Electromagnetic Duality and Optical Helicity

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Outline

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Outline

Proton Spin

- Proton spin and gluon spin ?
  Leader and Lorce, Phys. Rept. 541 (2014) 163

- Electron spin and photon spin ?
  Bliokh, et al, Phys. Rept. 690 (2017) 1-70

- Helicity of photon

Electromagnetic duality

- It has been long known [O. Heaviside (1892), J. Larmor (1897)] that duality transformations

$$\mathbf{E}' = \mathbf{E}\cos\theta + \mathbf{B}\sin\theta, \tag{1a}$$
$$\mathbf{B}' = \mathbf{B}\cos\theta - \mathbf{E}\sin\theta, \tag{1b}$$

leave the Maxwell equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = 0, \tag{2a}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{2b}$$

invariant.

- Elements of $T^{\mu\nu}$, e.g.,

$$T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad T^{0i} = (\mathbf{E}\times\mathbf{B})^i \tag{3}$$

are also invariant under (1).

- However, Maxwell action itself

$$S_M = \int d^4x \; L_M = -\frac{1}{4}\int d^4x \; F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\int d^4x \; (\mathbf{E}^2 - \mathbf{B}^2), \quad (4)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

transforms under finite transformations (1) as

$$S'_M = \frac{1}{2}(\cos^2\theta - \sin^2\theta)\int d^4x \; (\mathbf{E}^2 - \mathbf{B}^2) \; + 2\sin\theta\cos\theta \int d^4x \; \mathbf{E}\cdot\mathbf{B}.$$

- Infinitesimal transformations

$$\delta\mathbf{E} = \delta\theta\mathbf{B}, \quad \delta\mathbf{B} = -\delta\theta\mathbf{E} \quad (5)$$

yield a surface term

$$S'_M = S_M + 2\delta\theta\int \; d^4x\mathbf{E}\cdot\mathbf{B} = -\frac{1}{4}\delta\theta\int d^4x \; \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \quad (6)$$

Calkin/Noether theorem

- M. G. Calkin, [M. G. Calkin, (1965)], was the first to investigate the conserved charge/generator of (1) from the Lagrangian point of view.
- The action of (5) descends on the potentials $\mathbf{A}, \phi$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}, \tag{7}$$

as

$$\delta\phi = -\delta\theta(\partial_t \lambda), \quad \delta\mathbf{A} = \delta\theta(\nabla\lambda) - \delta\theta(\nabla \times \mathbf{Z}). \tag{8}$$

where λ is a gauge parameter and $\mathbf{Z}$ is the polarization potential/Hertz vector.

- Calkin substituted (8) in (4) and obtained the associated charge , χ_C:

$$\chi_C = \int d^3\mathbf{x}\,\Big((\frac{\partial \mathbf{Z}}{\partial t})\cdot \mathbf{B} - \mathbf{Z}\cdot(\frac{\partial \mathbf{B}}{\partial t})\Big), \quad \partial_t \mathbf{Z} = \mathbf{A}^T. \tag{9}$$

- Physical meaning and properties of χ_C is not clear in the form (9).

- However, box quantization with p.b.c. , e.g.,

$$\mathbf{Z} = \left(\frac{4\pi\hbar}{V}\right)^{1/2} \sum_{k,s} i(2\omega_k^3)^{-1/2} \boldsymbol{\epsilon}_{k,s} (b_{k,s} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} - h.c.), \ \ldots$$

  yields

$$\chi_C \sim \hbar\Big(N(R) - N(L)\Big) \tag{10}$$

  where $N(R), N(L)$ is the total number of right and left circularly polarized photons, respectively.

- The conclusion is that the difference $N(R) - N(L)$ is the conserved charge associated with the duality symmetry.

