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Efimov properties of some 3-body systems

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Thomas and Efimov effects

1. L.H. Thomas (1935) *Phys. Rev.47, 903 (1935).*

   a) 2-particle potentials are short-range
   b) Each of them supports only a single bound state even with an arbitrarily weak binding energy

   Nevertheless, the three-body ground state energy can go to (-∞) when the range of the two-body forces approaches zero!
   (Just this surprising phenomenon is called the Thomas effect.)
   I.e., such a three-body system should collapse! ⇒ The 3-body Hamiltonian with zero-range interaction is not semibounded from below.

2. V. Efimov (1970) *Phys. Lett.33, 563 (1970).*

   When one weakens the two-body potentials (supporting a single bound state) the number of 3-body bound states can increase to infinity! And this happens at the moment when the two-body bound states disappear.

   ⇒ Efimov states ⇔ the states which appear under weakening and disappear under strengthening of the two-body potentials.
   - Quite strange and "non-standard" behavior

The simplest situation for which Efimov states occurs correspond to three identical boson interacting via resonant short-range interaction.

Beyond the range of the interaction r_0 the relative motion of two particles is almost free. The wave function $\Psi\,(\vec{r})$ in the asymptotically free region has a phase shift δ_ℓ with respect to the non-interacting wave function. The partial-wave expansion of $\Psi\,(\vec{r})$

$$\Psi\,(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{f_\ell(r)}{r} P_\ell\,(\cos\theta)$$

where

$$f_\ell(r) = \begin{cases} \mathit{complicated} \text{ for } r \lesssim r_0 \quad \text{(interaction region)} \\ \propto \sin\left(kr - \ell\dfrac{\pi}{2} + \delta_\ell\right) \text{ for } r \gg r_0 \quad \text{(free region)} \end{cases}$$

k is relative wave number between two particles, P_ℓ - Legendre polynomials.

Efimov physics arises when the two-body interaction is near resonant in the s-wave partial wave ($\ell = 0$), which means that the phase shift δ_0 of the s-wave is closed to $\frac{\pi}{2}$.

At low scattering energy a phase shift can be written as

$$\delta_0 \sim -\operatorname{arctg}(ka) \ \text{ for } k \ll r_0^{-1}$$

where a is the scattering length, so $|a| \gg r_0$ in order to have resonant interaction.

Zero-range theory: assume, that short-range region can be neglected and only asymptotically free region (that is parametrized by scattering length) is relevant.

– *zero-range pseudopotential* [Fermi, *Ricerca Scientifica*, 7 (1936) 13-52] *with regularization by a cut-off in p-space or Lee-Huang-Yang pseudopoten.*

– *Bethe-Peierls boundary condition*

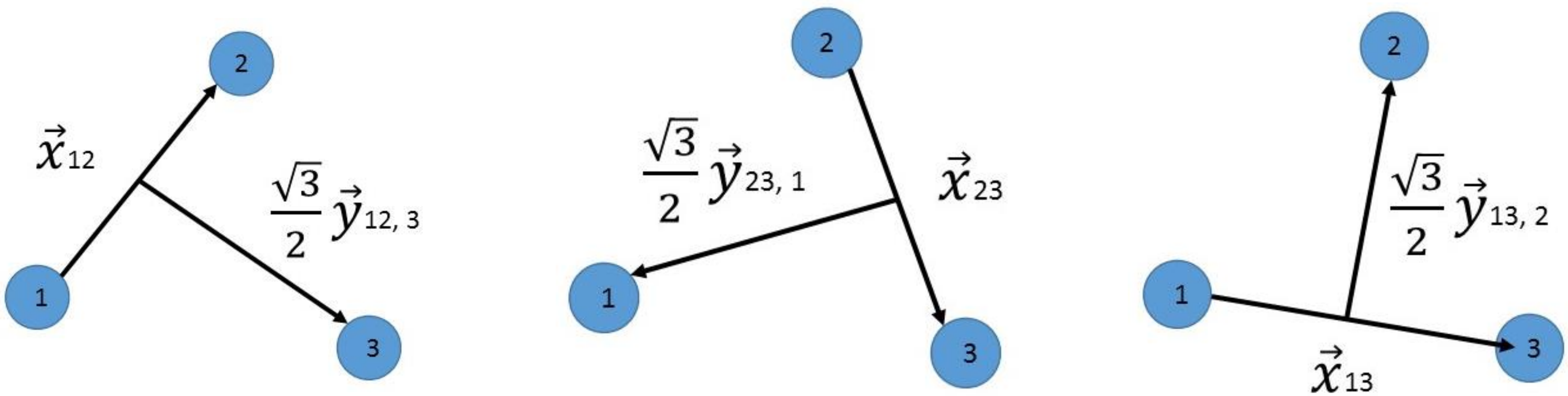
$$-\frac{1}{r\Psi}\frac{\partial}{\partial r}(r\Psi) \xrightarrow[r\to 0]{} \frac{1}{a}$$ [Bethe,Peierls, Proc. Royal Soc. London, (1935) 146-156]

All these methods correctly reproduce the form of the two-body w.f. In the region $r_0 \leq r \leq k^{-1}$

$$\Psi\left(\vec{r}\right) \propto \frac{1}{r} - \frac{1}{a}$$

The simplification which is used in the zero-range method keeps the same form of w.f. down to $r = 0$, although this is unphysical for $r \lesssim r_0$.

