Light Front Hamiltonian Approach

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- Light Front (LF) quantization, i.e. quantization in coordinates:

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}},\ x^{\perp},\quad \text{"time"}\ x^{+} = 0, \tag{1}$$

has the application not only for high energy physics but also as an approach to nonperturbative calculations in quantum field theory like QCD.
This approach encounter difficulties related to singularities of the LF as a quantization plane. These difficulties are related to the so called "zero mode" problem ($p_{-} = 0$) which can be responsible for difficulties with the description of vacuum effects and confinement.

• We considered two possibilities for the construction of LF Hamiltonian:
(1) The comparison of perturbation theory generated by the regularized canonical LF Hamiltonian and usual covariant perturbation theory in Lorentz coordinates, then the construction of renormalized LF Hamiltonian perturbatively equivalent to that in usual quantization.
(2) The investigation of the limit transition to LF quantization starting from Hamiltonians on the spacelike hyperplanes approaching to the LF. This allows to treat the singularity of the LF nonperturbatively and formulate more simple models like quark model or its generalization.

• Perturbative construction of LF Hamiltonian can be done with the extra fields like that in Pauli-Villars regularization. This allows to escape the differences in calculation of diagrams on the LF and in Lorentz coordinates. We have done this for nongauge theories and for Yang-Mills and QCD.

• On the second approach, with the limit transition to the LF Hamiltonian, we propose quark-antiquark model taking into account only zero mode of transverse gluon field and reducing the Fock space to only quark and antiquark. We have obtained the spectral equation for quark-antiquark bound state, which is UV finite, and calculated the meson mass spectrum which can be compared with experimental meson spectrum.

Further we will consider only this approach.

• Main advantage of LF quantization is a possibility to define physical vacuum as the state with the minimal eigenvalue of the momentum operator $P_- \geq 0$ (playing the role of the space component of the momentum),

$$P_-|0\rangle = 0. \quad (2)$$

One can define the "physical" Fock space on this vacuum. As an example, for scalar field in Heisenberg picture we can write the Fourier decomposition on the LF:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int\limits_{|p_-|\geqslant\varepsilon} \frac{dp_- dp_\perp}{\sqrt{2|p_-|}} \left(a(p_-, p_\perp; x^+) e^{-ip_- x^- - ip_\perp x^\perp} + \mathrm{H.c.} \right) . (3)$$

Then the action takes the form: $S = \int d^4x \, (\partial_+\varphi \partial_-\varphi - \mathcal{H}) =$

$$= \int dx^+ \int_\varepsilon^\infty dp_- \int dp_\perp \left(ia^+(p_-, p_\perp; x^+) \partial_+ a(p_-, p_\perp; x^+) - \mathcal{H} \right) . (4)$$

It follows from the canonical form of this action that $a(p_-, p_\perp; 0)$ and $a^+(p'_-, p'_\perp; 0)$ satisfy the commutation relation of creation and annihilation operators:

$$[a(p_-, p_\perp; 0), a^+(p'_-, p'_\perp; 0)] = \delta(p_- - p'_-)\delta(p_\perp - p'_\perp). \qquad (5)$$

For momentum operators we get:

$$P_- = \frac{P_0 - P_3}{\sqrt{2}} = \int_\varepsilon^\infty dp_- \int dp_\perp p_- a^+(p_-, p_\perp; x^+) a(p_-, p_\perp; x^+), (6)$$

$$a(p_-, p_\perp; x^+)|0\rangle = 0, \qquad P_-|0\rangle = 0, \qquad (7)$$

$$P_+ = \frac{P_0 + P_3}{\sqrt{2}} = \int dp_\perp \int_\varepsilon^\infty dp_- \frac{p_\perp^2 + m^2}{2p_-} a^+(p) a(p) + U(\varphi). \quad (8)$$

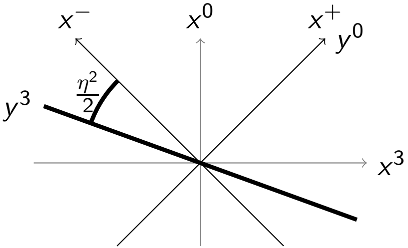
In this Fock space one can formulate the mass spectrum problem:

$$P_+|p_-,p_\perp\rangle = \frac{m^2+p_\perp^2}{2p_-}|p_-,p_\perp\rangle. \tag{9}$$

• To take into account zero mode $p_- = 0$ we choose the regularization $|x^-| \leq L$ plus periodic boundary conditions for fields in x^-. This discretizes the $p_- = p_n = \pi n/L$ and separates zero mode, which now can approximately describe the vicinity of $p_- = 0$. LF canonical formalism leads in this case to constraints relating zero and nonzero modes. However these constraints are nonlinear in fields and very complicated. Moreover the difficulty arises with their unique definition after quantization. So further we propose the other way of taking into account zero mode, considering the limit transition to the light front Hamiltonian from Hamiltonians on space-like hyperplanes approaching to the light front.

