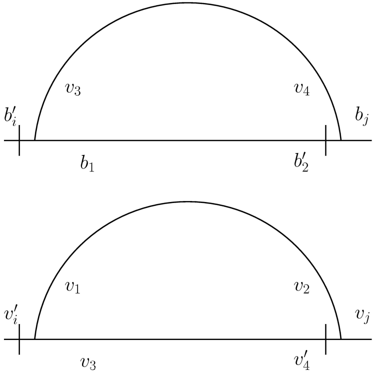

Figure 1: Feynman diagrams in one-loop approximation. Self energy operators $\Sigma_{b'b}$ and $\Sigma_{v'v}$.

Feynman diagrams, two-loop approximation

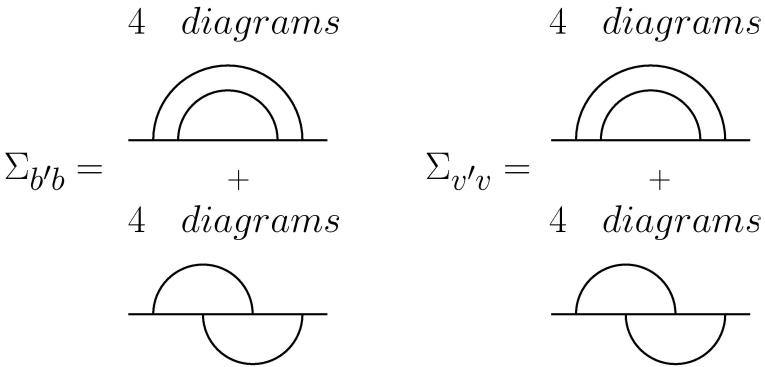

Figure 2: Feynman diagrams in two-loop approximation. Self energy operators $\Sigma_{b'b}$ and $\Sigma_{v'v}$.

Renormalization constants

Divergences are present only in the one-irreducible functions $\langle b'_i b_j \rangle$ and $\langle v'_i v_j \rangle$ thus we need only two independent renormalization constants
Renormalized action functional

$$\begin{aligned} S_R(\Phi) &= \mathbf{v}' D_{f^{\mathbf{v}}} \mathbf{v}'/2 + \mathbf{b}' D_{f^{\mathbf{b}}} \mathbf{b}'/2 + \mathbf{v}'[-\partial_t \mathbf{v} + \nu Z_1 \Delta \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v}] \\ &+ \mathbf{b}'[-\partial_t \mathbf{b} + \nu u Z_2 \Delta \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + \mathcal{A}(\mathbf{b}\cdot\partial)\mathbf{v}] \end{aligned} \tag{17}$$

By multiplicative renormalization of the parameters of the model we obtain

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u \tag{18}$$

where Z_g is related to Z_ν by relation

$$Z_g = Z_\nu^{-3} \tag{19}$$

Renormalization constants

Renormalization constants Z_1 and Z_2 relate to the renormalization constants Z_ν, Z_g and Z_u by the following relations

$$Z_\nu = Z_1, \qquad Z_g = Z_1^{-3}, \qquad Z_u = Z_2 Z_1^{-1} \tag{20}$$

General perturbation form of the renormalization constants, MS scheme

$$Z_1(g, d, \varepsilon, \mathcal{A}) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(1)}(d, \mathcal{A})}{\varepsilon^j} \tag{21}$$

$$Z_2(g, u, d, \varepsilon, \mathcal{A}) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(2)}(u, d, \mathcal{A})}{\varepsilon^j} \tag{22}$$

Turbulent vector Prandtl number in $\mathcal{A}$ model

The two-loop approximation for the inverse turbulent Prandtl number

L. Ts. Adzhemyan, J. Honkonen, T. L. Kim and L. Sladkoff, Phys. Rev. E 71, 056311, (2005)

$$u_{eff} = u_*^{(1)}\left(1+\varepsilon\left\{\frac{1+u_*^{(1)}}{1+2u_*^{(1)}}\left[\lambda-\frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right] + \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}[a_v - a_b(u_*^{(1)})]\right\}\right) \tag{23}$$

where a_v and a_b are integral functions of $\mathbf{k}$.
The one-loop value for the inverse magnetic Prandtl number is given by

