Introduction to Cosmology

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February 1, 2021

The Friedmann -Robertson-Walker (FRW) metrics reads

$$ds^2 = dt^2 - a^2(t)d\chi^2 - a^2(t)d\Omega^2$$

What it is possible to say about distances and velocities in an expanding Universe?
Tamara M. Davis, Charles H. Lineweaver, Expanding Confusion: common misconceptions of cosmological horizons and the superluminal expansion of the Universe, astro-ph/0310808.
Examples of misconceptions or easily misinterpreted statements in the literature:

1. Feynman, R. P. 1995, Feynman Lectures on Gravitation (1962/63), (Reading, Mass.: AddisonWesley) p. 181.
2. Rindler, W. 1956, MNRAS, 6, 662-667, Visual Horizons in World-Models.
3. McVittie, G. C. 1974, QJRAS, 15, 246-263.
4. Weinberg, S. 1977, The First Three Minutes, (New York: Bantum Books), p. 27.

...

25. McVittie, G. C. 1974, QJRAS, 15(1), 246–263.

Setting an observer in the origine of the coordinate system we get the proper distance between the observer and a distant point with the radial comoving coordinate χ for the $t = const$ hypersurface in the form $l = a\chi$.
Similary, we can define a velocity of the Hubble flow as

$$v = dl/dt = \dot{a}\chi = (\dot{a}/a)a\chi = Hl$$

so that for the proper distance and velocity "now" the Hubble law is en exact law for the Friedmann metric. The velocity v can not be measured directly, so it can be considered as an unphysical entity. In general, it is not bounded from above.
To choose correct and physically reasonable definition of $V^{(i)}$, we use the tetrad formalism. Then, motion of a massive particle is subluminal, so the absolute value of the vector $V^{(i)}$ is less than 1. If a particle has four-velocities u^{μ}, and the observer reference frame is described by the tetrad field $h_{(i)\mu}$, then there is a standard definition of three-velocity

$$V^{(i)} = -\frac{u^{\mu} h_{(i)\mu}}{h^{\mu(0)} u_{\mu}}.$$

In cosmology a velocity with respect to a comoving observer is usually called a peculiar velocity. Since the corresponding tetrad is

$$h_{(i)\mu} = diag(-1, a, a\chi, a\chi \sin\theta),$$

for the 3-velocity of a particle with 4-velocity u^{μ} we have for the radial component of a peculiar velocity

$$V_r = au^{\chi}/u^t = ad\chi/dt.$$

The consent of a peculiar velocity is a local one, however, the radial component allows for a nice non-local interpretation. Namely, the rate of change of a proper distance between the coordinate origin and a distant point $l = a\chi$ is

$$\frac{dl}{dt} = \frac{d(a\chi)}{dt} = \chi\frac{da}{dt} + a\frac{d\chi}{dt} = v_H + V_r$$

where $v_H = \chi\dot{a} = (\chi a)\dot{a}/a = lH$ is the velocity of the Hubble flow. The left hand side of this equation can be considered as a reasonable definition of a velocity of a *distant* object (more precisely, its radial part) which is an intrinsically non-local entity.

The above equation means that the overall change of a proper distance to a distant point is naturally decomposed into a sum of the velocity of the cosmological flow and the radial part of a peculiar velocity. Note that summation rule is of the Galileo type independently of velocity values. We see that in spite of non-local nature of this equation as a whole as well as the Hubble flow velocity v_H, the second term in the right hand side has a local interpretation.
The same statement can be done for the velocity of light which always equals to c locally, but for a light in a distant point we have $v_l = v_H + c$ or $v_l = v_H - c$ depending on the direction of light. It can be shown by the same chain of equalities noting that the coordinate velocity of light in the FRW metric $d\chi/dt = c/a$.
Let us go closer to observational perspective. We can not directly measure the coordinate χ. What we do measure is the redshift. If a light have been emitted when the scale factor was a_{em} and have been observed at a_0, we have $1 + z = a_0/a_{em}$.
The simplest way to obtain the redshift in the FRW metric is to use the conformal time, η. Defining $\eta(t) = \int dt/a$ we reduce the