Deser and Teitelboim/Hamiltonian formalism

- In [S. Deser and C. Teitelboim, (1975)], same problem was studied from a Hamiltonian point of view.
- Canonical variables $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{E} = \mathbf{E}^T$ transform under (5) as

$$\delta \mathbf{E} = \delta\theta \boldsymbol{\nabla} \times \mathbf{A}, \quad \delta \mathbf{A} = \delta\theta \boldsymbol{\nabla}^{-2} \boldsymbol{\nabla} \times \mathbf{E}, \tag{11}$$

where $\boldsymbol{\nabla}^{-2}$ is the inverse of the $3-$d Laplacian.

- They obtained the conserved charge χ_{DT} as,

$$\begin{aligned} \chi_{DT} &= \frac{1}{2} \int d^3\mathbf{x}\, (\mathbf{B} \cdot \boldsymbol{\nabla}^{-2} \boldsymbol{\nabla} \times \mathbf{B} + \mathbf{E} \cdot \boldsymbol{\nabla}^{-2} \boldsymbol{\nabla} \times \mathbf{E}), \\ &= \frac{1}{2} \int d^3\mathbf{x}\, (-\mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{A} + \mathbf{E} \cdot \boldsymbol{\nabla}^{-2} \boldsymbol{\nabla} \times \mathbf{E}), \end{aligned} \tag{12}$$

which is manifestly non-local.

- In the second line, they arrived a local expression by defining

$$\mathbf{A}(t,\mathbf{x}) = -\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}\times\mathbf{B} = \int d^3\mathbf{x}' \, \frac{\boldsymbol{\nabla}'\times\mathbf{B}(t,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{13}$$

- χ_{DT} (12) is explicitly gauge invariant. However, gauge independence of χ_C (9) is not so obvious.
- Q: Is it possible to find a local, gauge invariant and dual symmetric charge with an explicit clear meaning?
- They stated that the other non-local expression can also be made local by introducing another potential.
- Then, relation to Calkin's work should be:

$$\boldsymbol{\nabla}\times\mathbf{Z} = -\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}\times\mathbf{E}. \tag{14}$$

Dual vector potential

- Helicity of the electromagnetic field, without any relation to duality symmetry, is also studied by [G. N. Afanasiev and Yu. P. Stepanovsky, (1996)].
- In their formulation, A^μ co-exists with a second, dual vector potential C^μ:

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\times\mathbf{C} = -\partial_t\mathbf{A} - \boldsymbol{\nabla}A^0, \\ \mathbf{B} &= -\boldsymbol{\nabla}C^0 - \partial_t\mathbf{C} = \boldsymbol{\nabla}\times\mathbf{A}. \end{aligned}$$

- They have a relativistic current in terms of both of them

$$j^\mu = \tilde{F}^{\mu\nu}A_\nu - F^{\mu\nu}C_\nu, \tag{15}$$

where

$$\tilde{F}^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \star F^{\mu\nu} \tag{16}$$

is the dual field strength.

- Provided Maxwell equations (2) hold, (15) is conserved, $\partial_\mu j^\mu = 0$.
- Separately, divergence of each part is calculated to be

$$\partial_\mu(\tilde{F}^{\mu\nu} A_\nu) = \partial_\mu(F^{\mu\nu} C_\nu) = -2\mathbf{E}\cdot\mathbf{B}. \tag{17}$$

- Splitting temporal and spatial components, we get

$$j^0 = \mathbf{A}\cdot\mathbf{B} - \mathbf{C}\cdot\mathbf{E}, \tag{18a}$$

$$\mathbf{j} = A^0\mathbf{B} + \mathbf{E}\times\mathbf{A} - C^0\mathbf{E} + \mathbf{B}\times\mathbf{C}. \tag{18b}$$

- Integral of j^0 is called the optical helicity:

$$\chi_{AS} = \int d^3\mathbf{x}\, j^0 = \int d^3\mathbf{x}\, \mathbf{A}\cdot\mathbf{B} - \mathbf{C}\cdot\mathbf{E} \tag{19a}$$

$$= \int d^3\mathbf{x}\, \mathbf{A}\cdot\nabla\times\mathbf{A} + \mathbf{C}\cdot\nabla\times\mathbf{C}. \tag{19b}$$

- χ_{AS} is gauge invariant provided the fields vanish on the spatial boundary.