We introduce instead of LF the near LF coordinates y^μ ("η-coordinates"):

$$y^0 = x^+ + \frac{\eta^2}{2} x^-, \quad y^3 = x^-, \quad y^\perp = x^\perp,$$



and corresponding momenta:

$$q_0 = p_+, \quad q_3 = p_- - \frac{\eta^2}{2} p_+, \quad q_\perp = p_\perp,$$

where $\eta > 0$ is small parameter.

Our quark-antiquark model is based on $SU(N_c)$ QCD in η-coordinates. We take $A_3(y) = 0$ as the analog of the LF gauge $A_- = 0$. To descretize the momentum Q_3 we take $|y^3| \leq L$ plus corresponding periodic boundary conditions, so that $q_3 = \pi n/L$, $n \in Z$. Now we have two parameters, L and η. We perform the limit transition to the LF in two steps. At first step we fix parameter L and consider limit transition to the LF ($\eta \to 0$) for that part of the Hamiltonian in η-coordinates which contains nonzero modes, while for the part of this Hamiltonian, containing only zero modes, we fix the value of $\eta = \eta_0 > 0$. Further at the first step we introduce effective mass squared operator $M^2_{eff}(\eta_0)$ as a sum of contributions from both these separated parts of the Hamiltonian and write spectral equation for the eigenvalues of this operator. The second step is to perform $L \to \infty$, $\eta_0 \to 0$ limit in such a way that the spectrum remains UV finite.

We denote by $H_{(0)}$ the part of the Hamiltonian containing only zero modes and by $H_{(\emptyset)}$ the LF limit of the part of the Hamiltonian containing nonzero modes. We use the expressions for M^2 in LF and η_0-coordinates:

$$M^2 = 2P_+P_- - P_\perp^2 \qquad \text{in LF coordinates,}$$

$$M^2 = 2Q_0Q_3 + \eta_0^2 Q_0^2 - P_\perp^2 \qquad \text{in } \eta_0\text{-coordinates.}$$

Then the expression for the effective mass squared operator takes the form:

$$M_{eff}^2 = \eta_0^2 H_{(0)}^2 + 2P_- H_{(\emptyset)},$$

where we take $Q_3 = 0$, $Q_0 = H_{(0)}$ and the total $P_\perp = 0$.

• We introduce UV regularization using transverse space lattice. In this regularization the Hamiltonian corresponding to only zero mode has the following form:

$$H_{(0)} = \sum_{x^\perp} \left(\frac{g^2}{4L\,\eta_0^2}\,\pi_k^a \pi_k^a + \frac{4L}{g^2 a^2} Re\; Tr\Big(I - U_{12}\Big)\right), \qquad (10)$$

where $U_{12}(x) = U_1^+(x - ae_2)U_2^+(x)U_1(x)U_2(x - ae_1)$;
$N \times N$ unitary matrices $U_1(x)$, $U_2(x)$ are lattice link variables and $\pi_k^a(x)$ are corresponding momenta:

$$[\,\pi_k^a(x), U_{k'}(x')] = -\delta_{kk'}\delta_{x^\perp x^{\perp\prime}}\,\frac{\lambda^a}{2}\,U_k(x)\,,$$

$$[\,\pi_k^a(x), \pi_{k'}^b(x')] = i\,\delta_{kk'}\delta_{x^\perp x^{\perp\prime}} f^{abc}\pi_k^c(x)\,.$$

The part of the lattice Hamiltonian containing nonzero modes can be obtained as the result of the limit transition to the LF.

