Kinematical Fit(KinFit) in SPD(?)

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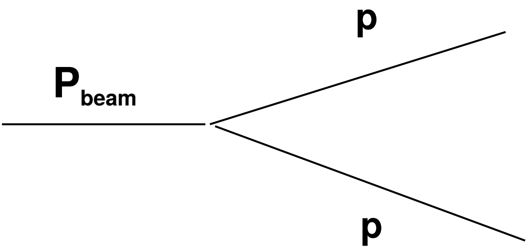
Introduction to KinFit

What is it needed for?



At least two hypotheses:

1. $pp \to pp$
2. $pp \to pp\pi^0$

Conservation laws should be used to select the correct hypothesis!

History

$$\chi^2 = \sum_{k=1,j=1}^{n_p,n_p} (X_i - X_i^m) Z_{i,j} (X_j - X_j^m) \tag{1}$$

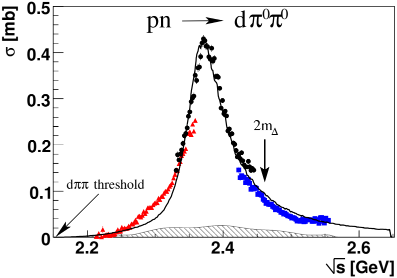
$$f_\lambda(\vec{X}) = 0; \quad \lambda = 1 \ldots n_c \tag{2}$$

[1] J.P. Berge, F.T. Solmitz and H.D. Taft, Rev. Sci. Instr. **32** 538 (1961);

[2] R. Bock, CERN 60-30 (1960).

The authors have shown that if X_j distributed according to Gaussian and the hypothesis is true, then form (1) has χ^2 distribution with number of degree of freedom (ndf) equal to nc after substitution those values X_j, which turn (1) to minimum and satisfies (2).

Example of an application: WASA discovery



Cross section of $pn \to d\pi^0\pi^0$ as a function of pn mass

An example of the employment of a similar technique is the WASA observation of the resonance-like cross section behavior for the $pd \to d\pi^0\pi^0 p_{\text{sp}}$ reaction.

The authors identifed the following chain of the processes:

$$p + d \to d + \pi^0 + \pi^0 + p_{\text{sp}} \to d + 2\gamma + 2\gamma + p_{\text{sp}}.$$

There were 12 equations, and $\vec{p}_{\text{sp}}$, $\vec{p}_{\pi_1^0}$, $\vec{p}_{\pi_2^0}$ were found with KinFit.

Realization and Restriction of proposed method

WASA approach used the method of Lagrange multipliers:

$$\chi^2 = \sum_{k=1,j=1}^{n_p,n_p} (X_i - X_i^m) Z_{i,j} (X_j - X_j^m) + 2 \sum_{\lambda=1}^{n_c} \alpha_\lambda \cdot f_\lambda(\vec{X}) \qquad (3)$$

Here α_λ are arbitrary multipliers to be found during minimization; both X_i and α_λ are varied.

Shortcoming of the method:

1. In (3) the kinematical parameters themselves are used. In the experiment we obtain a number of primary observables like hit coordinates. Often we have limited knowledge on their errors, and they may be far from Gaussian;
2. Thus, for applying the technique (3) one should somehow find the matrix $Z_{i,j}$;

<u>In other words we have the problem of error propagation!</u>

Generalization of the method

In order to bypass the propagation error problem we considered more general case(see (A.J. Ketikian, ..., V.S. Kurbatov, ..., NIM **A314** (1992) 572–577, *Generalized kinematical fit in event reconstruction*) when instead (3) we proposed to search for minimum, minimizing the form

$$\chi^2 = 1/2 \cdot \sum_{i=1,j=1}^{nf,nf} (C_i(\vec{X}) - C_i^m) Q_{i,j} (C_j(\vec{X}) - C_j^m) \tag{4}$$

and satisfying (2) Here $C_i(\vec{X})$ are observables,i.e. functions of kinematical parameters $\vec{X}$,C_i^m - their measured values, matrix $Q_{i,j}$ - inverse of error matrix.

