# Stratification of 2-qubit $X$-state space

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# Outline

1. Qubit
2. $SU(2)$ group action
3. Qubit-qubit
4. $X$-states
5. Group action on $X$-states
6. Global orbits
7. Local orbits
8. Concluding remarks

# Qubit

A generic mixed state [1] of an $n-$level quantum system is described by an $n \times n$ density matrix $\varrho$, such that: $\varrho = \varrho^\dagger, \ Tr(\varrho) = 1, \ \rho \geq 0$.

The state of a qubit is given by the density matrix

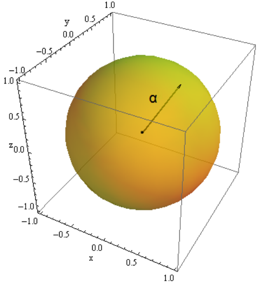
$$\varrho = \frac{1}{2}\left(\mathbb{I}_2 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}\right), \tag{1}$$

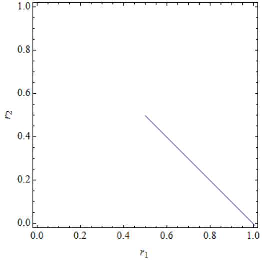
where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ , and $\boldsymbol{\sigma}$ is the basis of $su(2)$ algebra - the standart Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

[1] The special class of idempotent matrices, satisfying $\varrho^2 = \varrho$, corresponds to the so-called *pure states*. A mixed state is a mixture of pure states.

Since $\varrho \geq 0$, the parameters space is restricted to the unit ball ($\boldsymbol{\alpha}^2 \leq 1$), and pure states describe the so-called Bloch sphere ($\boldsymbol{\alpha}^2 = 1$).




The spectrum $\{r_1, r_2\}$ of the density matrix of a qubit forms the simplex $\Delta_1$: $r_1 + r_2 = 1$, $0 \leq r_2 \leq r_1 \leq 1$.

# $SU(2)$ group action

The adjoint action of $SU(2)$ group is

$$\varrho' = U\varrho U^\dagger, \tag{3}$$

where $U = \exp\left(i\sum_{i=1}^{3}\sigma_i\theta_i\right)$ is a general element of $SU(2)$ group.

When parameters $\theta_i$ are small, one can use the following infinitesimal transformations:

$$U = \mathbb{I} + i\theta_i\sigma_i, \qquad U^\dagger = \mathbb{I} - i\theta_i\sigma_i, \tag{4}$$

then

$$U\varrho U^\dagger = \varrho + i\theta_i[\sigma_i, \varrho] + O(\theta^2), \tag{5}$$

where $[\sigma_i, \varrho] = \sigma_i\varrho - \varrho\sigma_i$, and $\theta^2$ is negligible.

The group orbit of a group element $x$ is defined as

$$G(x) = \{gx \in X : g \in G\}, \tag{6}$$

where $g$ runs over all elements of the group $G$.

The dimension of an orbit $O_\varrho$ depends on a point $\varrho$ and is defined by the number of linearly independent vectors in a tangent space.

The tangent vectors are

$$t_i = \frac{\partial}{\partial \theta_i} \Big( U(\theta_1, \theta_2, \theta_3)\ \varrho\ U^\dagger(\theta_1, \theta_2, \theta_3) \Big) \Big|_{\theta_i = 0} = i[\sigma_i, \varrho]. \tag{7}$$

The number of linearly independent parameters equals the rank of a Gram matrix

$$G = \|G_{ij}\| = \frac{1}{2} \|Tr(t_i t_j)\|. \tag{8}$$

In this case

$$t_i = -\alpha_j \epsilon_{ijk} \sigma_k, \qquad G_{ij} = \alpha_s \alpha_s \delta_{ij} - \alpha_j \alpha_i, \qquad i, j, s = 1, 2, 3. \tag{9}$$

Explicitly,

$$G = \begin{pmatrix} \alpha_2^2 + \alpha_3^2 & -\alpha_2 \alpha_1 & -\alpha_3 \alpha_1 \\ -\alpha_1 \alpha_2 & \alpha_1^2 + \alpha_3^2 & -\alpha_3 \alpha_2 \\ -\alpha_1 \alpha_3 & -\alpha_2 \alpha_3 & \alpha_1^2 + \alpha_2^2 \end{pmatrix}, \tag{10}$$

and in diagonal form

$$G_{diag} = \begin{pmatrix} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & 0 & 0 \\ 0 & \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Therefore,

1. $dim(O_\rho) = 2$ in general case,
2. $dim(O_\rho) = 0$ for maximally mixed state (when $\alpha_1 = \alpha_2 = \alpha_3 = 0$).

