Diffraction of electromagnetic waves on a waveguide joint

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Maxwell's equations in waveguide $S \times \mathbb{R}$

Consider a waveguide of a constant simply connected section S with ideally conducting walls. The axis Oz is directed along the cylinder axis. By excluding the longitudinal components from Maxwell equations, we have

$$\begin{cases} \vec{e}_z \times \partial_z \vec{H}_\perp + \nabla \dfrac{1}{ik\mu} (\operatorname{rot} E_\perp)_z \times \vec{e}_z + ik\varepsilon \vec{E}_\perp = 0, \\ \vec{e}_z \times \partial_z \vec{E}_\perp - \nabla \dfrac{1}{ik\varepsilon} (\operatorname{rot} H_\perp)_z \times \vec{e}_z - ik\mu \vec{H}_\perp = 0, \end{cases} \tag{1}$$

On the boundary, conditions of ideal conductivity are satisfied:

$$\vec{E}_\perp \times \vec{n} = 0, \quad H_\perp \cdot \vec{n} = 0, \quad E_z = 0. \tag{2}$$

Reduction to Helmholtz equation

If ε and μ are constants then the investigation of the electromagnetic field reduces to solving of Helmholtz equation [Tikhonov&Samarski, 1949].

Main question

How the system of Maxwell's equations can be reduced two scalar equations of the Helmholtz type in the most general case of a waveguide with an arbitrary filling ε and μ?

The natural question turned out to be in the shadow of the theorem on the completeness of the system of normal modes, because the first papers had in mind the generalization of the Fourier method.

Helmholtz decomposition

We offer to seek a solution of Maxwell system in the form

$$\vec{E}_\perp = \nabla u_e + \nabla' v_e, \quad \vec{H}_\perp = \nabla v_h + \nabla' u_h,$$

where

$$\nabla = (\partial_x, \partial_y), \quad \nabla' = (-\partial_y, \partial_x).$$

The four scalar functions will be called potentials, and we'll always assume that they satisfy the boundary conditions

$$u_e = u_h = n \cdot \nabla v_e = n \cdot \nabla v_h = 0$$

on the boundary of the waveguide.

Maxwell's equations in potentials

Substituting these expressions into Maxwell's equations (1), we obtain

$$\begin{cases} \nabla' \left(\partial_z v_h - \dfrac{1}{ik\mu} \Delta v_e \right) + ik\varepsilon \nabla' v_e - \nabla \partial_z u_h + ik\varepsilon \nabla u_e = 0 \\ \nabla' \left(\partial_z u_e + \dfrac{1}{ik\varepsilon} \Delta u_h \right) - ik\mu \nabla' u_h - \nabla \partial_z v_e - ik\mu \nabla v_h = 0 \end{cases}$$

The boundary conditions (2) are automatically satisfied if

$$u_e = u_h = n \cdot \nabla v_e = n \cdot \nabla v_h = 0$$

on the boundary of the waveguide.

Weak form of the system

Consider the solution

$$w = (v_h, v_e, u_e, u_h)$$

as a map of $z \in \mathbb{R}$ to a Hilbert space

$$\mathfrak{H} = \left\{ v_h, v_e, u_e, u_h \in W_2^2(S) : \quad u_e, u_h, n \cdot \nabla v_e, n \cdot \nabla v_h|_{\partial S} = 0 \right\}.$$

Then the system can be written as follows

$$\partial_z b(\tilde{w}, w) + ika(\tilde{w}, w) + \frac{1}{ik} c(\tilde{w}, w) = 0 \quad \forall \tilde{w} \in \mathfrak{H}$$

or by Frigyes Riesz theorem in bounded operators as

$$B \frac{dw}{dz} + ikAw + \frac{1}{ik} Cw = 0.$$

Three bilinear forms

Here we introduce three bilinear forms

$$a(\tilde{w}, w) = \iint\limits_S \varepsilon(\nabla' v_e + \nabla u_e) \cdot (\nabla' \tilde{v}_h - \nabla \tilde{u}_h)\, dx\, dy +$$