It was shown that if errors are distributed by Gauss Law and hypothesis is true, then the form (4) has χ^2 distribution with ndf = nf - np + nc. Here np - the number of kinematical parameters or dimensionality of vector $\vec{X}$. In the case when nf=np, ndf = np as was in original proposal of the method.

KinFit in JINR in 1960-s

- FUMILI by S.N. Sokolov and I.N. Silin;
- Penalty function method by V.I. Moroz [V.I. Moroz, JINR, P-1958, 1965].

$$\chi^2 = \sum_{k=1,j=1}^{n_p,n_p} (X_i - X_i^m) Z_{i,j} (X_j - X_j^m) + T \sum_{\lambda=1}^{n_c} (f_\lambda / \Delta(f_\lambda))^2$$

T: large number;

$\Delta(f_\lambda)$: "error" of the constraint.

The idea is if $T \to \infty$ the parameter estimates approach the true ones.

Drawbacks of this method:

- Selection of the value T?
- The resulting value of χ^2 and parameters are distorted and one should control it.

Later in last half of 60-s JINR switched to the method of Lagrange multipliers used in CERN.

Method of FUMILI

At the time when we considered generalised case we came to the idea how to solve such a problem by other approach. This approach is valid in any other gradient method but because we implemented it into FUMILI let me remind its main features. Let us have a look once more on (4).

$$\chi^2 = 1/2 \cdot \sum_{i=1,j=1}^{no,no} (C_i(\vec{X}) - C_i^m) Q_{i,j} (C_j(\vec{X}) - C_j^m) \tag{5}$$

Parameter estimates are those which turn (4) into minimum and you find them by iteration method. In doing so you take some initial values for the parameters $\vec{X}_0$, expand (4) near $\vec{X}_0$, getting the form similar (6).

$$F = F_0 + \vec{G} \cdot \Delta\vec{X} + 1/2 \Delta\vec{X}^T \cdot Z \cdot \Delta\vec{X} \tag{6}$$

Then you apply standard requirement of the minimum-namely first derivatives of (6) over parameters in the minimum should be equal zeros.

The latter give you formula for the parameters increments leading to the minimum,namely:

$$\Delta\vec{X} = -Z^{-1} \cdot \vec{G} \tag{7}$$

But there is very serious problem - you will go to the real minimum(without problems!) if the matrix of the second derivatives(hessian) - is positively defined! If the function to be minimized is (4),then second derivatives matrix is (8)

$$Z_{i,j} = \sum_{k=1,l=1}^{no,no} [dC_k/dX_i \cdot dC_l/dX_j + d^2C_k/(dX_i \cdot dX_j) \cdot (C_l(\vec{X}) - C_l^m)] \cdot Q_{k,l} \tag{8}$$

If you in (8) drop the second term, then matrix $Z_{i,j}$ becomes ever positively defined. That is exactly what is done in a FUMILI method. Real practice during many years of FUMILI exploitation showed its simplicity, reliability over enormous quantity of the problems, though it has its defects.

Change Of Parameter Increments(COPI) Method

Let's get back to the search for the minimum of the form (9)

$$\chi^2 = 1/2 \cdot \sum_{i=1,j=1}^{no,no} (C_i(\vec{X}) - C_i^m) Q_{i,j} (C_j(\vec{X}) - C_j^m) \tag{9}$$

It is performed by iteration procedure, when near some beginning values of parameters $\vec{X} = \vec{X}_0$ the function (9) is approximated by quadratic form (6),i.e.

$$F = F_0 + \vec{G} \cdot \Delta\vec{X} + 1/2 \Delta\vec{X}^T \cdot Z \cdot \Delta\vec{X} \tag{10}$$

where $\vec{G}$ is vector of derivatives and Z matrix of the second derivatives over parameters $\vec{X}$.