# Qubit-qubit

The space of states of the system, obtained by joining two systems 1 and 2, is a subspace of the tensor product of their individual Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

The state of a qubit-qubit system is given by the density matrix

$$\varrho = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\boldsymbol{\xi} \cdot \boldsymbol{\lambda}\right), \tag{12}$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{15}) \in \mathbb{R}^{15}$, and $\boldsymbol{\lambda}$ is the basis of $su(4)$ algebra - the Gell-Mann matrices.

The state of two qubits system in the Fano form:

$$\varrho = \frac{1}{4}\left(\mathbb{I}_4 + a_i\sigma_i \otimes \mathbb{I}_2 + b_i\mathbb{I}_2 \otimes \sigma_i + c_{ij}\sigma_i \otimes \sigma_j\right), \tag{13}$$

with 15 parameters $a_i, b_i, c_{ji}$ $(i, j = 1, 2, 3)$.

# $X$-states

The density matrices of the form:

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix}, \qquad \begin{cases} \varrho_{ii} \in \mathbb{R}, \\ \varrho_{14} = \bar{\varrho}_{41}, \ \varrho_{23} = \bar{\varrho}_{32}, \\ tr\,\varrho_X = 1, \end{cases} \tag{14}$$

are called the $X-$states.

The $4 \times 4$ density matrix $\varrho_X$ can be block-diagonalized:

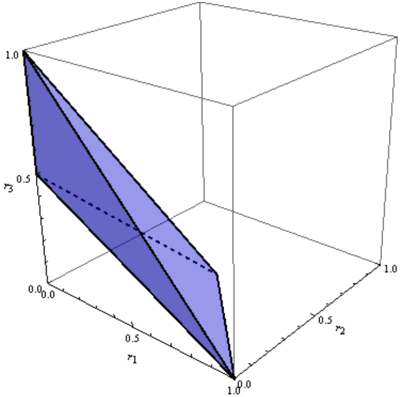
$$\varrho_X = W \left( \begin{array}{c|c} diag\,(r_1, r_2) & 0 \\ \hline 0 & diag\,(r_3, r_4) \end{array} \right) W^\dagger, \tag{15}$$

using the special unitary matrix $W = P_\pi \left( \begin{array}{c|c} e^{i\omega} U & 0 \\ \hline 0 & e^{-i\omega} V \end{array} \right) P_\pi$, where

$U = e^{i\frac{\psi_1}{2}\sigma_3} e^{i\frac{\phi_1}{2}\sigma_2}$, $V = e^{i\frac{\psi_2}{2}\sigma_3} e^{i\frac{\phi_2}{2}\sigma_2}$, and $\phi_1, \phi_2 \in [0, \pi]$, $\psi_1, \psi_2 \in [0, 2\pi]$.

The spectrum $\{r_1, r_2, r_3, r_4\}$ of $\varrho_X$ forms the partially ordered simplex $\Delta_3$:

$$\begin{cases} \sum_{i=1}^{4} r_i = 1, \\ 0 \leq r_2 \leq r_1 \leq 1, \\ 0 \leq r_4 \leq r_3 \leq 1. \end{cases}$$




Denote

$$x_1 = c_{11} - c_{22}, \quad x_2 = c_{12} + c_{21}, \quad x_3 = a_3 + b_3, \tag{16}$$

$$y_1 = c_{11} + c_{22}, \quad y_2 = c_{12} - c_{21}, \quad y_3 = -a_3 + b_3, \tag{17}$$

where $a_3 := r_A$ and $b_3 := r_B$ are the Bloch radii of constituent qubits.

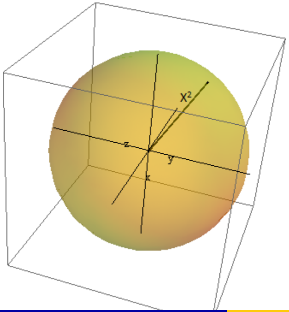
Therefore,

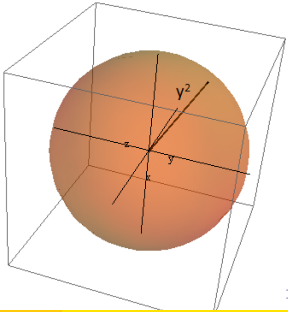
$$\begin{cases} x_1 = -2\,(r_1 - r_4)\cos(\psi_1)\sin(\phi_1)\,, \\ x_2 = 2\,(r_1 - r_4)\sin(\psi_1)\sin(\phi_1)\,, \\ x_3 = 2\,(r_1 - r_4)\cos(\phi_1)\,, \end{cases} \quad \text{and} \quad \begin{cases} y_1 = 2\,(r_2 - r_3)\cos(\psi_2)\sin(\phi_2)\,, \\ y_2 = -2\,(r_2 - r_3)\sin(\psi_2)\sin(\phi_2)\,, \\ y_3 = -2\,(r_2 - r_3)\cos(\phi_2)\,. \end{cases}$$

One can see that those are exactly the spherical coordinates.

