Quantum fields in Keldysh-Schwinger diagram technique and semiclassical expansion

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In collaboration with

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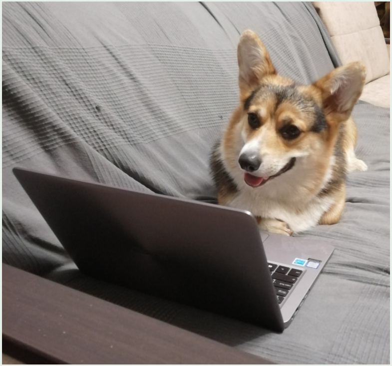
Feynman,
my dog



What to expect from the talk

- based on arXiv:2003.06395v3
- pure theoretical (no nice pictures, no experimental data, but a lot of diagrams φ^4)
- example of nonequilibrium shear viscosity
- aim -> to marry standard Keldysh diagram technique and classical simulations

Keldysh technique 1

- heavy ions collision $\rightarrow$ nonequilibrium matter, relaxation of the matter from a defined initial state $\rightarrow$ Keldysh technique
- $\langle \hat{O} \rangle = Tr(\hat{\rho}\hat{O})$

General formula

$$\langle \hat{O} \rangle =$$

$$\int \mathfrak{D}\Pi(x)\mathfrak{D}\alpha(x)\ \mathcal{W}[\alpha(x), \Pi(x)] \int_{i.c.} \mathcal{D}\varphi_{cl}(t, x) \int \mathcal{D}\varphi_q(t, x) O[\varphi_{cl}] e^{\frac{i}{\hbar} S_K[\varphi_{cl}, \varphi_q]}$$

$i.c.$ $\rightarrow$ initial values for φ_{cl} field sre: $\varphi_{cl}(t_0, x) = \alpha(x)$, $\partial_t \varphi_{cl}(t_0, x) = \Pi(x)$; the initial values for the φ_q are not fixed.

- The Wigner functions defines the initial state of the system

$$W[\alpha(x), \Pi(x)] = \int \mathcal{D}\beta(x) e^{i \int d^{d-1} x\, \beta(x)\Pi(x)} \langle \alpha(x) + \tfrac{\hbar}{2}\beta(x) | \hat{\rho}(t_0) | \alpha(x) - \tfrac{\hbar}{2}\beta(x) \rangle.$$

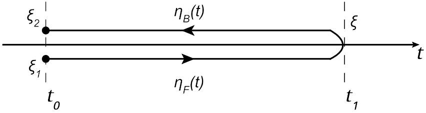
Keldysh technique 2

Scalar field theory φ^4

$$S[\varphi(x)] = \frac{1}{2}\int d^dx\,\left(\partial_\mu\varphi(x)\partial^\mu\varphi(x) - m^2\varphi^2(x) - \frac{g}{2}\varphi^4(x) + 2J(x)\varphi(x)\right)$$

Keldysh action

$$S_K[\varphi_F,\varphi_B] = S[\varphi_F] - S[\varphi_B]$$



Rotation

$$\varphi_{cl}(x) \equiv \phi_r = \frac{1}{2}\left(\varphi_F(x) + \varphi_B(x)\right), \quad \hbar\varphi_q(x) \equiv \phi_a = \varphi_F(x) - \varphi_B(x)$$

$$S_K[\varphi_{cl},\varphi_q] = -\hbar\int_{t_0}^{\infty} dt \int d^{d-1}\mathrm{x}\;\varphi_q(t,\mathrm{x})\left(\partial_t^2 - \nabla^2 + m^2\right)\varphi_{cl}(t,\mathrm{x})$$

$$-g\hbar\int_{t_0}^{\infty} dt \int d^{d-1}\mathrm{x}\left(\varphi_{cl}^3(t,\mathrm{x})\varphi_q(t,\mathrm{x}) + \frac{\hbar^2}{4}\varphi_{cl}(t,\mathrm{x})\varphi_q^3(t,\mathrm{x})\right).$$