- χ_{AS} is in the double Chern-Simons form.

- Physical meaning becomes clear in momentum space (in Coulomb gauge)

$$\chi_{AS} \sim \int d^3\mathbf{k}\ (|f_R|^2 - |f_L|^2). \tag{20}$$

where $|f_{R,L}|^2$ are the number densities for the right and left circularly polarized photons, respectively.

- Relation with the previous expressions, i.e, χ_C and χ_{AS}:

$$-\nabla \times \mathbf{Z} = -\nabla^{-2}\nabla \times \mathbf{E} = \mathbf{C}^T. \tag{21}$$

- Choosing the Coulomb gauge

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{C} = 0, \quad A^0 = C^0 = 0$$

also simplifies the spatial term which is called the helicity flow,

$$\mathbf{j} = \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}. \tag{22}$$

- In momentum space, it becomes

$$\mathbf{s} = \int d^3\mathbf{x}\, \mathbf{j} \sim \int d^3\mathbf{k}\, (|f_R|^2 - |f_L|^2)\frac{\mathbf{k}}{|\mathbf{k}|}. \tag{23}$$

- Helicity χ_{AS} is divided into magnetic part χ_m and electric part χ_e as

$$\chi = \chi_m + \chi_e, \tag{24a}$$

$$\chi_m = \frac{1}{2}\int d^3\mathbf{x}\, \mathbf{A} \cdot \nabla \times \mathbf{A}, \quad \chi_e = \frac{1}{2}\int d^3\mathbf{x}\, \mathbf{C} \cdot \nabla \times \mathbf{C}, \tag{24b}$$

- Separately, they obey

$$\frac{d\chi_m}{dt} = -\int d^3\mathbf{x}\ \mathbf{E}\cdot\mathbf{B}, \quad \frac{d\chi_e}{dt} = \int d^3\mathbf{x}\ \mathbf{E}\cdot\mathbf{B}. \tag{25}$$

- Unlike χ_m and χ_e, χ_{AS} is the constant of the free Maxwell theory.
- Duality transformations (1) descend to the transverse vector potentials as

$$\mathbf{A}' = \mathbf{A}\cos\theta + \mathbf{C}\sin\theta, \tag{26a}$$

$$\mathbf{C}' = \mathbf{C}\cos\theta - \mathbf{A}\sin\theta. \tag{26b}$$

- One can observe that neither χ_m nor χ_e are invariant under (26).
- Only their combination χ_{AS} is invariant.

- In plasma physics χ_m is called the magnetic helicity, a measure of twisting of magnetic field lines in plasma physics [L. Woltjer (1958)]....
- Knot theory: χ_m and χ_e are related to the linking numbers of the magnetic and electric field lines, respectively [A. F. Rañada, (1991). M. Arrayás, D. Bouwmeester and J. L. Trueba (2017)].
- R. H. S. of (25) $\mathbf{E} \cdot \mathbf{B} = -1/8\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is the Chern-Pontryagin density [S. S. Chern (1979)], a metric independent quantity.
- Therefore, in relation with the chiral anomaly of massless fermions in QGP [C. Manuel and J. M. Torres-Rincon (2015)].
- Fluid mechanics' analogue is the vortex helicity [...R. Jackiw, (2000)]....

- We worked out this problem from a symplectic point of view, as duality transformations being canonical transformations (1608.01131).

- Later, we tried to construct various Lagrangians based on the photon wave function (1608.08573).