$$u_*^{(1)}[1+u_*^{(1)}] = 2(d+2)/d \tag{24}$$

and for $d=3$ it is $u_*^{(1)} = 1.393$

Turbulent vector Prandtl number in $\mathcal{A}$ model

Turbulent vector Prandtl number in one-loop approximation

$$Pr^{(1)}_{\mathcal{A},t} = \frac{1}{u^{(1)}_*} \tag{25}$$

where $u^{(1)}_*$ has the form

$$u^{(1)}_* = \frac{1}{3a_2}\Big[-2a_2 - \frac{2^{1/3}b_1}{(b_2+b_3)^{1/3}} + \frac{(b_2+b_3)^{1/3}}{2^{1/3}}\Big] \tag{26}$$

Turbulent vector Prandtl number in $\mathcal{A}$ model

Turbulent vector Prandtl number in two-loop approximation

$$Pr_{\mathcal{A},t} = \frac{1}{u_{eff}} \tag{27}$$

where

$$\begin{aligned} u_{eff} &= u_*^{(1)}\left(1+\varepsilon\left\{\frac{1+u_*^{(1)}}{1+2u_*^{(1)}}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right]\right.\right. \\ &\left.\left. + \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}[a_v - a_b(u_*^{(1)})]\right\}\right) \end{aligned} \tag{28}$$

E. Jurčišinová, etal., Phys. Rev. E 93, 033106 (2016)

Results

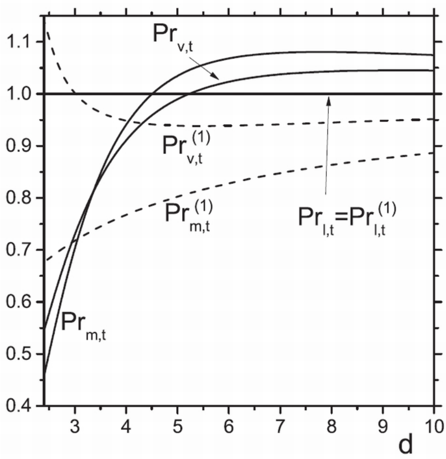

Figure 3: The behavior of the two-loop $Pr_{A,t}$ for three special cases, namely, $\mathcal{A} = -1 \cdots Pr_{l,t}$, $\mathcal{A} = 0 \cdots Pr_{v,t}$ and $\mathcal{A} = 1 \cdots Pr_{m,t}$.

Results

Comparison of the two-loop and and one-loop approximation of the model

- $\mathcal{A} = -1$ and $\mathcal{A} = 1$ models are very well described at the one-loop approximation
- Two-loop corrections to the turbulent Prandtl number are significant for models inside the interval $-1 < \mathcal{A} < 1$, especially for the model of a passively advected vector field $\mathcal{A} = 0$

Results

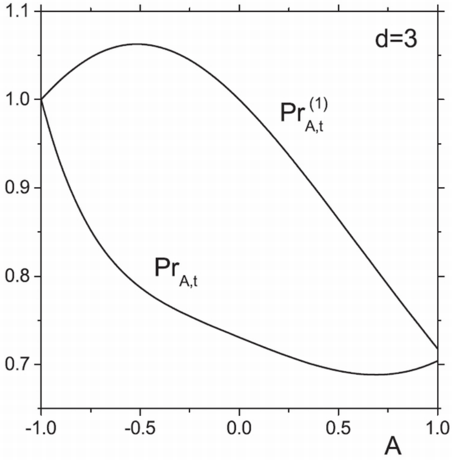

Figure 4: The dependence of the one-loop $Pr^{(1)}_{A,t}$ and the two-loop $Pr_{A,t}$ turbulent vector Prandtl number on the parameter $\mathcal{A}$ for the spatial dimension $d = 3$.

Conclusion

- We were interested in Prandtl numbers of different passive vector models in fully developed turbulence driven by stochastic Navier-Stokes equation
- Three physically interesting models were analyzed, namely
  - $\mathcal{A} = -1$ model of linearized Navier-Stokes equation
  - $\mathcal{A} = 0$ vector impurity by Navier-Stokes equation
  - $\mathcal{A} = 1$ kinematic MHD turbulence
- Turbulent magnetic Prandtl numbers for the models were established and their dependece on parameter $\mathcal{A}$ and on spatial dimension d was shown