Such zero-range method can be applied to three-boson system, described by Jacobi coordinates after elimination the center of mass motion



$$\vec{x}_{ij} = \vec{r}_j - \vec{r}_i \qquad \vec{y}_{ij,k} = \frac{2}{\sqrt{3}}\left(\vec{r}_k - \frac{\vec{r}_i - \vec{r}_j}{2}\right)$$

Each Jacobi coordinate sets related to each other by rotation matrix

$$\vec{x}_{23} = -\frac{1}{2}\vec{x}_{12} + \frac{\sqrt{3}}{2}\vec{y}_{12,3} \qquad \vec{x}_{31} = -\frac{1}{2}\vec{x}_{12} - \frac{\sqrt{3}}{2}\vec{y}_{12,3}$$

$$\vec{y}_{23,1} = -\frac{\sqrt{3}}{2}\vec{x}_{12} - \frac{1}{2}\vec{y}_{12,3} \qquad \vec{y}_{31,2} = \frac{\sqrt{3}}{2}\vec{x}_{12} - \frac{1}{2}\vec{y}_{12,3}$$

Choosing one set for Jacobi coordinates, the three-body w.f. satisfies free Schrödinger equation at total energy $E = \frac{\hbar^2}{m}k^2$

$$\left(-\nabla_{x_{12}}^2 - \nabla_{y_{12,3}}^2 - k^2\right)\Psi = 0$$

With the Bethe-Peierls boundary conditions for all pair of bosons $-\frac{1}{r\Psi}\frac{\partial}{\partial r}(r\Psi) \xrightarrow[r\to 0]{} \frac{1}{a}$. The w.f. Ψ could be decompose to Faddeev components:

$$\Psi = F(\vec{x}_{12}, \vec{y}_{12,3}) + F(\vec{x}_{23}, \vec{y}_{23,1}) + F(\vec{x}_{31}, \vec{y}_{31,2})$$

which is satisfies the equations

$$\left(-\nabla_x^2 - \nabla_y^2 - k^2\right)F(\vec{x}, \vec{y}) = 0 \qquad (1)$$

with boudary condition for pair

$$\begin{aligned}&[\frac{\partial}{\partial x}\left(xF(\vec{x}, \vec{y})\right)]_{x\to 0} + F\left(\frac{\sqrt{3}}{2}\vec{y}, -\frac{1}{2}\vec{y}\right) + F\left(-\frac{\sqrt{3}}{2}\vec{y}, -\frac{1}{2}\vec{y}\right)\\ &= [-\frac{x}{a}(F(\vec{x}, \vec{y}) + F\left(\frac{\sqrt{3}}{2}\vec{y}, -\frac{1}{2}\vec{y}\right) + F\left(-\frac{\sqrt{3}}{2}\vec{y}, -\frac{1}{2}\vec{y}\right))]_{x\to 0}\end{aligned} \qquad (2)$$

The Efimov effect for bosons occurs in the partial wave channel with total angular momentum $L = 0$. In this channel F is independent of the direction of $\vec{x}$ and $\vec{y}$ and can be written as

$$F(\vec{x}, \vec{y}) = \frac{\chi_0(x, y)}{xy} \tag{3}$$

χ_0 is finite for $x \to 0$ and $\chi_0(x, y) \xrightarrow[y \to 0]{} 0$, inserting (3) into (1) & (2)

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - k^2\right)\chi_0(x, y) = 0$$

with boudary condition for $x \to 0$

$$[\frac{\partial}{\partial x}\left(\chi_0(x, y)\right)]_{x \to 0} + 2\frac{1}{\frac{\sqrt{3}}{4}y}\chi_0\left(\tfrac{\sqrt{3}}{2}y, \tfrac{1}{2}y\right) = -\frac{1}{a}\chi_0(0, y)$$

Rewrite these equations in polar (hyperspherical) coordinates

$$x = R\sin\alpha$$

$$y = R\ \cos\alpha \qquad \mathrm{R} = \sqrt{x^2 + y^2} \equiv \sqrt{\tfrac{2}{3}(x_{12}^2 + x_{23}^2 + x_{31}^2)}$$

$$\left(-\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2} - k^2\right)\chi_0(R,\alpha) = 0 \qquad (4)$$

with boudary condition for $\alpha \to 0$

$$[\frac{\partial}{\partial \alpha}\big(\chi_0(R,\alpha)\big)]_{\alpha\to 0} + \frac{8}{\sqrt{3}}\chi_0\big(R,\tfrac{\pi}{3}\big) = -\frac{R}{a}\chi_0(R,0) \qquad (5)$$

The problem then become separable in R and α for the case $a \to \pm\infty$

Thus one can find solution of eq. (4) in the form

$$\chi_0(R,\alpha) = f(R)\phi(\alpha) \qquad -\frac{d^2}{d\alpha^2}\phi(\alpha) = s_n^2\phi(\alpha)$$

with boudary condition at $\alpha = 0$ and $\alpha = \pi/2$

This gives the solution

$$\phi_n(\alpha) = \sin\left(s_n\left(\tfrac{\pi}{2} - \alpha\right)\right)$$

where s_n is solution of the equation

$$-s_n \cos\left(s_n \tfrac{\pi}{2}\right) + \frac{8}{\sqrt{3}} \sin\left(s_n \tfrac{\pi}{6}\right) = 0 \qquad (6)$$

Each n solution constitutes a channel for hyperradial motion, for each solution ϕ_n there is a corresponding hyperradial function $f_n(R)$ such that $f_n(R)\,\phi_n(\alpha)$ is a solution of equation (4):

$$\left(-\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{s_n}{R^2} - k^2\right) f_n(R) = 0$$

which can be written as one-dimensional Schrödinger equation

$$\left(-\frac{\partial^2}{\partial R^2} + V_n(R) - k^2\right)\sqrt{R}\, f_n(R) = 0 \qquad (7)$$

with hyperspherical potential

$$V_n(R) = \frac{s_n^2 - \frac{1}{4}}{R^2}$$

All solutions of the eq. (6) are real, except one $s_0 = \pm 1.0062378i$ which is purely imaginary, as a result, the effective potential is attractive for the channel $n = 0$. This is in contrast with non-interacting three-body problem, where the boundary condition (5) replaced by $\chi_0(R, \alpha) \underset{x \to 0}{\longrightarrow} 0$, leading to eigenvalues s_n =2(n+1) that are all real, and so repulsive effective potential for all n (a generalized centrifugal barrier).

