PT-symmetric quantum cosmology of the phantom fields

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Outline

- Motivation - cosmology of phantom fields
- Pseudo-Hermitian quantum mechanics
- Minisuperspace PT-symmetric quantum cosmology
- Inhomogeneous fluctuations

Dark Energy

- According to observations after inflation the Universe was spatially-flat, homogeneous and isotropic with fine structure representing small fluctuations over a flat space,

$$ds^2 = d\tau^2 - a^2(\tau) d\vec{x}^2 \tag{1}$$

induced by a diagonal energy-momentum tensor

$$T_{\mu\nu} = \mathrm{diag}(\epsilon, p, p, p) \tag{2}$$

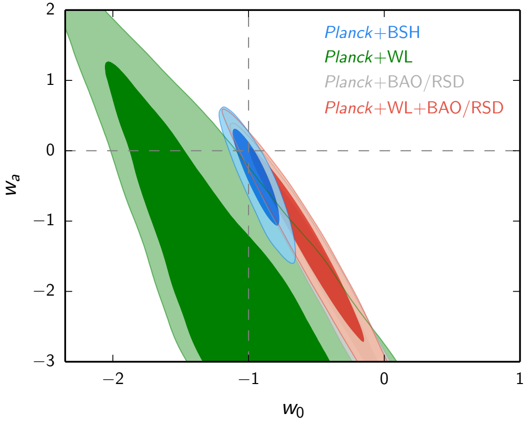
Hubble variable $h \equiv \frac{\dot{a}}{a}$ characterizes the Universe expansion and satisfies the Friedmann equation

$$h^2 = \frac{\kappa^2}{3}\epsilon, \quad p = -\frac{2\dot{h}}{\kappa^2} - \frac{3h^2}{\kappa^2} \quad , \quad \kappa^2 = 8\pi G = M_{Pl}^{-2} \tag{3}$$

- The average energy density ϵ and pressure p are somewhat unusual: dark and visible matter are dominated by something called *dark energy* with equation of state $w = p/\epsilon \sim -1$.

Dark Energy equation of state

The simplest explanation is cosmological constant with $w = -1$ however the so-called *phantom matter* with $w < -1$ has not been excluded. For ansatz $w = w_0 + w_a(1-a) + O\Big((1-a)^2\Big)$ according to PLANCK, 1502.01590



Phantom scalar matter

The phantom matter can be e.g. a scalar field ξ possessing negative kinetic energy,

$$L_{phantom} = -\frac{1}{2}\partial_\mu \xi \partial^\mu \xi - V(\tilde{\xi}) \tag{4}$$

However this leads to severe instabilities because its energy is not bounded from below and cosmological evolution may end up in the Big Rip. We propose to describe them with classically equivalent model,

$$L_{PTom} = \frac{1}{2}\partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} - V(i\tilde{\Phi}) \tag{5}$$

with PT symmetry $\tilde{\Phi} \mapsto -\tilde{\Phi}$, $i \mapsto -i$. The perturbations should be considered along the real axis,

$$\tilde{\Phi} = i\xi_{class} + \delta\Phi \tag{6}$$

Such perturbations near classical trajectory happen to possess positively definite effective Hamiltonian. To separate the fields with PT-symmetry from usual phantoms we coin a new name - PTom. The aim of this work is to explore the possibility to construct quantum model of PToms.

PT-symmetric non-Hermitian Hamiltonians

Consider the following Hamiltonian,

$$H = p^2 + x^2(ix)^\epsilon \tag{7}$$

While it's not Hermitian for $\epsilon > 2$ it is symmetric under PT-symmetry,

$$\mathcal{PT} : x \mapsto -x,\, p \mapsto p\, i \mapsto -i \tag{8}$$

It happens that for this Hamiltonian and many other PT-symmetric Hamiltonians all eigenvalues happen to be real and positive even for $\epsilon = 2$ (Bender, Boettcher, 1997). Its eigenfunctions form a complete set but of course are not orthogonal in terms of the initial norm but rather some new norm,

$$\langle \phi_n | \mathcal{CPT} | \phi_m \rangle = \delta_{nm}, \quad \mathcal{C}^\dagger = -\mathcal{C}, \quad [\mathcal{C}, H] = 0, \quad [\mathcal{C}, \mathcal{PT}] = 0 \tag{9}$$

From the point of view of this new norm the Hamiltonian can be considered Hermitian and generates unitary evolution.