Assumptions

$$\langle\hat{O}\rangle = \int \mathfrak{D}\Pi(x)\mathfrak{D}\alpha(x)\ \mathcal{W}[\alpha(x),\Pi(x)] \int_{i.c.} \mathcal{D}\varphi_{cl}(t,x)\int \mathcal{D}\varphi_q(t,x) O[\varphi_{cl}] e^{\frac{i}{\hbar}S_K[\varphi_{cl},\varphi_q]}$$

Diagram techique $g \ll 1$

- the Wigner function is Gaussian $\rightarrow$ the Wick theorem is valid
- Perturbative expansion $g \ll 1$

$$\langle\hat{O}\rangle = \left\langle O[\varphi_{cl}] e^{-ig\int d^dx\left(\varphi_{cl}^3(x)\varphi_q(x)+\frac{\hbar^2}{4}\varphi_{cl}(x)\varphi_q^3(x)\right)}\right\rangle_0$$

Classical simulations $\hbar \rightarrow 0$

- arbitrary initial Wigner function
- semiclassical expansion (works for arbitrary g)

$$\langle\hat{O}\rangle = \int \mathfrak{D}\alpha(x)\mathfrak{D}\Pi(x) W[\alpha(x),\Pi(x))] O(\phi_c)$$

ϕ_c - solution of the classical EoM

Perturbative technique

- Consider full retarded Green function

$$G_R(x,x') = -i\langle \varphi_{cl}(x)\varphi_q(x')\rangle$$

- Perturbative approach

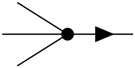
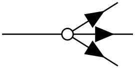
$$\langle \hat{O}\rangle = \left\langle O[\varphi_{cl}]e^{-ig\int d^dx\left(\varphi_{cl}^3(x)\varphi_q(x)+\frac{\hbar^2}{4}\varphi_{cl}(x)\varphi_q^3(x)\right)}\right\rangle_0$$

- Noninteracting Green functions

- Retarded $iG_R^0(x_1;x_2)$ x_2  x_1

  Keldysh $iG_K^0(x_1;x_2)$ x_2 x_1

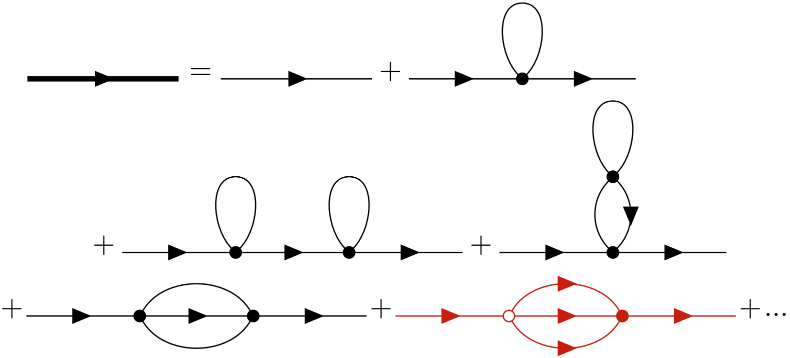
- Vertices

 $-ig$,  $-\frac{ig\hbar^2}{4}$

"black" and "white" vertices differ by the power of $\hbar^2$

Perturbative technique

The first two orders of the coupling constant expansion for the full retarded Green function



Perturbative technique

Zero order Green functions

$$G_R^0(x;x') = -\theta(t-t')\int \frac{d^{d-1}\mathrm{p}}{(2\pi)^{d-1}}\frac{\sin(\omega_p(t-t'))}{\omega_p}e^{-i\mathrm{p}\cdot\mathrm{x}},$$

$\omega_p^2 = \mathrm{p}^2 + m^2$

$$G_K^0(x;x') = -i\hbar\int \frac{d^{d-1}\mathrm{p}}{(2\pi)^{d-1}}\frac{\cos(\omega_p(t-t'))}{4\omega_p}(2f_p+1)e^{-i\mathrm{p}\cdot\mathrm{x}}$$