Photon wave function

- The history of the photon wave function goes back to the work of E. Majorana [E. Majorana (1928-1932)].
- In this work, E. Majorana wrote a Dirac-like equation for the photon [R. Mignani, E. Racami and M. Baldo (1974)].
- It is also advocated by I. Bialynicki-Birula [I. Bailynicki-Birula (1996)...] as a link between classical electromagnetism and quantum electrodynamics.
- As in E. Majorana's work, the key element is the Riemann-Silberstein vector [L. Silberstein (1907), see also H. Bateman (1915)]:

$$\mathbf{F}_{\pm} = \frac{1}{\sqrt{2}}(\mathbf{E} \pm i\mathbf{B}). \quad (27)$$

- I. Bialynicki-Birula's derivation is based on taking the square root of the Klein-Gordon equation.

- Bialynicki-Birula used the $SO(3)$ generators for spin-1 particles

$$(S_i)_{ab} = -i\varepsilon_{iab}, \quad i, a, b = 1, 2, 3. \tag{28}$$

- Complicated anti-commutation relation of $S-$matrices

$$\{S_i, S_j\}_{ab} = 2\delta_{ij}\delta_{ab} - \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi} \tag{29}$$

dictates to work out divergenceless $\mathbf{F}_\pm$, i.e.,

$$\boldsymbol{\nabla} \cdot \mathbf{F}_\pm = 0. \tag{30}$$

- Then, the first order equations become

$$i\partial_t \mathbf{F}_\pm = \mp i(\mathbf{S} \cdot \boldsymbol{\nabla})\mathbf{F}_\pm = \pm\boldsymbol{\nabla} \times \mathbf{F}_\pm, \tag{31}$$

where $\pm$ stands for the helicity of the photon.

- (30) and (31) provide a compact form of vacuum Maxwell eqns (2).

Dirac/Weyl Lagrangian

By using (31), we would like to obtain a Dirac type Lagrangian with both helicities.

We begin with

$$\mathcal{F} = \begin{pmatrix} \mathbf{F}_+ \\ \mathbf{F}_- \end{pmatrix} \quad \Sigma^\mu = \begin{pmatrix} 0 & \overline{S}^\mu \\ S^\mu & 0 \end{pmatrix}. \tag{32}$$

$\mu = 0, \ldots, 3$ and $S^\mu = (1, \mathbf{S})$ and $\overline{S}^\mu = (1, -\mathbf{S})$ where $\left(S_j\right)_{ab}$ are (28).

Then, (31) can be written in a similar form to the Dirac equation,

$$\Sigma^\mu \, \partial_\mu \mathcal{F} = 0 \tag{33a}$$

$$\boldsymbol{\nabla} \cdot \mathcal{F} = 0. \tag{33b}$$

supplemented with the divergence constraint, Naturally, we propose a Dirac type Lagrangian

$$\mathcal{L}_{\mathcal{F}} = \overline{\mathcal{F}}\,(\Sigma^\mu \partial_\mu)\,\mathcal{F} = \left(\mathbf{F}_-^\dagger\,\overline{S}^\mu \partial_\mu \mathbf{F}_- \;+\; \mathbf{F}_+^\dagger\,S^\mu \partial_\mu \mathbf{F}_+\right), \quad \overline{F} = \mathcal{F}^\dagger \Sigma^0. \quad (34)$$

Treating $\mathcal{F}$ and $\overline{\mathcal{F}}$ as independent quantities in (34), variational calculus yields (33a).

(34) is invariant under the duality transformations (1):

$$\mathcal{F} \to e^{-i\theta\rho_3}\mathcal{F}, \quad \rho_3 = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_3 \end{pmatrix} \quad (35)$$

Noether theorem yields the associated conserved current,

$$k^\mu = \overline{\mathcal{F}}\,\Sigma^\mu \rho_3\,\mathcal{F} = \mathbf{F}_+^\dagger S^\mu \mathbf{F}_+ - \mathbf{F}_-^\dagger \overline{S}^\mu \mathbf{F}_-\,, \qquad \partial_\mu k^\mu = 0, \quad (36)$$

which is similar to the current of chiral fermions in form.