In the interacting problem at $a \to \pm\infty$ limit the channel $n = 0$ leads to an effective three-body attraction

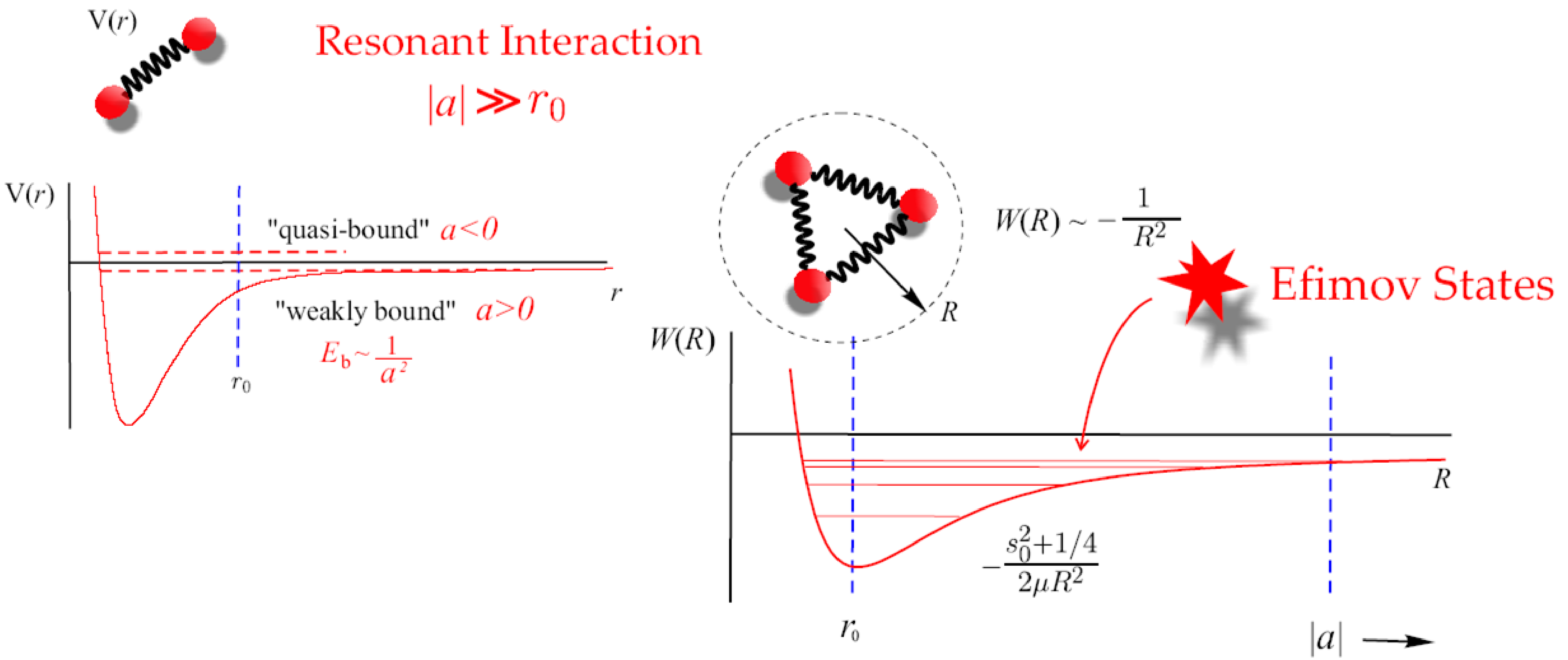
$$V_n(R) = -\frac{|s_n|^2 + \frac{1}{4}}{R^2}$$

which is the basis for the Efimov Physics.

One of the fundamentally new findings of V.Efimov is that the three-body problem with the boundary conditions not only yield just one bound state, but infinitely many bound states.

The equation is invariant under scaling transformation $R \to \lambda R$ and still invariant under a discrete set of scale transformations $R \to \lambda_0^n R$, with scaling factors that are integral powers of $\lambda_0 = e^{\pi/|s_0|} \approx 22.7$. Thus, if boundary condition gives a solution at some energy $E < 0$, it also gives solutions with energies $E/\lambda_0^{2n} < 0$.

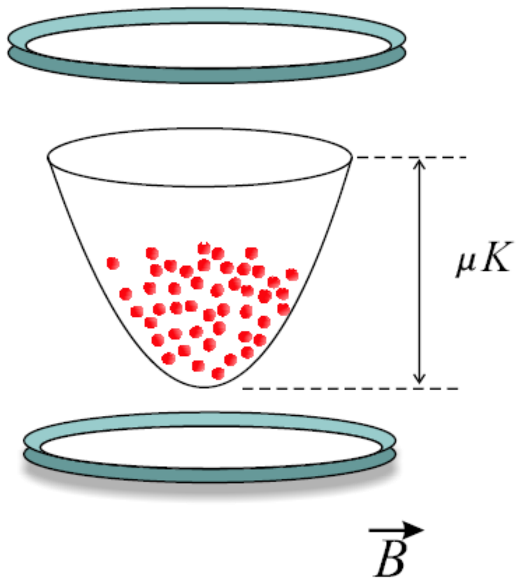
Therefore, an infinite number of bound states forming a geometric series of energies accumulating at zero energy with scaling factor $\lambda_0^2 \approx 515.035$.



