Finite difference schemes as algebraic correspondences between layers

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Introd.

Consider a ode

$$\frac{dy}{dx} = f(x, y), \quad f \in \mathbb{Q}(x, y).$$

Finite differences method suggests to replace this equation with the equation of the form

$$F(y, \hat{y}; x, \hat{x}) = 0,$$

for ex.

$$\hat{y} - y = f(x, y)\Delta x.$$

This equation defines correspondence between neighboring layers y and $\hat{y}$, which usual investigated as points on two affine straight lines.

Singularities

If a singular point of the solution to Cauchy problem

$$\frac{dy}{dx} = f(x, y), \quad y|_{x=a} = y_0$$

depends on initial data then it called movable singularity of ode.

Theorem (Painlevé, 1897)

Movable singularities of the solution of Cauchy problem are always algebraic, that is, the equation in the neighbourhood of such singularity can be expanded into Puiseux series

$$y = a_0(x - c)^p + \dots, \quad p \in \mathbb{Q}.$$

Detecting singularities

In general we don't know the exact solution $y = \varphi(x)$ of given Cauchy problem but can calculate approximate solution by finite difference method.

Problem

For given Cauchy problem and the interval $a < x < b$ we want detect mobile singularities on this interval by analysis of one or several approximate solutions.

Many authors think that this problem can't be solved because finite difference method describes mistakenly the solution in neighborhood of singularities.

Euler's thinking

1. If exact solution of Riccati equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

   has singularity at $x = c$ on given interval $[a, b]$ then this singularity is pole (property of Riccati equation).
2. We can't solve this equation in elementary functions (Liouville) and can try solve it by Euler scheme with constant step Δx.
3. Step by step difference Δy increases and thus error of changing ode to fde also increases without any limit.
4. Result: fdm doesn't suit for the description of singularities.

Practical solution by E. Alshina et al., 2005

1. There is such scheme (ex. CROS) that approximate solution goes to a finite value when exact solution has a pole.
2. In regular points approximate solution can be decomposed in asymptotic series

$$y(x_n, \Delta x) = \varphi(x_n) + r(x_n)\Delta x^r,$$

   where r is approximation order of the scheme.
3. Thus in regular points the ratio

$$\frac{y(x_n, \Delta x) - y(x_n, \Delta x/2)}{y(x_n, \Delta x/2) - y(x_n, \Delta x/2^2)} \simeq \frac{1 - \frac{1}{2^r}}{\frac{1}{2^r} - \frac{1}{2^{2r}}} = 2^r$$

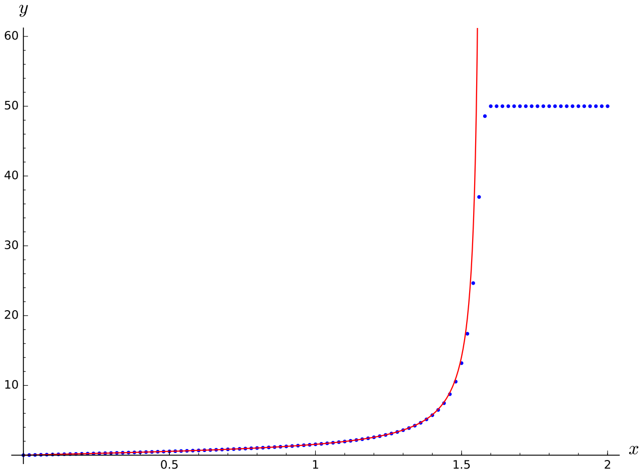
   but in neighborhood of singularity

$$\frac{y(x_n, \Delta x) - y(x_n, \Delta x/2)}{y(x_n, \Delta x/2) - y(x_n, \Delta x/2^2)} = 2^p,$$

   where p is order of algebraic singularity.

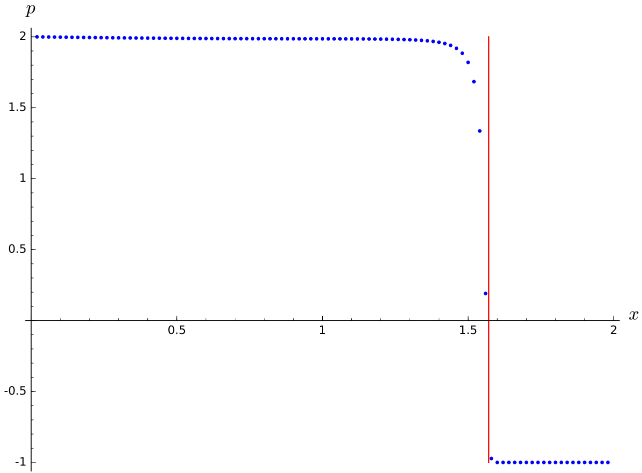
Example: graph of solution

$$\frac{dy}{dx} = 1 + y^2, \quad y|_{x=0} = 0 \quad \Rightarrow \quad y = \tan x.$$



Example: effective order

$$\frac{dy}{dx} = 1 + y^2, \quad y|_{x=0} = 0 \quad \Rightarrow \quad y = \tan x \quad \Rightarrow \quad p = -1.$$



Full solution of the problem for Riccati eq.