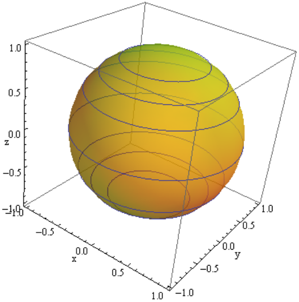
The state space is given by the Bloch spheres, where he Bloch radii

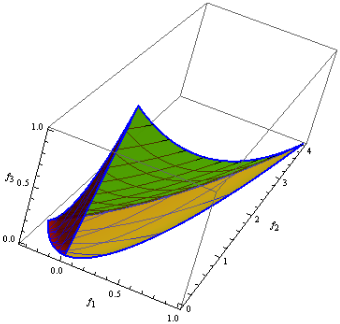
$$r_A = \frac{x_3 - y_3}{2}, \qquad r_B = \frac{x_3 + y_3}{2}. \tag{18}$$



The polynomials $\{f_1 = c_{33},\ g_1 = x_3 + y_3,\ g_2 = x_3 - y_3,\ g_3 = x_1^2 + x_2^2,\ g_4 = y_1^2 + y_2^2\}$ and $\{f_1 = c_{33},\ f_2 = x_1^2 + x_2^2 + x_3^2,\ f_3 = y_1^2 + y_2^2 + y_3^2\}$ represent algebraically independent local and global invariants for the $X$-states respectively.




Hence the eigenvalues of the density matrix $\varrho_X$ are $r_1 = \frac{1}{4}\left(1 + f_1 + \sqrt{f_2}\right)$, $r_2 = \frac{1}{4}\left(1 - f_1 + \sqrt{f_3}\right)$, $r_3 = \frac{1}{4}\left(1 - f_1 - \sqrt{f_3}\right)$, and one can map the simplex $\underline{\Delta}_3$ onto the $(f_1, f_2, f_3)$ coordinates space.

# Group action on $X$-states

$X$-states expand over the subset $\alpha_X = \{\lambda_3, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, -\lambda_{11}, \lambda_{15}\}$ of the introduced basis of $su(4)$: $\varrho_X = \frac{1}{4}\left(I_4 + 2i\sum_{\lambda_k \in \alpha_X} h_k \lambda_k\right)$.
The set $\alpha_X$ generates the subalgebra $\mathfrak{g}_X := su(2) \oplus su(2) \oplus u(1) \in su(4)$.

The exponentiation of the algebra $\mathfrak{g}_X$ results in the 7-parametric subgroup $G_X := exp(\mathfrak{g}_X) \in SU(4)$, whose action preserves the $X-$states space $\mathfrak{P}_X$, i.e., $G_X\, \varrho_X\, G_X^\dagger \in \mathfrak{P}_X$.

Using the expansion $\mathfrak{g}_X = \sum_i \omega_i \lambda_i$ over the 7-tuple $\lambda_i \in \alpha_X$, one can verify that the group

$$G_X = P_\pi \left(\begin{array}{c|c} e^{-i\omega_{15}} SU(2) & 0 \\ \hline 0 & e^{i\omega_{15}} SU(2)' \end{array}\right) P_\pi\,, \quad \text{where} \tag{19}$$

$$SU(2), SU(2)' = \exp\left[i\,(\pm\omega_4 + \omega_7)\,\sigma_1 + i\,(\pm\omega_2 + \omega_5)\,\sigma_2 + i\,(\omega_3 \pm \omega_6)\,\sigma_3\right].$$

# Global orbits

The dimensionality of a tangent space of $G_X$-orbits is given by the rank of the $7 \times 7$ Gram matrix

$$G = \| G_{kl} \| = \frac{1}{2} \| Tr(t_k t_l) \| \,, \tag{20}$$

where the tangent vectors $t_k = \frac{\partial}{\partial v_k} \left( g(\boldsymbol{v}) \, \varrho_X \, g^\dagger(\boldsymbol{v}) \right) \Big|_{v_k = 0} = [\lambda_k, \varrho_X]$,

elements $g(\boldsymbol{v}) = \exp \left( \sum_{\lambda_k \in \alpha_X} v_k \lambda_k \right) \in G_X \,, \; k = 3, 6, 7, 8, 10, 11, 15.$