If one-particle distribution function $f_p \gg 1$ (highly excited initial state, heavy ion collisions)

$G_K^0(x;x') \gg G_R^0(x;x')$

Red diagram is less important

Semiclassical expansion

$$S_K[\varphi_{cl},\varphi_q,J] = -\hbar\int_{t_0}^{\infty} dt \int d^{d-1}x\ \left(\varphi_q A[\varphi_{cl}] + \frac{g\hbar^2}{4}\varphi_{cl}\varphi_q^3\right),$$

$$A[\varphi_{cl}] = (\partial_\mu\partial_\mu + m^2)\varphi_{cl} + g\varphi_{cl}^3 - J.$$

- $A[\varphi_{cl}] = 0$ corresponds to projecting onto the classical equation of motion of the Lagrangian.

$$e^{-i\frac{g\hbar^2}{4}\int_{t_0}^{\infty} dt \int d^{d-1}x\, \varphi_{cl}\varphi_q^3} = 1 - i\frac{g\hbar^2}{4}\int_{t_0}^{\infty} dt \int d^{d-1}x\ \varphi_{cl}\varphi_q^3 + \cdots$$

Classical Statistical Approximation

$$\langle \hat{O} \rangle = \int \mathfrak{D}\alpha(\mathrm{x})\mathfrak{D}\Pi(\mathrm{x}) W[\alpha(\mathrm{x}), \Pi(\mathrm{x}))] O(\phi_c) \equiv \left\langle O[\phi_c] \right\rangle_{i.c.}$$

where ϕ_c is the solution of the classical equation of motion = key element of the approach

$$\partial_\mu \partial^\mu \phi_c + g\phi_c^3 = J$$

with initial conditions given by

$$\phi_c(t_0, \mathrm{x}) = \alpha(\mathrm{x}), \quad \partial_t \phi_c(t_0, \mathrm{x}) = \Pi(\mathrm{x})$$

Numerical simulation

- find the classical trajectory as a function of the initial conditions;
- calculate observables on this trajectory;
- average over the initial conditions with the Wigner function corresponding the considered problem.

Beyond the CSA

- $\varphi_q(x)e^{iS_K[\varphi_c,\varphi_q,J]} = -i\frac{\delta}{\delta J(x)}e^{iS_K[\varphi_c,\varphi_q,J]}$.
- $\langle\hat{O}\rangle = \left\langle O[\phi_c(x)] + \frac{g\hbar^2}{4}\int dy\ \phi_c(y)\frac{\delta^3 O[\phi_c(x)]}{\delta J(y)^3}\Big|_{J=0}\right\rangle_{i.c}$.
- $\Phi_n(x;x_1,x_2,\ldots x_n) = \frac{\delta^n\phi_c(x)}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}$
- can be calculated by variation of the classical equation of motion

$$\hat{L}_\phi\Phi_1(x;x_1) = \delta^{(4)}(x-x_1),$$
$$\hat{L}_\phi\Phi_2(x;x_1,x_2) = -6g\phi_c(x)\Phi_1(x;x_1)\Phi_1(x;x_2),$$
$$\hat{L}_\phi = \partial_\mu\partial^\mu + m^2 + 3g\phi_c^2(x) \equiv \hat{L}_0 + 3g\phi_c^2(x)$$

Comparison of g^2 and $\hbar^2$ expansions

Let's perform semiclassical expansion of the $G_R(x_1, x_2)$ up to $\hbar^2$ terms, then decompose further up to g^2 and see if it reproduce the perturbative answer.
Semiclassical expansion of the full retarded Green function

$$
\begin{aligned}
G_R(x_1, x_2) = -i\langle\varphi_{cl}(x_1)\varphi_q(x_2)\rangle = -\langle\Phi_1(x_1; x_2)\rangle_{i.c.} \\
+ \frac{g\hbar^2}{4}\Big\langle \int dy \big(\Phi_1(y; x_2)\Phi_3(x_1; y, y, y) \\
+ \phi_c(y)\Phi_4(x_1; y, y, y, x_2)\big)\Big\rangle_{i.c.}.
\end{aligned}
$$