$$+ \iint\limits_S \mu(\nabla' u_h + \nabla v_h) \cdot (\nabla \tilde{v}_e - \nabla' \tilde{u}_e) v\, dx\, dy,$$

$$b(\tilde{w}, w) = \iint\limits_S (\nabla' v_h \cdot \nabla' \tilde{v}_h + \nabla' u_e \cdot \nabla' \tilde{u}_e + \nabla v_e \cdot \nabla \tilde{v}_e + \nabla u_h \cdot \nabla \tilde{u}_h)\, dx$$

$$c(\tilde{w}, w) = \iint\limits_S \left(\frac{\Delta v_e \Delta \tilde{v}_h}{\mu} - \frac{\Delta u_h \Delta \tilde{u}_e}{\varepsilon} \right) dx\, dy.$$

The normal modes

If solution depends on z as $e^{ik\beta z}$ then it called normal wave.

Problem

For given waveguide and functions $\varepsilon(x, y)$ and $\mu(x, y)$ to solve eigenvalue problem

$$\beta B w = A w - \frac{1}{k^2} C w$$

with respect to the spectral parameter β.

The discreteness of the spectrum and the completeness of the system of normal waves are obtained as a corollary of the theorem of M.V. Keldysh about the operator bundle [Delicyn, 1999].

The example

Consider an eigenvalue problem in a waveguide whose cross section is the unit square $Q = \{0 < x < 1, \quad 0 < y < 1\}$, with the filling

$$\varepsilon = 1 + 16 \cdot \delta \cdot xy(1-x)(1-y), \quad \mu = 1.$$

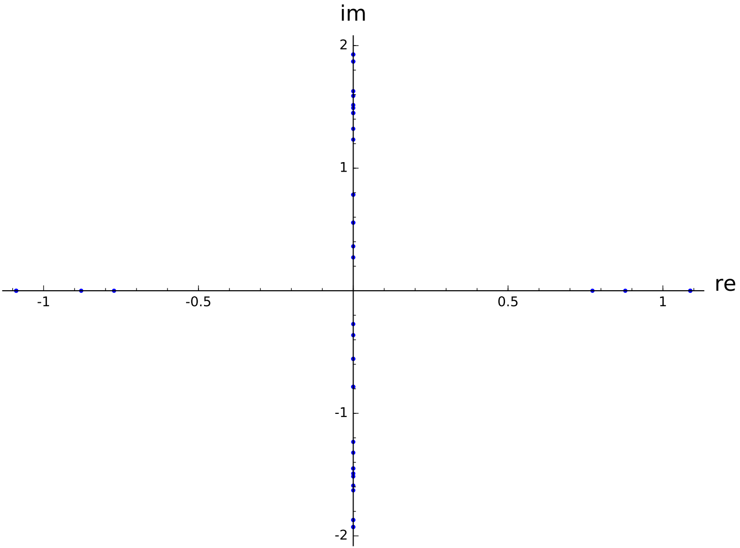
Numerical solution of eigenvalue problem:

- discretization, that is, truncation of the infinite matrices (Galerkin method),
- solving of algebraic eigenvalue problem