Especial for Riccati equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

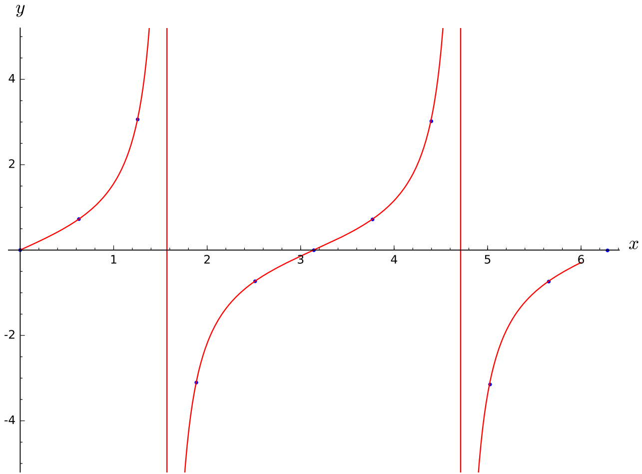
we can use scheme

$$\frac{\Delta y}{\Delta x} = p + qy + ry\hat{y}.$$

This is scheme 1st order but calculation can be prolongate after pole without any error accumulation.

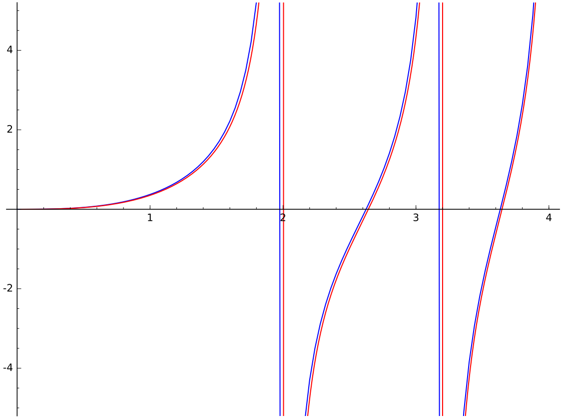
Example: graph of solution

$$\frac{dy}{dx} = 1 + y^2, \quad y|_{x=0} = 0 \quad \Rightarrow \quad y = \tan x.$$



Example: graph of solution

$$\frac{dy}{dx} = x^2 + y^2, \quad y|_{x=0} = 0 \quad \Rightarrow \quad y = -\frac{x\left(J_{-\frac{3}{4}}\left(\frac{1}{2}x^2\right) - Y_{-\frac{3}{4}}\left(\frac{1}{2}x^2\right)\right)}{J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) - Y_{\frac{1}{4}}\left(\frac{1}{2}x^2\right)}$$



What topology is natural for a error estimate?

Note

The conversation in 'affine' C-norm is a unnatural concept if we use $\mathbb{P}$ as layers.

If y, y', y'', y''' are approximate solutions with steps $h, 2h, 2^2h, 2^3h$ respectively then by Richardson

$$y = \varphi(x) + r(x)h^r + \dots,$$

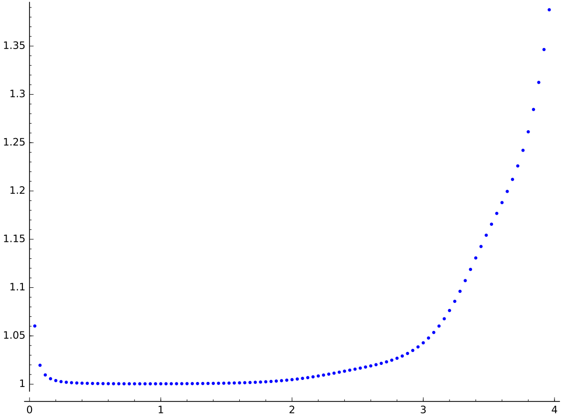
and thus

$$(y, y', y'', y''') \simeq (1, 2^r, 2^{2r}, 2^{3r}).$$

We can check the constancy of this ratio and then calculate effective anharmoncal order r.