This part depends on fermion field $\chi(x)$ on the LF coupled to $U_k(x)$:

$$\chi_r^i(x) = \frac{1}{a\sqrt{2L}} \sum_{n>0} \left(b_{nr}^{\,i}(x^\perp)\, e^{-ip_n x^-} + d_{nr}^{\,i+}(x^\perp)\, e^{ip_n x^-} \right), \quad (11)$$

$$H_{(\emptyset)} = \sum_{x^\perp} \Bigg\{ \int_{-L}^{L} dx^- \Bigg[a^2\, Tr\left(F_{+-}^2\right) -$$

$$-\frac{i}{8}\left(\chi^+(x - ae_{k'})\, \sigma_{k'}\, U_{k'}^{-1}(x) - \chi^+(x + ae_{k'})\, U_{k'}(x + ae_{k'})\, \sigma_{k'} + 2m_q a \chi^+ \right) \cdot$$

$$\cdot \partial_-^{-1}\left(U_k(x) \sigma_k \chi(x - ae_k) - \sigma_k\, U_k^{-1}(x + ae_k)\, \chi(x + ae_k) + 2m_q a \chi \right) \Bigg] +$$

$$+ \left(\frac{g^2\,(N - 1/N)}{16La^2} \sum_{m>0} \frac{1}{p_m} \right) \left(\sum_{n \neq m} \frac{\chi_n^+ \chi_n}{p_{m-n}} + \left(\sum_{n>m} + \sum_{n<-m} \right) \frac{\chi_n^+ \chi_n}{p_n} \right) \Bigg\} +$$

$$+ const,$$

where $F_{+-} = -\frac{g}{2}\partial_-^{-1}\left(\chi^+\lambda^a\chi\frac{\lambda^a}{2}\right)$ due to gauge constraint.
We take the states with quark and antiquark, connected by a gluon zero mode "string", predominantly along the shortest path in the transverse plane:

$$\left\{\sum_{x^\perp} b^\dagger(x^\perp, p_- = \pi m/L) U^{\mathcal{S}}_{x,x+\Delta x} d^\dagger(x^\perp + \Delta x, p_- = (n-m)\pi/L)\,|0\rangle\right\}.$$

Here $U^{\mathcal{S}}_{x,x'}$ denotes chains of matrices $U_k^\dagger$ and U_k $(k = 1, 2)$, that connect points x and x' of 2-dimensional transverse space along the path $\mathcal{S}$; the $b^\dagger$ and $d^\dagger$ are creation operators of quark and antiquark respectively.

Next assumption is related to the form of four-fermion operator in the Hamiltonian. This operator is the result of the solution of the gauge constraint on the LF:

$$\sum_{x^{\perp}} \int dx^{-} dx'^{-} \left(\chi^{\dagger}(x^{-}, x^{\perp}) \frac{\lambda^{a}}{2} \chi(x^{-}, x^{\perp}) \right) |x^{-} - x'^{-}| \cdot$$

$$\cdot \left(\chi^{\dagger}(x'^{-}, x^{\perp}) \frac{\lambda^{a}}{2} \chi(x'^{-}, x^{\perp}) \right).$$

In QCD(1+1) at $N \to \infty$ ('t Hooft model) such a term leads to quark confinement on the LF due to the nonlocal factor $|x^{-} - x'^{-}|$. However in (3+1)-dimensions the action of this operator on our quark-antiquark states gives zero except for the state in which quark and antiquark are not separated in $x^{\perp}$. This occurs due to the locality of this operator in $x^{\perp}$ and the truncation of Fock space by only one quark and only one antiquark. So the local in $x^{\perp}$ four-fermion operator can not act nontrivially on states with quark and antiquark separated in $x^{\perp}$.

To overcome this difficulty we propose the nonlocal in $x^{\perp}$ modification of this four-fermion operator such that its action on our states becomes nonzero also for separated in $x^{\perp}$ quark and antiquark. We introduce the nonlocality in gauge invariant way:

$$\frac{a^2}{L_{\text{had}}^2}\sum_{x^{\perp},\Delta x,b}\int dx^{-}dx^{\prime -}\left(\chi^{\dagger}(x^{-},x^{\perp})U^{\mathcal{S}}_{x,x'}\frac{\lambda^b}{2}U^{\mathcal{S}}_{x',x}\chi(x^{-},x^{\perp})\right)\cdot$$

$$\cdot|x^{-}-x^{\prime -}|\left(\chi^{\dagger}(x^{\prime -},x^{\perp}+\Delta x)U^{\mathcal{S}'}_{x+\Delta x,x'}\frac{\lambda^b}{2}U^{\mathcal{S}'}_{x',x+\Delta x}\chi(x^{\prime -},x^{\perp}+\Delta x)\right),$$

where $\mathcal{S}$ and $\mathcal{S}'$ are the shortest paths leading to the point x' from points x and $x+\Delta x$ respectively, and the point x' itself lies on the line connecting the points x and $x+\Delta x$. The factor a^2/L_{had}^2 corresponds to averaging over paths inside of the domain $|\Delta x|\leq L_{\text{had}}$ (the L_{had} is a scale of order Λ_{QCD}^{-1}). On the Lagrangian level, this modification corresponds to introducing a nonlocal interaction of the field A_{+} with the fermion current.