Pseudo-Hermitian quantum mechanics

PT-symmetric Hamiltonians are a particular case of the Pseudo-Hermitian quantum mechanics.

$$H = \eta^{-1} h \eta, \quad h = h^\dagger, \quad \eta^\dagger \eta \neq 1 \tag{10}$$

One may either use the Hermitian Hamiltonian h and the corresponding norm $\langle \psi | \phi \rangle$ to compute probabilities or the equivalent description with non-Hermitian Hamiltonian H and new norm $\langle \psi | \eta^\dagger \eta | \phi \rangle$.
However e.g. for $H = p^2 + x^2 (ix)^\epsilon$ the similarity transformation η is known only in perturbation theory and the corresponding h is highly nonlocal.
Thus naively non-Hermitian or unbounded from below Hamiltonian may describe unitary evolution of some stable but very complicated system.

For time-dependent Hamiltonian $h(t)$ and similarity operator $\eta(t)$ the equivalent non-Hermitian Hamiltonian no longer is Pseudo-Hermitian and may have imaginary eigenvalues,

$$H = \eta^{-1} h \eta - i \eta^{-1} \dot{\eta} \tag{11}$$

Quintessence + PTom model

In order to fit observations we need a composition of two scalar fields: quintessence and PTom ones. Let's consider the following model,

$$
\begin{aligned}
S &= \int d^4x\,\sqrt{-g}\Big(-\frac{1}{2\kappa^2}R + \frac{1}{2}M_{\Phi\Phi}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}M_{\tilde\Phi\tilde\Phi}\partial_\mu\tilde\Phi\partial^\mu\tilde\Phi \\
&+ iM_{\Phi\tilde\Phi}\partial_\mu\Phi\partial^\mu\tilde\Phi - Ve^{\lambda\Phi} + \tilde V e^{i\tilde\lambda\tilde\Phi}\Big),
\end{aligned}
\tag{12}
$$

where all parameters are real to preserve the following symmetry,

$$
t \mapsto -t, \quad i \mapsto -i, \quad \Phi \mapsto \Phi, \quad \tilde\Phi \mapsto -\tilde\Phi. \tag{13}
$$

Let us restrict ourselves at first to spatially flat Friedman-Robertson-Walker *minisuperspace*,

$$
ds^2 = N^2(t)\,dt^2 - e^{2\rho(t)}\,d\vec{x}^2, \qquad \Phi = \Phi(t), \quad \tilde\Phi = \tilde\Phi(t). \tag{14}
$$

with $N(t)$ being a *lapse* variable providing the time-reparametrization invariance.

Hamiltonian

The Hamiltonian is $\mathcal{H} = Ne^{-3\rho}H_0$, proportional to constraint. After the following transformation:

$$\chi = \lambda\Phi + 6\rho, \quad \pi = \frac{1}{\lambda}p_\Phi, \quad \tilde{\chi} = \tilde{\lambda}\tilde{\Phi} - 6i\rho, \quad \tilde{\pi} = \frac{1}{\tilde{\lambda}}p_{\tilde{\Phi}}, \tag{15}$$

$$\omega = p_\rho - \frac{6}{\lambda}p_\Phi + \frac{6i}{\tilde{\lambda}}p_{\tilde{\Phi}}, \tag{16}$$

For special form of kinetic matrix,

$$\lambda\tilde{\lambda}\frac{M_{\Phi\tilde{\Phi}}}{\mathcal{D}} = 6\kappa^2, \; D = \lambda^2\frac{M_{\tilde{\Phi}\tilde{\Phi}}}{\mathcal{D}} - 6\kappa^2, \; \tilde{D} = \tilde{\lambda}^2\frac{M_{\Phi\Phi}}{\mathcal{D}} + 6\kappa^2 \tag{17}$$

where $\mathcal{D} = M_{\Phi\Phi}M_{\tilde{\Phi}\tilde{\Phi}} + M^2_{\Phi\tilde{\Phi}}$, the variables separate and one can obtain exact classical trajectories,

$$H = Ne^{-3\rho}\Big[-\frac{\kappa^2}{12L^3}\omega^2 - \frac{\kappa^2}{L^3}\omega\pi + i\frac{\kappa^2}{L^3}\omega\tilde{\pi} + \frac{1}{2L^3}D\pi^2 + \frac{1}{2L^3}\tilde{D}\tilde{\pi}^2 + Ve^{\chi} - \tilde{V}e^{i\tilde{\chi}}\Big]. \tag{18}$$