To perform expansion in g we need to express $\phi_c(x)$ and $\Phi_1(x_1; x_2)$ through the non-interacting counterparts

$$\phi_c(x) = \phi_0(x) + g\int dy\ G_R^0(x, y)\phi_c^3(y),$$

$$\Phi_1(x_1; x_2) = -G_R^0(x_1, x_2) + 3g\int dy\ G_R^0(x_1, y)\phi_c^2(y)\Phi_1(y, x_2),$$

Comparison of g^2 and $\hbar^2$ expansions

The iterative expansion of the $\phi_c(x)$ and $\Phi_1(x_1; x_2)$ up to g^2 is

$$\phi_c(x) = \phi_0(x) + g \int dy \; G_R^0(x,y)\phi_0^3(y)$$

$$+3g^2 \int dy \; G_R^0(x,y)\phi_0^2(y) \int dz \; G_R^0(y,z)\phi_0^3(z) + O(g^3),$$

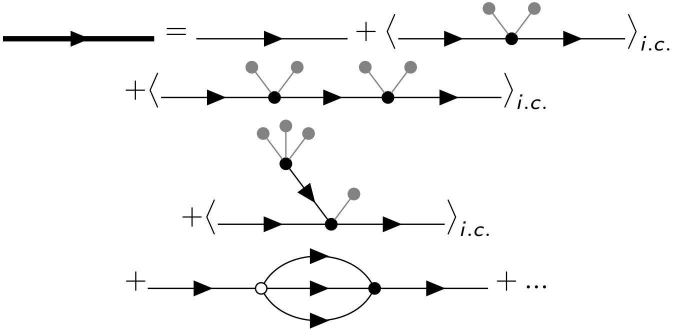
$$\Phi_1(x_1; x_2) = -G_R^0(x_1,x_2) - 3g \int dy \; G_R^0(x_1,y)\phi_0^2(y) G_R^0(y,x_2)$$

$$-9g^2 \int dy \; G_R^0(x_1,y)\phi_0^2(y) \int dz \; G_R^0(y,z)\phi_0^2(z) G_R^0(z,x_2)$$

$$-6g^2 \int dy \; G_R^0(x_1,y)\phi_0(y) G_R^0(y,x_2) \int dz \; G_R^0(y,z)\phi_0^3(z).$$

Comparison of g^2 and $\hbar^2$ expansions

Let us draw the contributions to the full retarded Green function $G_R(x_1, x_2)$ pictorially.



All lines and vertices have the same meaning as in perturbative technique. The only new element - the grey blob - denotes the free field $\phi_0(x)$.

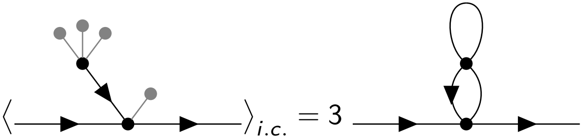
Comparison of g^2 and $\hbar^2$ expansions

Since only $\phi_0(x)$ depends on the initial conditions in the above expansion, it is straightforward to perform averaging according to the rule

$$\langle \phi_0(x)\phi_0(y)\rangle_{i.c.} = \Big\langle \text{ x } \text{—}\bullet \quad \bullet\text{— y } \Big\rangle_{i.c.} =$$