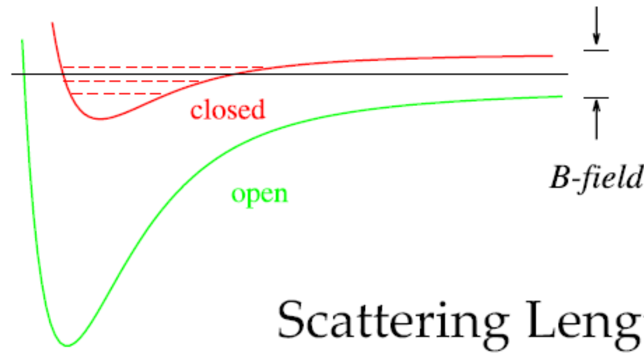
- Even though the pair-wise interactions are a short-range, the three particles feel the long-range attraction.
- Intuitively it can be explain by the fact that an effective interaction is mediated between two particles by the third particle moving back and forth between the two.
- It is thus possible for the three particles to feel their influence at distances much larger than the range of interactions, typically up to distances on the order of the scattering length.

Ultracold Quantum Gases

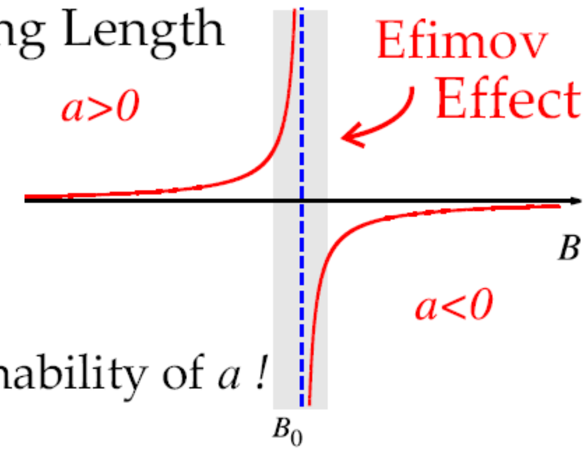
... after evaporative cooling ...



Feshbach Resonances



Scattering Length

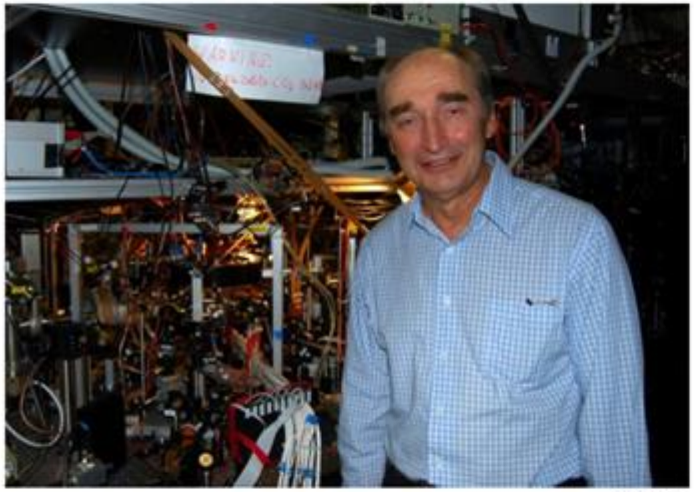


Tunability of a !

2-Body Collisions → Suppressed !

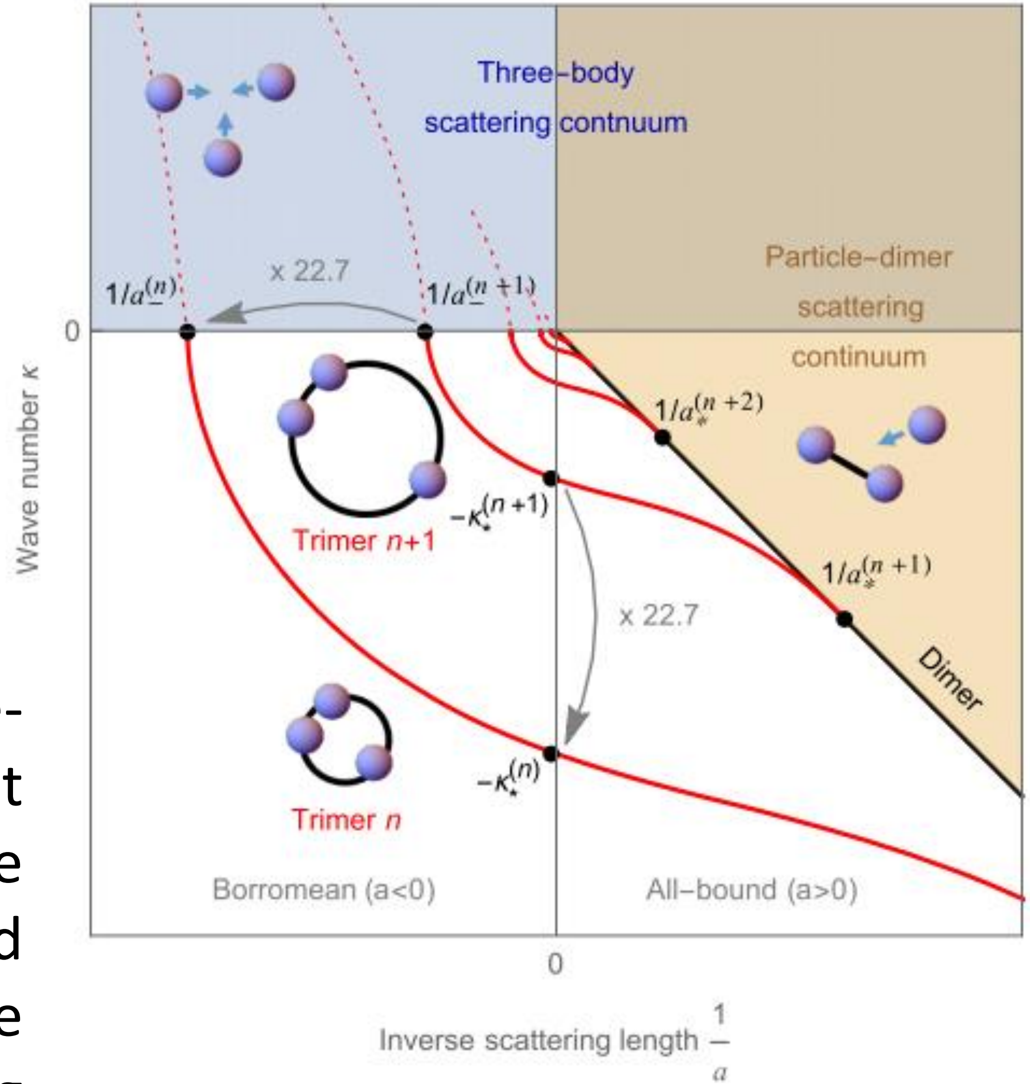
3-Body Collisions → Lifetime, Stability ... Efimov Effect !