Wheeler-DeWitt equation

By canonical quantization of the constraint we obtain WdW equation $\hat{H}\Psi = 0$

$$\Psi(\rho,\chi,\tilde{\chi}) = \sum_C \int d\omega\, e^{iL^{3/2}\omega\rho}\psi(\omega,C,\chi)\tilde{\psi}(\omega,C,\tilde{\chi}), \tag{19}$$

For Hermitian sector we get,

$$\left[-\frac{\kappa^2}{12}\omega^2(\frac{1}{2}+C) + i\frac{\kappa^2}{L^{3/2}}\omega\,\partial_\chi - \frac{D}{2L^3}\,\partial_\chi^2 + Ve^\chi\right]\psi = 0, \tag{20}$$

We expect that the norm can be constructed using L^2 on χ. Therefore one should select oscillating solutions with $\nu = \frac{L^{3/2}\kappa\omega}{D}\sqrt{\frac{2}{3}(6\kappa^2 + D/2 + CD)}$

$$\psi(\omega,C,\chi) = \exp\left[iL^{3/2}\frac{\kappa^2\omega}{D}\chi\right]K_{i\nu}\left(2L^{3/2}\sqrt{\frac{2V}{D}}e^{\chi/2}\right) \tag{21}$$

PTom sector in minisuperspace

In PT sector the equation reads,

$$\left[-\frac{\kappa^2}{12}\omega^2(\frac{1}{2}-C)+\frac{\kappa^2}{L^{3/2}}\omega\,\partial_{\tilde{\chi}}-\frac{\tilde{D}}{2L^3}\,\partial^2_{\tilde{\chi}}-\tilde{V}e^{i\tilde{\chi}}\right]\tilde{\psi}=0, \qquad (22)$$

with solutions, $\tilde{\nu}=\frac{L^{3/2}\kappa\omega}{\tilde{D}}\sqrt{\frac{2}{3}(6\kappa^2-\tilde{D}/2+C\tilde{D})}$

$$\tilde{\psi}(\omega,C,\chi)=\exp\left[L^{3/2}\frac{\kappa^2\omega}{\tilde{D}}\chi\right]J_{\tilde{\nu}}\left(2L^{3/2}\sqrt{\frac{2\tilde{V}}{\tilde{D}}}e^{i\tilde{\chi}/2}\right) \qquad (23)$$

Two natural options are to require regular behaviour on $e^{i\tilde{\chi}}=1$ circle after or before applying exponential factor. In the first option we get to the system considered in (Curtright, Mezincescu, 2007) with biorthogonal system constructed from Neumann polynomials, requiring $\tilde{\nu}$ to be integer. Then ω is real however this way we lose regular behaviour for the function Ψ. If we require regularity of the whole ψ we get instead, $\tilde{\nu}=n+2iL^{3/2}\frac{\kappa^2\omega}{\tilde{D}}$. This however would require complex ω except when $n=0$ which fixes $C=1/2$.

Longitudinal inhomogeneous modes

Longitudinal (3d scalar) decouple from transverse (tensor and vector) ones at quadratic order.

$$g_{\mu\nu} = (N^2(t) + s(t,x))dt^2 + 2(\partial_k v(t,x))dtdx^k - e^{2\rho}\Big(\delta_{ij} + h(t,x)\delta_{ij} + \partial_i\partial_j E(t,x)\Big)dx^i dx^j \tag{24}$$

$$\Phi = \Phi(t) + \phi(t,x), \quad \tilde{\Phi} = \tilde{\Phi}(t) + \tilde{\phi}(t,x) \tag{25}$$

Let us choose the gauge $h = E = 0$ and decompose into eigenfunctions of Laplace operator,

$$\phi(t,x) = \phi(t,\Omega)f(\Omega,x), \quad -\Delta f(\Omega,x) = \Omega^2 f(\Omega,x) \tag{26}$$

$$H_0 + \sum_{\Omega} H_2(\Omega) = 0, \quad p_s = 0, \quad p_v = 0 \tag{27}$$

Thus s and v are undynamical variables that can be get rid of after solving secondary constraints.