However, this current is identically vanishing:

$$k^\mu \equiv 0 \quad \Rightarrow \quad \chi_F = \int d^3\mathbf{r}\, k^0 = \int d^3\mathbf{r}\big(\mathbf{F}_+^\dagger \mathbf{F}_+ - \mathbf{F}_-^\dagger \mathbf{F}_-\big) = 0. \tag{37}$$

Therefore, we conclude that (34) is unsuitable to derive optical helicity χ.

Moreover, if we substitute $\mathbf{F}_\pm = \frac{1}{\sqrt{2}}\big(\mathbf{E} \pm i\mathbf{B}\big)$ into (34), we get

$$\mathcal{L}_\mathcal{F} = \mathbf{E}\cdot\big(\partial_t\mathbf{E} - \boldsymbol{\nabla}\times\mathbf{B}\big) + \mathbf{B}\cdot\big(\partial_t\mathbf{B} + \boldsymbol{\nabla}\times\mathbf{E}\big) \tag{38a}$$

$$= \partial_t\big(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\big) + \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}), \tag{38b}$$

observing that $\mathcal{L}_\mathcal{F}$ is different from (4), rather it is the divergence of the $\mathcal{T}^{\mu 0}$ associated with it.

Klein-Gordon Lagrangian

We observe that, (31) can also be satisfied if we replace $\mathbf{B}, \mathbf{E}$ with $\mathbf{A}, \mathbf{C}$ and define a new RS vector:

$$\mathbf{V}_{\pm} = \frac{1}{\sqrt{2}}(\mathbf{A} \pm i\,\mathbf{C}). \tag{39}$$

Then we have the following set of wave equations:

$$i\partial_t \mathbf{V}_{\pm} = \mp i(\mathbf{S} \cdot \nabla)\mathbf{V}_{\pm}, \tag{40a}$$

$$\nabla \cdot \mathbf{V}_{\pm} = 0. \tag{40b}$$

Subsidiary condition (40b) fixes the gauge choice as Coulomb gauge,

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{C} = 0, \quad A^0 = C^0 = 0. \tag{41}$$

On the other hand, (40a) turn out to be

$$\nabla \times \mathbf{A} = -\partial_t \mathbf{C}\ (= \mathbf{B}), \qquad \nabla \times \mathbf{C} = \partial_t \mathbf{A}\ (= -\mathbf{E}), \tag{42}$$

Iterating (40a), we obtain two massless Klein-Gordon equations,

$$\partial_\mu \partial^\mu \mathbf{V}_\pm \equiv \left[\partial_t^2 - \nabla^2\right] \mathbf{V}_\pm = 0, \tag{43}$$

which can be naturally derived from the following K-G Lagrangian

$$L_V = \frac{1}{2}(\partial_\mu \mathbf{V}_-) \cdot (\partial^\mu \mathbf{V}_+), \tag{44}$$

by considering $\mathbf{V}_+$ and $\mathbf{V}_-$ as independent variables.

(43) are actually Maxwell equations in the Coulomb gauge:

$$\partial_\mu \partial^\mu \mathbf{A} = \partial_\mu \partial^\mu \mathbf{C} = 0.$$

Action of duality transformations (26) i.e.,

$$\mathbf{V}_\pm \to \mathbf{V}_\pm e^{\mp i\theta}. \tag{45}$$

leave (44) invariant.

With the infinitesimal version of (45), $\delta \mathbf{V}_\pm = \mp i\theta \mathbf{V}_\pm,$ we derive the Noether current

$$j^\mu = \frac{1}{2}\Big(\partial^\mu \mathbf{V}_+ \cdot \delta \mathbf{V}_- + \partial^\mu \mathbf{V}_- \cdot \delta \mathbf{V}_+\Big) = \frac{1}{2}\Big((\partial^\mu \mathbf{A}) \cdot \mathbf{C} - (\partial^\mu \mathbf{C}) \cdot \mathbf{A}\Big), \quad (46)$$

whose conservation, $\partial_\mu j^\mu = 0$, can also be checked directly using (43).