Efimov scenario



2005

Vitaly Efimov in front of the experimental setup (group of Prof. Grimm, Uni Innsbruck), on which the "Efimov" effect was observed 35 years after the theoretical prediction



A trimer appears from three-body scattering threshold at $1/a_-$, has a binding wave number k_* at unitarity, and disappears below the particle−dimer sctattering threshold at $1/a_*$

Experimental observations

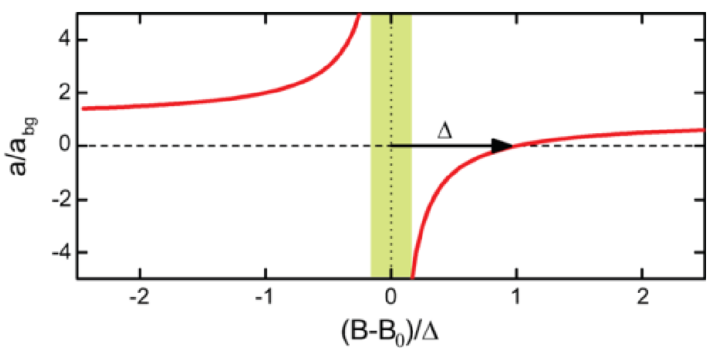
Evidence for Efimov quantum states in an ultracold gas of caesium atoms

NATURE|Vol 440|16 March 2006

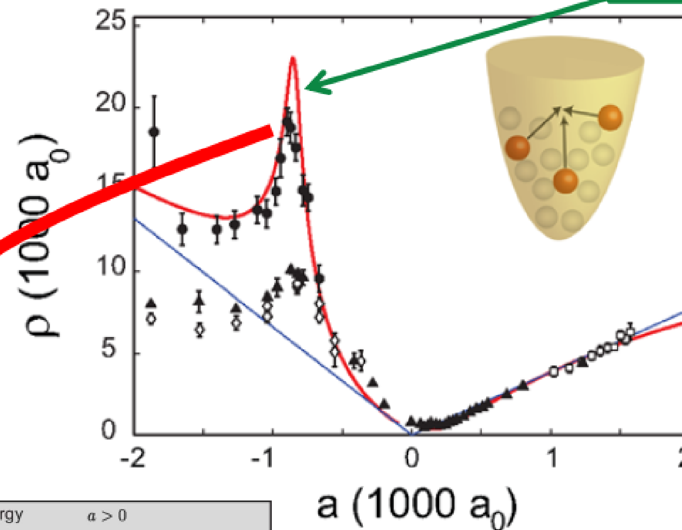
T. Kraemer[1], M. Mark[1], P. Waldburger[1], J. G. Danzl[1], C. Chin[1,2], B. Engeser[1], A. D. Lange[1], K. Pilch[1], A. Jaakkola[1] H.-C. Nägerl[1] & R. Grimm[1,3]