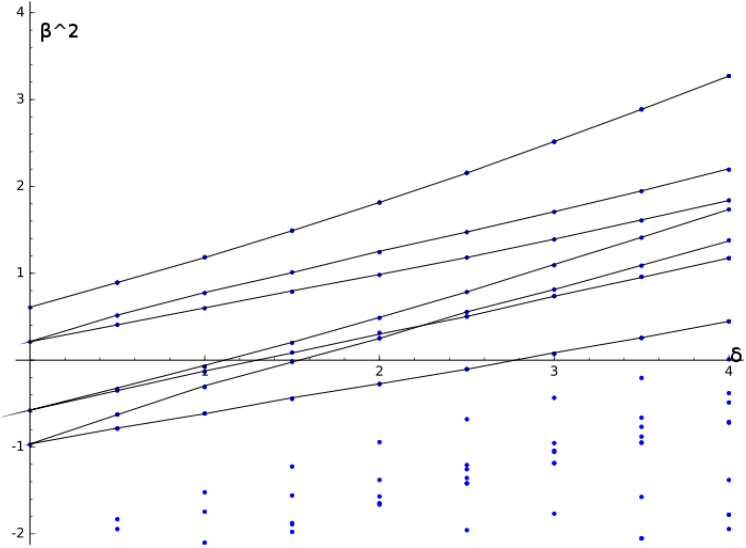
Note

Standard approach suggests usage Krylov Subspace Methods (ex. gmres by Yousef Saad) on the 2nd step, but we work over $\overline{\mathbb{Q}}$ without any numerical errors. So multiplicities of eigenvalues are conserve.

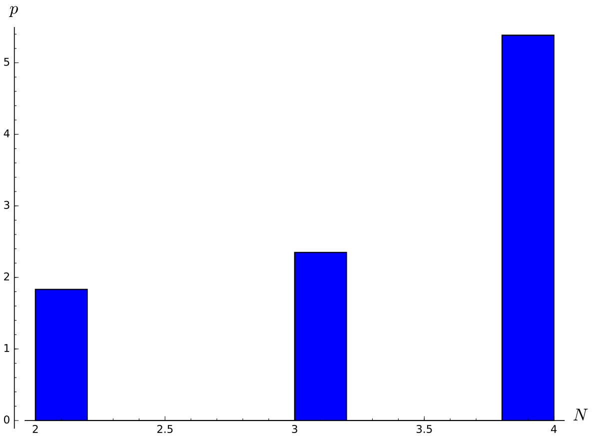
The complex β-plane at $\delta = 1$



Square of β as functions of δ



Effective order $p^{(N)} = -\lg\left|\beta_1^{(N)} - \beta_1^{(5)}\right|$.



The joint of two waveguide

Let the filling ε, μ of this waveguide have the jump at $z = 0$ so this plane is the joint of two waveguide.
Any solution of the equation

$$B\frac{dw}{dz} + ikAw - \frac{1}{ik}Cw = 0,$$

at $z < 0$ and at $z > 0$ can be represented as the superposition of normal modes.

Diffraction on joint

The problem about diffraction the wave

$$\sum_{\beta>0} F_n w_n^{(1)} e^{ik\beta_n^{(1)} z}$$

on joint $z = 0$ is reduced to follows.

Problem

For given waveguide, functions $\varepsilon_{1,2}(x, y)$ and $\mu_{1,2}(x, y)$ and numbers $F_1, \ldots$ to find such numbers R_n, T_n that

$$\sum_{\beta>0} F_n w_n^{(1)} e^{ik\beta_n^{(1)} z} + \sum_{\Re\beta<0} R_n w_n^{(1)} e^{ik\beta_n^{(1)} z} = \sum_{\Re\beta>0} T_n w_n^{(2)} e^{ik\beta_n^{(2)} z}$$

at $z = 0$.

Example

If

$$\varepsilon = \begin{cases} 1, & z < 0 \\ 1 + x(1-x)y(1-y) & z > 0 \end{cases}$$

then we can calculate normal waves as above and solve linear system with respect R and T. At $N = 2$ we have

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R0 == -0.0127723736238675* F0 ,
R1 == -0.013882235760583 8* F1 + 0.00215389099104622* F6 ,
R2 == -0.013882235760583 8* F2 - 0.00215389099104622* F5 ,
T0 == 0.987227626376132* F0 ,
T1 == 1.00349725340806* F3 ,
T2 == 0.988765396785862* F4 .
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The conclusion

- We can introduce four potentials, by means of which the system of Maxwell's equations in the waveguide reduces to an infinite system of linear ODEs and convenient for numerical analysis by Galerkin method.
- To compute the normal waves we can use CAS and safe multiplicities and other qualitative properties of eigenvalues.
- The problem about diffraction on joint is reduced to system of linear algebraic equation which can be solved also symbolically.

The end.