The associated conserved charge is the space integral of the zeroth component,

$$\chi = \int d^3\mathbf{r}\ \frac{1}{2}\Big(\partial_t \mathbf{A} \cdot \mathbf{C} - \partial_t \mathbf{C} \cdot \mathbf{A}\Big) = \int d^3\mathbf{r}\ \frac{1}{2}\Big(\mathbf{B} \cdot \mathbf{A} - \mathbf{E} \cdot \mathbf{C}\Big), \qquad (47)$$

where we recognize the double Chern-Simons expression of helicity.

Conversely, the charge (47) generates the duality action (45).

The constraint $\mathbf{V}_- = \mathbf{V}_+^*$ does not now imply the vanishing of (47).

χ (47) matches with (10, 20).

Components of the symmetric energy-momentum tensor $T^{\mu\nu}$ are,

$$T^{00} = \frac{1}{4}\big(\partial_t \mathbf{A}\cdot\partial_t \mathbf{A} + \partial_t \mathbf{C}\cdot\partial_t \mathbf{C} + \partial_i \mathbf{A}\cdot\partial_i \mathbf{A} + \partial_i \mathbf{C}\cdot\partial_i \mathbf{C}\big), \tag{48a}$$

$$T^{0i} = \frac{1}{2}(\partial_t \mathbf{A}\cdot\partial^i \mathbf{A} + \partial_t \mathbf{C}\cdot\partial^i \mathbf{C}), \tag{48b}$$

Its conservation, $\partial_\nu T^{\mu\nu} = 0$, can be checked also easily.

T^{00} and T^{0i} are, up to surface terms, the usual expressions of the energy and momentum densities, respectively.

We note also that the helicity flow is [up to a surface term] the spin angular momentum density,

$$\mathbf{j} = \mathbf{s} = \frac{1}{2}(\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C}). \tag{49}$$

However, $\mathbf{s}$ does not satisfy the $SO(3)$ algebra but it is a measurable quantity in optics [S. J. V Enk and G. Nienhuis, (1994)].

We realized that (44) is equivalent to the dual symmetric Lagrangian in [R. P. Cameron, S. M. Barnett (2012)...]:

$$\underbrace{\frac{1}{2}(\partial_\mu \mathbf{V}_-)\cdot(\partial^\mu \mathbf{V}_+)}_{our\ \ L_V} = \underbrace{-\frac{1}{8}\left[F_{\mu\nu}F^{\mu\nu} + \star F_{\mu\nu} \star F^{\mu\nu}\right]}_{Barnett\ \ et\ \ al-Bliokh\ \ et\ \ al} \underbrace{-\frac{1}{4}\partial_i\left(A_j\partial_j A_i + C_j\partial_j C_i\right)}_{surface\ \ term}.$$

Warning: As stated by in Barnett *et al.*, one should not attach any physical interpretation to $\mathbf{A}$, $\mathbf{C}$ such that (42) *a priori*.

Likewise, in (44), we treat $\mathbf{V}_+$ and $\mathbf{V}_-$ as independent and derive (43) before inserting the constraint $\mathbf{V}_- = \mathbf{V}_+^*$.

Dirac Lagrangian with potentials

$\mathbf{V}_{\pm}$ (39) could again be unified into a 6-component system by putting

$$\mathcal{V} = \begin{pmatrix} \mathbf{V}_{+} \\ \mathbf{V}_{-}. \end{pmatrix}$$

Then, (40a) and (40b) becomes,

$$\Sigma^{\mu}\,\partial_{\mu}\mathcal{V} = 0, \tag{50a}$$
$$\boldsymbol{\nabla}\cdot\mathcal{V} = 0, \tag{50b}$$

as in (33) : we get Dirac / Weyl type theory.