We construct the expression for the M^2_{eff} operator reduced to our restricted state space. This allows to write the equation for the quark-antiquark bound state mass spectrum wave function $f(r,l)$:

$$m^2_{\text{eff}} f(r,l) = \left[\left(\frac{g^2}{8L\eta_0 a}\left(N - \frac{1}{N}\right)\right)^2 r^2 + \left(\frac{p_n}{p_l} + \frac{p_n}{p_{n-l}}\right)\left(-\nabla^2 + m_q^2\right)\right] \cdot$$

$$\cdot f(r,l) + \frac{g^2 p_n}{2L^2_{\text{had}}}\frac{1}{L}\left(N - \frac{1}{N}\right) \sum_{k=1/2, k\neq l}^{n-1/2} \frac{f(r,l) - f(r,k)}{(p_l - p_k)^2},$$

where m_{eff} is the bound state mass, ∇^2 is the lattice analog of Laplace operator in transverse coordinates, r is the distance between quark and antiquark, m_q is the quark mass, $p_- = p_n = \pi n/L$ is the momentum of the state.
We introduce dimensionless variables

$$\mu = m_{\text{eff}} L_{\text{had}}, \quad \rho = \frac{r}{L_{\text{had}}}, \quad \overline{m}_q = m_q L_{\text{had}}, \quad \xi = \frac{p_l}{p_n}, \quad \alpha = \frac{L\eta_0 a}{L^2_{\text{had}}},$$

where the ξ is the relative momentum of the quark.

In the regularization removal limit $L \to \infty$, $\eta_0 \to 0$, $a \to 0$ we obtain UV finite result if we take the parameter $\alpha = L\eta_0 a/L_{\mathrm{had}}^2$ finite, for example, of order one. In this limit the spectral equation takes the form:

$$\mu^2 f(\rho,\xi) = \left[\left(\frac{g^2}{8\alpha^2}\left(N - \frac{1}{N}\right)\right)^2 \rho^2 + \left(\frac{1}{\xi} + \frac{1}{1-\xi}\right)\left(-\nabla^2 + \overline{m}_q^2\right)\right] \cdot$$

$$\cdot f(\rho,\xi) + \frac{g^2}{2\pi}\left(N - \frac{1}{N}\right) P \int_0^1 \mathrm{d}\xi' \, \frac{f(\rho,\xi) - f(\rho,\xi')}{(\xi - \xi')^2},$$

where $f(\rho,\xi)$ is the wave function of a quark-antiquark bound state, and the Cauchy principal value is assumed for the integral. In this limit the variable ξ becomes continuous and takes values in the interval $0 \le \xi \le 1$. The first term in the equation, proportional to ρ^2, is due to zero mode contribution and the last one has the form of the 't Hooft equation in 2-dimensional QCD. The eigenvalues and wave functions of our spectral equation cannot be described analytically, but they can be obtained numerically.

Qualitatively, the spectrum resembles the spectrum of the harmonic oscillator in three dimensions. A notable feature of our model is the possibility of degenerate equidistant energy levels that correspond to linear Regge trajectories. This is an important result because a significant degeneration is in fact observed in the experimental meson spectrum.

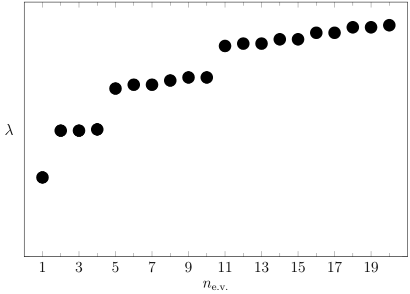


Figure 1: The spectrum of equation for a certan choice of parameters. Here the horizontal axis represents the number $n_{\text{e.v.}}$ of eigenvalue.