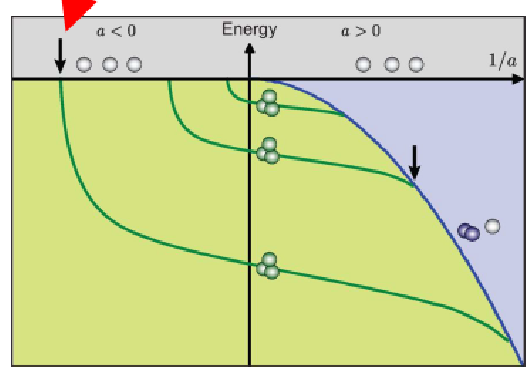
Efimov resonances

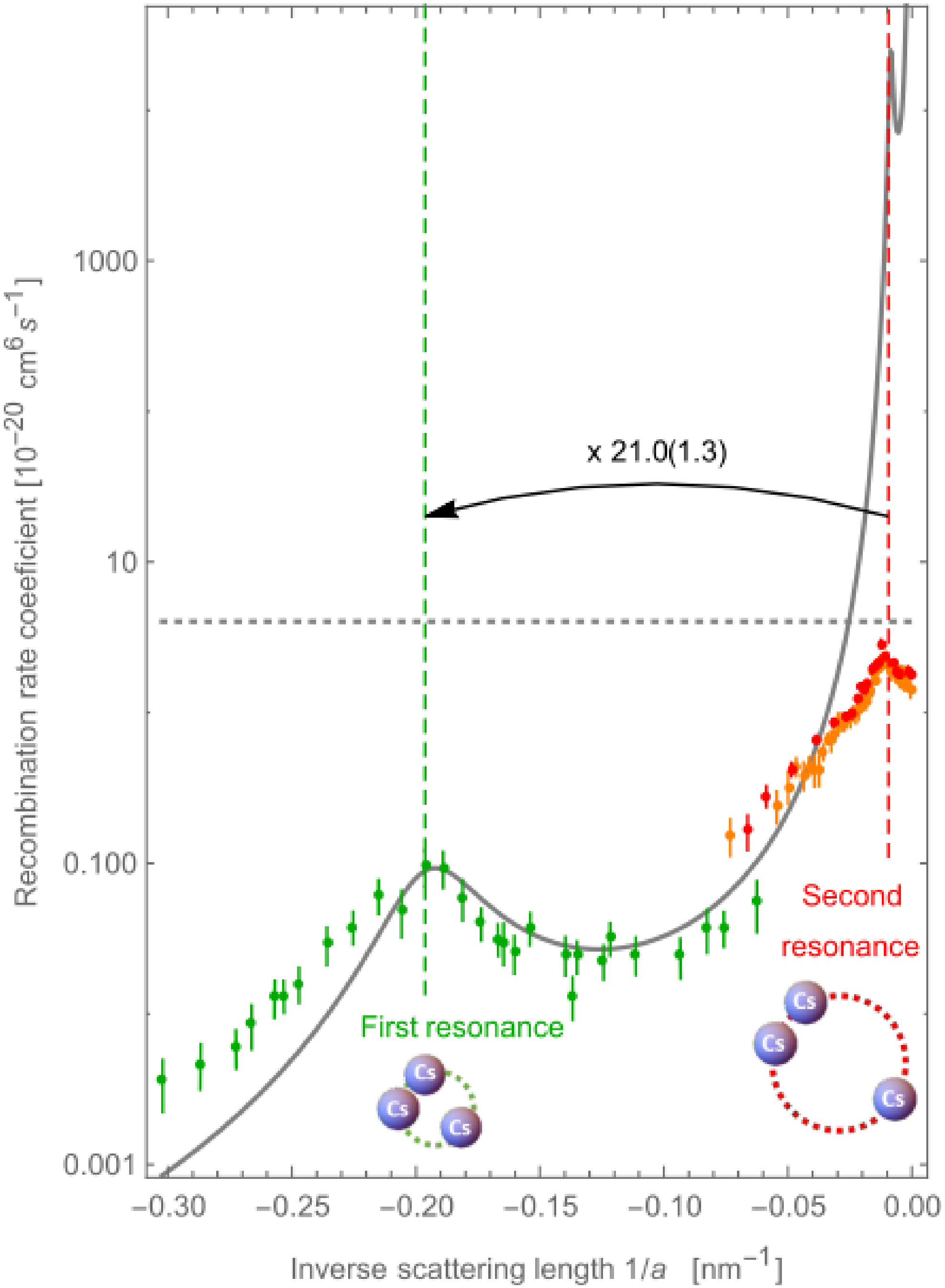
Feshbach resonance



Tuning scattering length **a** with magnetic field **B**







Experimental observations

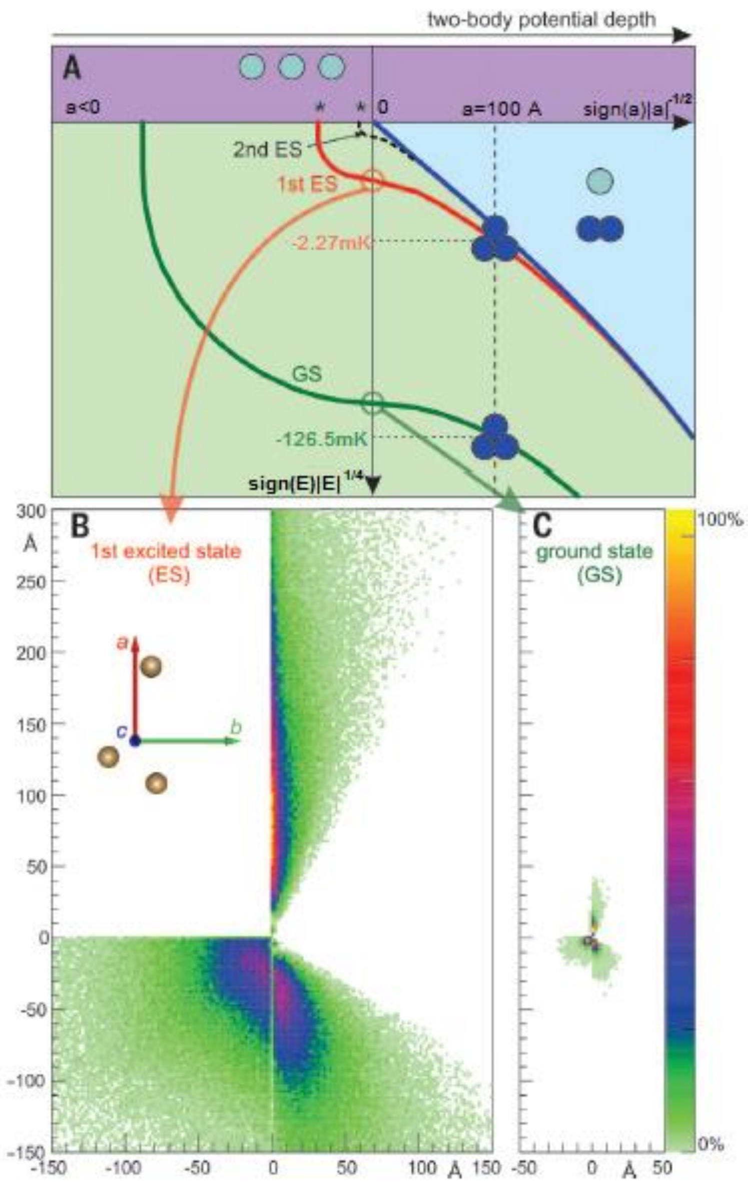
B. Huang, L. A. Sidorenkov, R. Grimm, and J. M. Hutson, "*Observation of the Second Triatomic Resonance in Efimov's Scenario*" Phys. Rev. Lett., **112** (2014) 190401.

R. Pires, J. Ulmanis, S. Häfner, M. Repp, A. Arias, E. D. Kuhnle, and M. Weidemüller, "*Observation of Efimov Resonances in a Mixture with Extreme Mass Imbalance*" Phys. Rev. Lett.,**112** (2014) 250404.