Similar to (34), we propose,

$$\mathcal{L}_{\mathcal{V}} = \overline{\mathcal{V}}(\Sigma^{\mu}\partial_{\mu})\mathcal{V} = \left(\mathbf{V}_{-}^{\dagger}\overline{S}^{\mu}\partial_{\mu}\mathbf{V}_{-} + \mathbf{V}_{+}^{\dagger}S^{\mu}\partial_{\mu}\mathbf{V}_{+}\right), \qquad \overline{\mathbf{V}} = \mathbf{V}^{\dagger}\Sigma^{0}. \tag{51}$$

This Lagrangian is invariant w.r.t. duality transformations (45), and yields a Noether current similar to (36),

$$\ell^\mu = \overline{\mathcal{V}}\,\Sigma^\mu \rho_3\,\mathcal{V} = \mathbf{V}_+^\dagger S^\mu \mathbf{V}_+ - \mathbf{V}_-^\dagger \bar{S}^\mu \mathbf{V}_- \,. \tag{52}$$

However the current vanishes again due to $\mathbf{V}_+^* = \mathbf{V}_-$,

$$\ell^\mu \equiv 0 \quad \Rightarrow \quad \chi_V = \int \ell^0 d^3\mathbf{r} = \int \left(\mathbf{V}_+^\dagger \mathbf{V}_+ - \mathbf{V}_-^\dagger \mathbf{V}_-\right) d^3\mathbf{r} = 0. \tag{53}$$

We conclude that the Dirac-type approach yields, once again, trivial current and charge.

K-G Lagrangian with fields

Since the original RS vector $\mathbf{F}_{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{E} \pm i\mathbf{B}\right)$ satisfies the K-G equation

$$\partial_\mu \partial^\mu \mathbf{F}_{\pm} \equiv \left[\partial_t^2 - \boldsymbol{\nabla}^2\right]\mathbf{F}_{\pm} = 0, \tag{54}$$

We would like to investigate the corresponding K-G Lagrangian

$$L_F = \frac{1}{2}(\partial_\mu \mathbf{F}_-) \cdot (\partial^\mu \mathbf{F}_+). \tag{55}$$

This Lagrangian is plainly symmetric under duality (1) with associated Noether current

$$z_\mu = \frac{1}{2}\Big((\partial_\mu \mathbf{E}) \cdot \mathbf{B} - (\partial_\mu \mathbf{B}) \cdot \mathbf{E}\Big). \tag{56}$$

Conserved charge, the integral of z_0 becomes

$$\begin{aligned} Z &= \int d^3\mathbf{r}\,\frac{1}{2}\Big((\partial_t\mathbf{E})\cdot\mathbf{B}-(\partial_t\mathbf{B})\cdot\mathbf{E}\Big), \\ &= \int d^3\mathbf{r}\,\frac{1}{2}\Big(\mathbf{B}\cdot\nabla\times\mathbf{B}+\mathbf{E}\cdot\nabla\times\mathbf{E}\Big). \end{aligned} \tag{57}$$

This expression, is again in the form of double Chern-Simons form.

In optics, this is known as the Lipkin's Z^{000}-zilch [D. M. Lipkin, (1964)] or optical chirality.

Its space part,

$$\mathbf{z}=\frac{1}{2}\int d^3\mathbf{r}\,\Big(\mathbf{E}\times(\nabla\times\mathbf{B})-\mathbf{B}\times(\nabla\times\mathbf{E})\Big), \tag{58}$$

is in turn Lipkin's $Z^{0i0}=Z^{00i}$, identified as the optical chirality flow.

Conclusions

- We used the photon wave function as a trick to rewrite electromagnetism in a Dirac/Weyl resp. Klein-Gordon-type form, allowing us to use field theoretical tools.
- Our trick of replacing the e.m. fields by the respective potentials works because all components satisfy, in the Coulomb gauge, the wave equation, allowing for the "square root trick"
- In our framework, zilch seems to be associated with duality symmetry.
- Our findings fit perfectly into the hierarchy pattern [M. G. Calkin, (1965). D. J. Candlin, (1965)...]