Fluctuation evolution

After quantization we assume Born-Oppenheimer approximation,

$$\Psi = \Psi_0(\rho, \Phi, \tilde{\Phi})\psi_2[\phi, \tilde{\phi}|\rho, \Phi, \tilde{\Phi}] \tag{28}$$

If we take the limit $L \to +\infty$ we assume that $\Psi_0 = \psi_0 e^{iL^3 S}$ then we get equation,

$$\frac{i\kappa^2}{6}(\partial_\rho S)(\partial_\rho \psi_2) - i(M^{-1})_{ab}(\partial_{\Phi_a} S)(\partial_{\Phi_b}\psi_b) = \sum_\Omega H_2(\Omega)\psi_2,$$

where indices $a, b = \Phi, \tilde{\Phi}$. The lhs is usually interpreted as derivative $\frac{i}{L^3}\partial_\tau$ in so-called WKB-time that is taken along the classical trajectory,

$$\frac{i\kappa^2}{6}(\partial_\rho S)(\partial_\rho \tau) - i(M^{-1})_{ab}(\partial_{\Phi_a} S)(\partial_{\Phi_b}\tau) = 1 \tag{29}$$

The effective Hamiltonian $\sum_\Omega H_2$ takes the form,

$$H_2 = \frac{1}{2}(M^{-1})_{ab} p_{\phi,a} p_{\phi,b} + \frac{e^{6\rho}}{2}(\partial^2 V)_{ab}\phi_a\phi_b + \frac{3}{4}\kappa^2\Big((M^{-1})_{ab}\phi_a p_{\Phi,b}\Big)^2$$
$$- \frac{3}{p_\rho}\Big((M^{-1})_{ab}\phi_a p_{\Phi_b})\Big)\Big[(M^{-1})_{ab} p_{\Phi,a} p_{\phi,b} + e^{6\rho}(\partial V)_a \phi_a\Big]$$

$\kappa \to 0$ limit

The problem is greatly simplified when $\kappa \to 0$ and quintessence and PTom fluctuations are separated, in $\kappa \to 0$ limit it greatly simplifies,

$$H_2 = \frac{D}{2} p_\phi p_\phi + \frac{\tilde{D}}{2} p_{\tilde{\phi}} p_{\tilde{\phi}} + \frac{V}{2} e^{6\rho + \Phi} \phi^2 - \frac{\tilde{V}}{2} e^{6\rho + i\tilde{\Phi}} \phi^2 + O(\kappa) \qquad (30)$$

The Hamiltonian is Hermitian on the purely imaginary classical trajectory. For stability the wavepacket should be concentrated so that $-\frac{\tilde{V}}{2} e^{6\rho + i\tilde{\Phi}} > 0$.
For $\mathrm{Im}\tilde{\Phi} \neq 0$ the similarity operator η should be introduced but due to the variable separation it should act nontrivially only on $\tilde{\phi}$ and $p_{\tilde{\phi}}$ i.e.

$$\eta = \exp\left[\alpha(t) p_{\tilde{\phi}}^2 + \beta(t) \tilde{\phi}^2 + \gamma(t)(p_{\tilde{\phi}} \tilde{\phi} + \tilde{\phi} p_{\tilde{\phi}})\right] \qquad (31)$$

Then one has to deal with non-Hermitian time-dependent Swanson Hamiltonian - the problem considered in (Fring, Moussa, 2016) and (Maamache et al, 2017)
For trajectories close to $\mathrm{Im}\tilde{\Phi} = 0$ one can obtain this operator easily using perturbation theory.

Conclusions

- The pseudo-Hermitian quantum mechanics is a way to deal with apparent instabilities of the phantom models
- The fluctuation evolution in WKB time for trajectories with nonzero real part of $\tilde{\Phi}$ can be obtained using time-dependent Hermitian similarity operator at least within $\kappa \to 0$ approximation and close enough trajectories.
- This result can be used to construct the probability distributions and obtain the observable consequences of the model.

Thank you for your attention!