The spectrum of the Gram matrix $\sigma(G_7) = \{\mu_1, \mu_1, \mu_2, \mu_2, 0, 0, 0\}$.
Two double multiplicity eigenvalues are the following:

$$\mu_1, \mu_2 = -\frac{1}{8} \left( (h_3 \pm h_6)^{\,2} + (h_8 \pm h_{10})^{\,2} + (h_7 \pm h_{11})^{\,2} \right) . \tag{21}$$

Therefore, one can classify the $G_X$-orbits as follows:

- **dim $\mathcal{O}$ =4**, the generic orbits, when $\mu_1, \mu_2 \neq 0$, that corresponds to the generic $X$-state block-diagonal matrix:

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22} \end{pmatrix} P_\pi , \qquad dim(\mathcal{O})_{Generic} = 7 - 3 = 4.$$

- **dim $\mathcal{O}$ =2**, the degenerate orbits defined by the equations $h_3 = \pm h_6$, $h_8 = \pm h_{10}$, $h_7 = \pm h_{11}$, are generated by X-matrix with double multiplicity of the following forms with $dim(\mathcal{O})_{Degenerate} = 7 - 5 = 2$:

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{22} & 0 \\ 0 & 0 & 0 & \varrho_{22} \end{pmatrix} P_\pi \text{ and } \varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & 0 & 0 & 0 \\ 0 & \varrho_{11} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22} \end{pmatrix} P_\pi .$$

- **dim $\mathcal{O}$=0**, the single orbit $\varrho_X = \frac{1}{4} I_4$ - the maximally mixed state.

# Local orbits

We define the group $LG_X$ as the subgroup of the global invariance group $G_X$ of $X$-states such that each its element $g \in LG_X$ has the tensor product form $g = g_1 \times g_2$ with $g_1, g_2 \in SU(2)$. The local unitary group can be written as

$$LG_X = P_\pi exp\left(i\frac{\phi_1}{2}\sigma_3\right) \times exp\left(i\frac{\phi_2}{2}\sigma_3\right) P_\pi\,. \tag{22}$$

The spectrum of the $2 \times 2$ Gram matrix $\sigma(G_2) = \{\mu_1, \mu_2\}$, with the following eigenvalues:

$$\mu_1, \mu_2 = -\frac{1}{8}\left((h_8 \pm h_{10})^{\,2} + (h_7 \pm h_{11})^{\,2}\right). \tag{23}$$

Thus, there exist four types of $LG_X$-orbits:

- **dim $\mathcal{O}$ =2**, the generic orbits, when $\mu_1, \mu_2 \neq 0$ with

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22}\,. \end{pmatrix} P_\pi\,, \qquad dim(\mathcal{O})_{Generic} = 2. \tag{24}$$

- **dim $\mathcal{O}$ =1**, the degenerate orbits defined by the equations $h_8 = \pm h_{10}$, $h_7 = \pm h_{11}$, and hence generated by X-matrix of the following forms with $dim(\mathcal{O})_{Degenerate} = 1$:

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & 0 \\ 0 & 0 & 0 & \varrho_{22} \end{pmatrix} P_\pi \text{ and } \varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & 0 & 0 & 0 \\ 0 & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22} \end{pmatrix} P_\pi \tag{25}$$

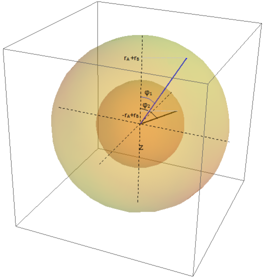
- **dim $\mathcal{O}$ =0**, the single orbit $\varrho_X = \frac{1}{4} I_4$ - the maximally mixed state.

# Concluding remarks

The Fano parameter $c_{33}$ is invariant under the action of global and local transformations. The adjoint action of the 7-parametric group $G_X$ induces the transformations of Fano parameters that are unitary equivalent to the block-diagonal actions of two copies of $SO(3)$.

$$\begin{pmatrix} SO(3) & & \\ & SO(3)' & \\ & & 1 \end{pmatrix}$$

Local transformations act as rotations by the angle $\psi_1$, $\psi_2$ respectively.



Thank you for attention