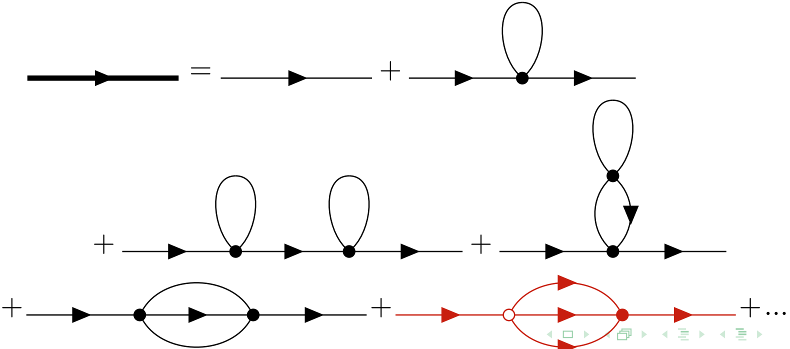
$$= \text{ x } \text{———} \text{ y } = iG_K^0(x;y).$$

For example, the "cactus" diagram appears as



Comparison of g^2 and $\hbar^2$ expansions

We reproduced all the terms of g^2 expansion and the only terms which is not included in the CSA ($\hbar^0$ semiclassical) One can see, that the Leading Order semiclassical term (the CSA) reproduce all the contributions of the g^2 terms of the is the red one, which is small if $f_p \gg 1$. That's why even the CSA include all loop g diagrams, valid for $f_p \gg 1$.



Viscosity

In order to evaluate viscosity one can use the Kubo linear response theory, where transport coefficients can be expressed through the retarded correlator $R^{\mu\nu}_{\alpha\beta}$ of two components of the stress-energy tensor $T^{\mu\nu}$ as

$$R^{\mu\nu}_{\alpha\beta}(x;x') = -\frac{i}{\hbar}\theta(t-t')\langle[\hat{T}^{\mu\nu}(x),\, \hat{T}_{\alpha\beta}(x')]\rangle.$$

In particular for shear viscosity one has

$$\eta(x) = -\frac{1}{10}\Delta^{\mu\nu}_{\alpha\beta}\int d^4y\ u^\rho y_\rho R^{\alpha\beta}_{\mu\nu}(x+y;x),$$

where (u^μ is energy flow velocity, $u_\mu u^\mu = 1$)

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\nu_\alpha\Delta^\mu_\beta - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right),$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Viscosity

$$\eta(x) = -\frac{1}{10}\int d^4y\ u^\rho y_\rho R(x+y;x),$$

Keldysh technique

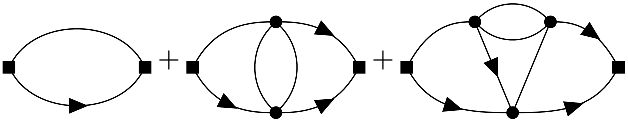
$$R(x;x') = -2i\Delta^{\alpha\beta}_{\mu\nu}\langle\partial^\mu\varphi_{cl}(x)\partial^\nu\varphi_{cl}(x)\partial'_\alpha\varphi_{cl}(x')\partial'_\beta\varphi_q(x')\rangle$$

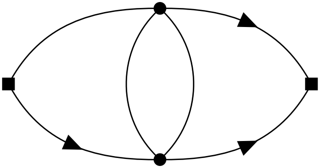
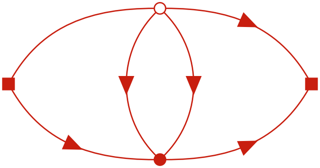
Semiclassical expansion

$$R(x;x') = -4\Delta^{\alpha\beta}_{\mu\nu}\langle\partial^\mu\phi_c(x)\partial'_\alpha\phi_c(x')\partial^\nu\partial'_\beta\Phi_1(x;x')\rangle_{i.c.}.$$

Viscosity

Semiclassical result is an infinite sum of multiloop diagrams in perturbative Keldysh technique

$R(x, x') =$  $+ \dots$

 $\sim \hbar^0, f_p^2,$  $\sim \hbar^2, f_p^0.$

Thank you for attention!