Example: eff. anharmonical order $r = 1$

$$\frac{dy}{dx} = x^2 + y^2, \quad y|_{x=0} = 0 \quad \Rightarrow \quad (y, y', y'', y''') \simeq (1, 2, 2^2, 2^3)$$



The conclusion no. 1

For Riccati equation the existence of pole on investigated interval isn't the reason for accumulation of errors in fdm. The reason is the bad choice of finite difference scheme.

What is the diff. scheme in general?

Definition

We map any ode to a pair (V, F), where

1. V is a layer, that is, an algebraic variety which can depend on x,
2. F is a difference scheme, that is, an algebraic correspondence between $V(x)$ and $V(\hat{x})$.

We can define a notion of approximation and approximate solutions by help of Weierstrass Vorbereitungssatz.

Classification of dif. schemes

Classification of algebraic correspondences gives us the natural classification of dif. schemes.
Simplest case:

- layer is projective straight line $\mathbb{P}$,
- correspondence is projective (birational) one-to-one transformation.

Theorem

Differential eq. can be approximated by projective one-to-one dif. scheme iff this eq. is Riccati equation.

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2 \quad \rightarrow \quad \frac{\Delta y}{\Delta x} = p + qy + ry\hat{y}.$$

Projective dif. schemes

Natural generalization of projective one-to-one transformation is n-to-n projective correspondence on algebraic varieties.

Theorem

There is projective n-to-n scheme for given ode iff general solution of ode depends on constant algebraically.

This class is the same what was investigated in early works of Painlevé. So classical transcendents are good functions not only from power series viewpoint.

Theorem

Calculation by projective n-to-n scheme can be prolongate after singularities without any error accumulation.

Example

Eq.

$$(x^4 - 2x^3y + 2x^2y^2 + 2xy^3 + y^4 - 2y^2)dx + (x^2 + 2xy - y^2)dy = 0$$

reduces to Riccati equation

$$\frac{dz}{dx} = x^2 + z^2$$

by substitution

$$z = \frac{y(y+x)}{y-x}.$$

There are no commonly used algorithms for work with classical transcendents. Thus Maxima, Maple 2016.1 and WolframAlpha can't integrate this equation.
Our method [Dubna'2016, Mephi'2016] leads to very hard symbolic calculations.

Example

Diff. scheme for eq.

$$pdx + qdy = 0$$

has a form

$$\mu(p\Delta x + q\Delta y) + (\alpha\Delta x + \beta\Delta y)\Delta y + \cdots = 0.$$

If the scheme defines projective 2-to-2 correspondence between layers then

$$\partial\mu p \leq 4, \quad \partial\mu q \leq 3.$$

In our case

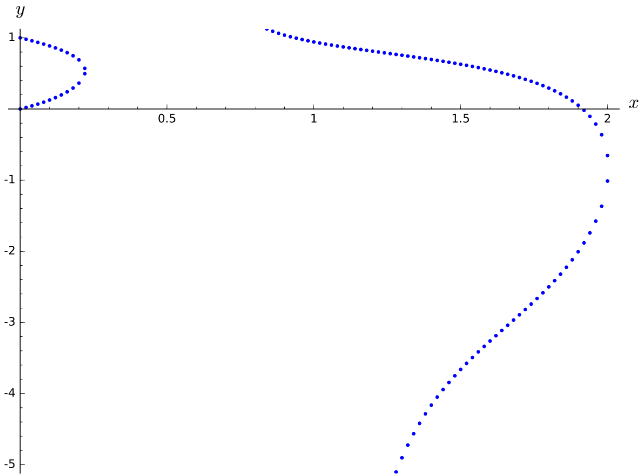
$$\partial p = 4 \quad \Rightarrow \quad \mu = 1.$$

After some calculations we have the scheme

$$(y\hat{y} - yx - \hat{y}x - x^2)(\hat{y} - y) + (\dots)\Delta x + (\dots)\Delta x^2.$$

Example: graph of solution

$$(x^4 - 2x^3y + 2x^2y^2 + 2xy^3 + y^4 - 2y^2)dx + (x^2 + 2xy - y^2)dy = 0.$$



The conclusions

1. For Riccati equation the existence of pole on investigated interval isn't the reason for accumulation of errors in fdm. The reason is the bad choice of finite difference scheme.
2. For ode which can be integrated in classical transcendents there are projective dif. schemes, so calculation by fdm can be prolongate after singularities without any error accumulation.
3. For detecting of integration in classical transcendents it is convenient to use also fdm.

The end.

Calculations made in `SageMath version 7.5.1, Release Date: 2017-01-15`.
See additional materials on `http://malykhmd.neocities.org`.