S.-K. Tung, K. Jim´enez-Garc´ıa, J. Johansen, C. V. Parker, and C. Chin, "*Geometric Scaling of Efimov States in a ^{6}Li- ^{133}Cs Mixture*" Phys.Rev. Lett., **113** (2014) 240402.

$^4He_3$

Three-Body Efimov States



Measurements:

$$|E^*_{1ES} - \varepsilon_d| = 0.98 \pm 0.2\text{mK}$$

Kunitski M. et al. // *Science* 348 (2015) 551.

Brühl R. et al. // Phys. Rev. Lett. **95** (2005), 06002.

Calculations:

$$|E^*_{1ES} - \varepsilon_d| = 0.972\text{mK}$$

$$\varepsilon_d = -1.3035\text{mK} \qquad l_{sc} = 100.23\,\overset{\circ}{\text{A}}$$

$$E_{GS} = -126.507\text{mK} \quad E^*_{1ES} = -2.276\text{mK}$$

Motovilov A.K., Sofianos S.A, K. E.A. // *Chem.Phys.Lett* **275** (1997) 168.

Motovilov A.K., Sandhas W., Sofianos S.A, K. E.A. // *Eur. Phys. J. D* **13** (2001) 33.

A Dependence of the binding energies of the ground (GS) and first excited (1ST) states of the He trimer on the scattering length calculated by scaling He-He potential.

B and **C** Structure of the excited and ground states of $^4He_3$

Observations in Nuclear Physics

All known multicomponent systems in nuclear physics related to Efimov physics involve two neutrons as two of the three particles:

- the interaction between two neutrons is resonant - the basic requirement
- they can form a spin singlet state – no centrifugal repulsion
- no electrical charge – no competition Coulomb repulsion *vs* Efimov attraction

The simplest case is the triton

the binding of the triton is consistent with the Efimov scenario.

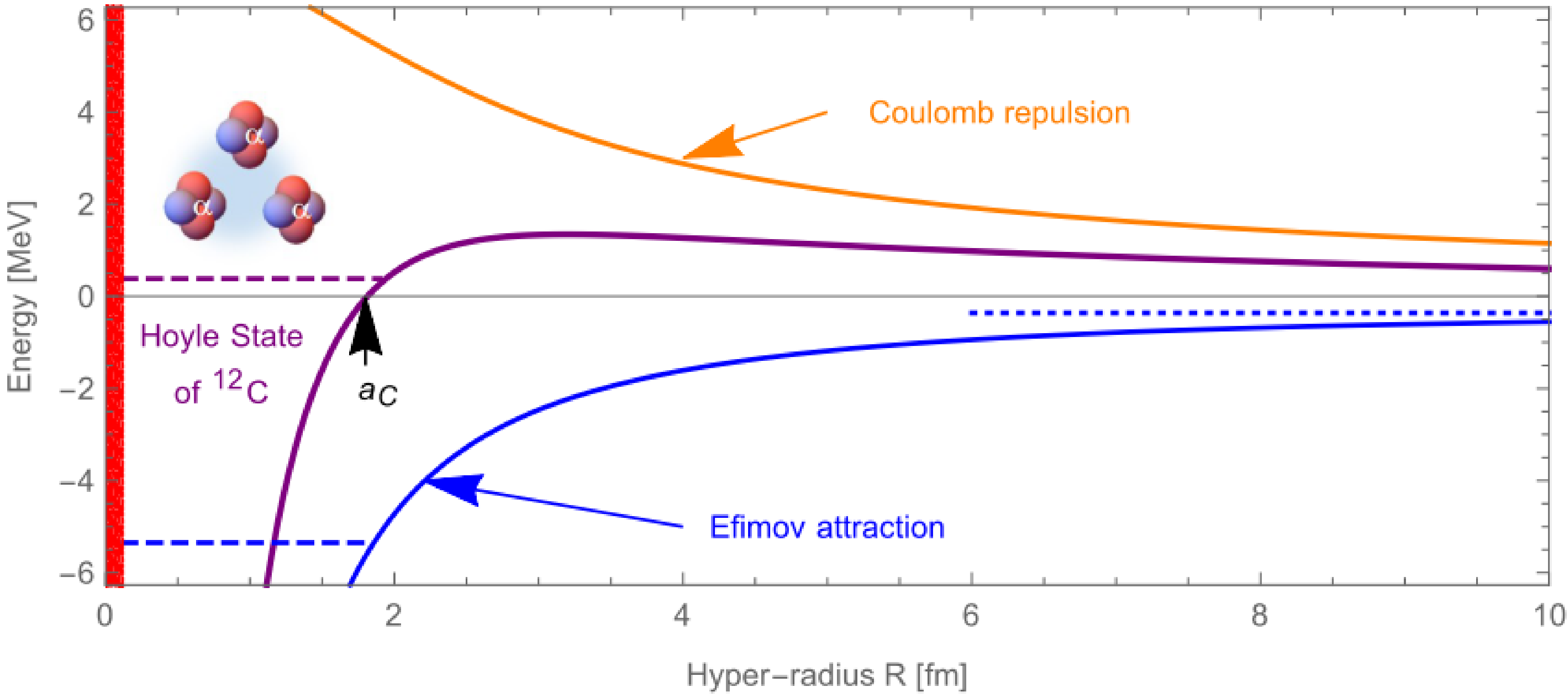
E.A.K. and A.K. Motovilov, *On the mechanism of formation of the Efimov states in the helium 4He trimer,* Phys. Atom. Nucl. **62**, No. 7 (1999), 1179–1192

A. Kievsky and M. Gattobigio, Universal nature and finite-range corrections in elastic atom-dimer scattering below the dimer breakup threshold, Phys.Rev. A**, 87** (2013) 052719

E.A. K. *Ultracold Scattering and Universal Correlations, Few-Body Systems* **55** (2014), 957–960

P. F. Bedaque, G. Rupak, H. W. Grießhammer, and H.-W. Hammer*, Low energy expansion in the three body system to all orders and the triton channel,* Nuclear Physics A, **714**, 589 – 610, 2003

R. W. Hackenburg, *Neutron-proton effective range parameters and zero-energy shape dependence,* Phys. Rev. C, **73** (2006) 044002



Schematic picture of the conjecture that the Hoyle state of ^{12}C is bound by the Efimov attraction. Here, the Hoyle state appears as a resonant state supported by the sum of two potentials as a function of the hyper-radius between the three alpha particles.