Near the same values $\vec{X_0}$ you can expand constraints (2) into :

$$\vec{f} = \vec{f}(\vec{X_0}) + \mathbf{DF} \cdot \Delta(\vec{X}) = 0 \tag{11}$$

where **DF** rectangular matrix of the constraint derivatives over parameters(nc rows and np columns). We may rewrite (11) as follows

$$\vec{f} = \vec{f}(\vec{X_0}) + \mathbf{DF_1} \cdot \Delta\vec{X_f} + \mathbf{DF_2} \cdot \Delta\vec{X_c} = 0 \tag{12}$$

Here DF_1 and DF_2 - submatrices of **DF**, 1-st has nc rows and (np - nc) columns, second nc rows and nc columns. Having (12) we may express $\Delta\vec{X_c}$ as a function of $\Delta\vec{X_f}$ in the form

$$\Delta\vec{X_c} = \vec{R} + \vec{S} \cdot \Delta\vec{X_f} \tag{13}$$

..Continued

Subvector $\Delta\vec{X}_c$ may be changed according to (13) and we come the new quadratic form depending only on nf= np -nc increments $\Delta\vec{X}_f$:

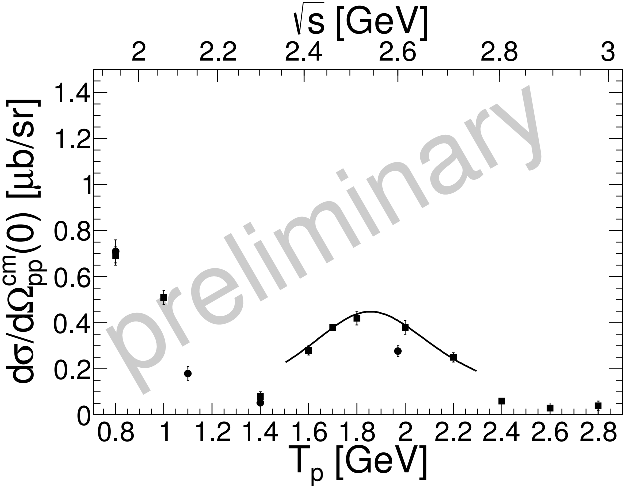
$$F \to F' = F_0' + \vec{G'} \cdot \Delta\vec{X}_f + 1/2\Delta\vec{X}_f^{\,T} \cdot Z' \cdot \Delta\vec{X}_f \tag{14}$$

$$F_0^{'} = F_0 + \sum_{k=1}^{nc} R_k \cdot [G_{nf+k} + 1/2 \cdot \sum_{l=1}^{nc} Z_{nf+k,nf+l} R_l] \tag{15}$$

$$G_i^{'} = G_i + \sum_{k=1}^{nc} G_{nf+k} \cdot S_{k,i} + \sum_{k=1}^{nc} R_k [Z_{nf+k,i} + \sum_{l=1}^{nc} S_{l,i} Z_{nf+l,nf+k}] \tag{16}$$

$$Z_{i,j}^{'} = Z_{i,j} + \sum_{k=1}^{nc} [S_{k,i} \cdot Z_{nf+k,j} + S_{k,j} \cdot Z_{i,nf+k}] + \sum_{k=1,l=1}^{nc,nc} S_{k,i} Z_{nf+k,nf+l} S_{l,j} \tag{17}$$

ANKE,COPI Method,$pp \to pp_S\pi^0 (pp_S = {}^1S_0 \text{ state})$



Tutorial Example of COPI Method

$$\mathrm{PDF}(x, y) = (1 + \alpha_1 \cdot x + \alpha_2 \cdot y)/(1 + 0.5 \cdot \alpha_1 + 0.5 \cdot \alpha_2)$$

Area: $0 < x < 1$ and $0 < y < 1$;
True values: $\alpha_1 = 0.5$ and $\alpha_2 = 0.8$;
Events: 10^5;
Constraint: $\alpha_1 + \alpha_2 = 1.3$.

The values of the estimates for the constrained and unconstrained cases. Errors cited are those calculated by the program.

parameter	constrained option	unconstrained option
α_1	0.501 ± 0.013	0.515 ± 0.023
α_2	0.799 ± 0.013	0.815 ± 0.026

1. In both cases the estimates are within one calculated error of true values;
2. Calculated errors in constrained option are two times less than in unconstrained;
3. The values of estimates in constrained option are much nearer to the true one.

Strictly opinion of Presenter:

I think that all of the three methods
must be incorporated in SPD
software and it is up to user
to select which is preferrable

Thank You!