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A, **809** (2008) 171 – 188

H. Suno, Y. Suzuki, and P. Descouvemont, Phys. Rev. C, **91** (2015) 014004.

P. Naidon, Sh. Endo, Rep.Prog.Phys. 80 (5) (2017) 056001;

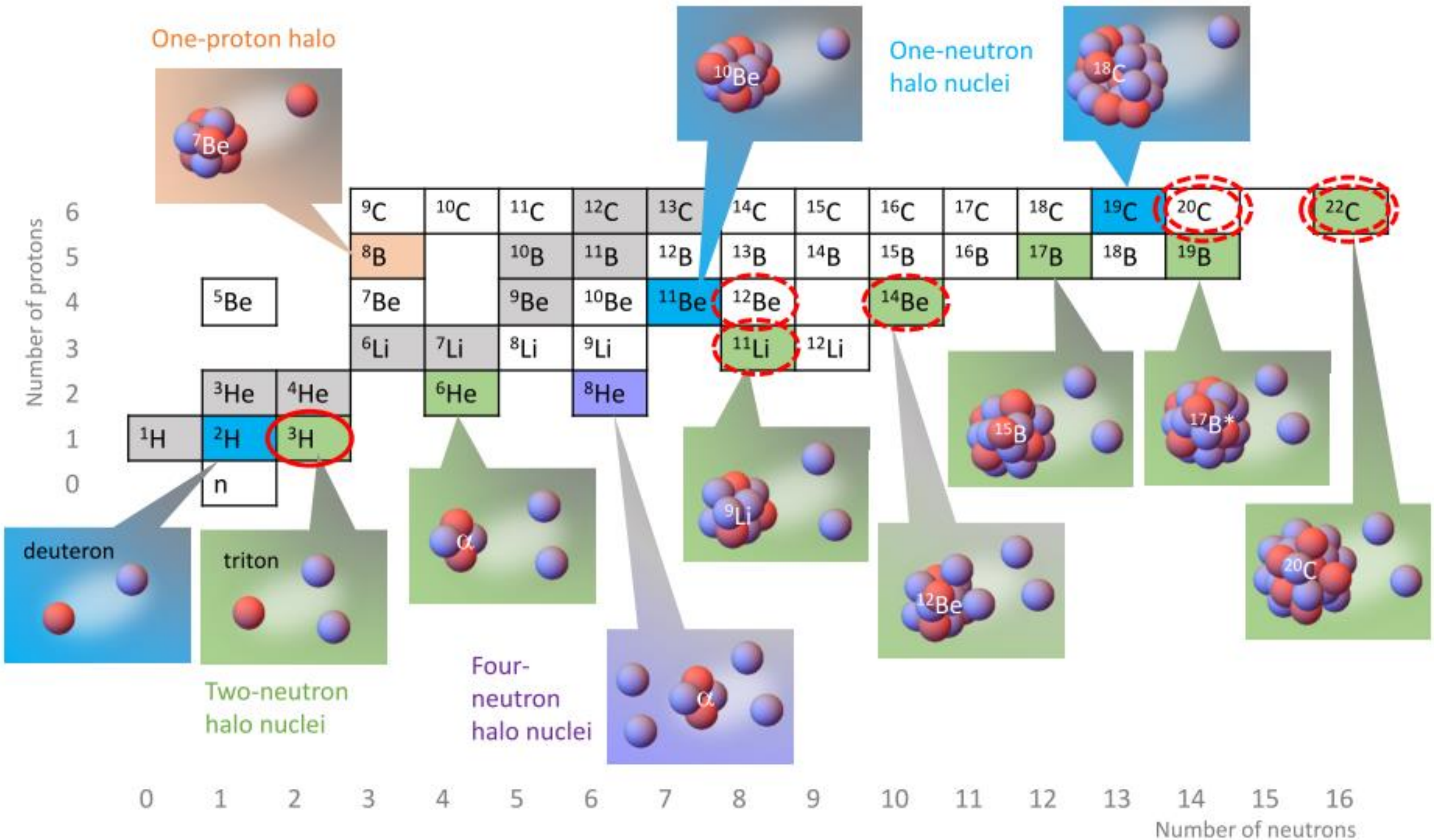


Table of nuclides, as a function of the number of protons and number of neutrons, showing stable elements (grey) and observed light halo nuclei: one-proton halo (orange), one-neutron halo (blue), two-neutron halo (green), and four-neutron halo (purple).

Summary

- The recent experimental observations with ultra-cold atoms and theoretical developments have opened a rich variety of systems related to Efimov physics.
- Nowdays, Efimov physics is studied in a much broader sense: N-body systems, condensed matter, mixed-dimensions.

Reviews:

P. Naidon, Sh. Endo, Reports on Progress in Physics 80 (5) (2017) 056001;

F. Ferlaino, R. Grimm, Physics, 3, 9 (2010);

H.-W. Hammer, L. Platter, Ann. Rev. Nucl. Part. Sc. 60 (2010) 207-236;

E.A. K., A.K. Motovilov, W. Sandhas, Phys.Part.Nucl. 40 (2009), 206-235;

E. Braaten, H.-W. Hammer, Physics Reports 428 (5-6) (2006) 259-392;

Thank you for your attention!