metric to the form

$$ds^2 = a^2(d\chi^2 - d\eta^2).$$

Assuming $\chi_{obs} = 0$ we obtain $\chi_{em} = \eta_{obs} - \eta_{em}$. Here, subscripts "obs" and "em" refer to an observer and emitter, respectively. If a signal was emitted during a period $\Delta\eta$, then it is observed during the same interval of conformal time. However, physical time intervals at two moments are different: $\Delta t_{em} = a(t_{em})\Delta\eta$ and $\Delta t_{obs} = a(t_{obs})\Delta\eta$. Thus, $\Delta t_{em}/\Delta t_{obs} = a(t_{em})/a(t_{obs})$, where we neglected the change of the scale factor during the processes of emission and observation.

This method uses particular properties of the FRW metric and can not be generalized. The most general method one can often find in textbooks uses null geodesics.

Let us then consider two pulses of an electromagnetic wave. They are emitted in two moments of time t_{em1}, t_{em2} and observed, correspondingly, at t_{obs1} and t_{obs2}. Both are emitted by a galaxy at χ_{em}. The observer is situated at χ_{obs}.

If an emitter and an observer follow the geodesic lines, their values of χ remain fixed. World lines of the two photons are:

$$\chi_{em} - \chi_{obs} = c \int_{t_{em1}}^{t_{obs1}} a^{-1} dt,$$

$$\chi_{em} - \chi_{obs} = c \int_{t_{em2}}^{t_{obs2}} a^{-1} dt.$$

Then:

$$\int_{t_{em1}}^{t_{obs1}} a^{-1} dt - \int_{t_{em2}}^{t_{obs2}} a^{-1} dt = 0.$$

This equation can be rewritten in the equivalent form

$$\int_{t_{obs1}}^{t_{obs2}} a^{-1} dt - \int_{t_{em1}}^{t_{em2}} a^{-1} dt = 0.$$

Neglecting changes in a during the interval of emission of the two pulses, and during the interval of their observation (i.e. putting $a(t_{em1}) = a(t_{em2})$ and $a(t_{obs1}) = a(t_{obs2})$), we obtain

$$\frac{t_{obs2} - t_{obs1}}{a(t_{obs})} - \frac{t_{em2} - t_{em1}}{a(t_{em})} = 0.$$

At emission $\lambda_{em} = c(t_{em2} - t_{em1})$. At observation $\lambda_{obs} = c(t_{obs2} - t_{obs1})$. Therefore,

$$\lambda_{em}/a(t_{em}) = \lambda_{obs}/a(t_{obs}).$$

Finally, we come to the usual expression for the cosmological redshift.

Now let us derive the equation for the cosmological redshift using velocities. Here we will use the fact that two light pulses, being at slightly different distances from an observer, have different velocities with respect to this observer since the light approaches the observer with the velocity $v_l = c - v_H$. Therefore, let us consider two pulses of a light wave directed toward an observer, separated by a spatial distance $\Delta r = \lambda$. This radial difference results in the velocity difference with respect to the observer $\Delta v = H\Delta r$. This means that the velocity of the pulse which is closer to the observer is less than the velocity of the pulse which is behind it. Due to this difference, the distance between these pulses changes and we can construct a differential equation

$$d\Delta r/dt = H\Delta r.$$

After obvious calculations using the definition of the Hubble parameter $H = \dot{a}/a$ we obtain that the ratio of wavelengths at t_{em} and the moment of observation, t_{obs}, is equal to the ratio of scale factors at these moments: $\lambda_{em}/\lambda_{obs} = a_{em}/a_{obs}$. This is the standard result for the cosmological redshift.
The advantage of this method is that it can be applied to inhomogeneous metrics also, if we work in synchronous coordinate system. In the paper
A. Toporensky, O. Zaslavskii, S. Popov, Unified approach to redshift in cosmological /black hole spacetimes and synchronous frame, Eur. J. Phys. 39, 015601 (2018)
this method have been used to derive a redshift for a spherically symmetric black hole in a "cosmological" matter, without any reference to the gravitational time delay. This means that it is not necessary to consider the cosmological redshift as a special class of redshifts, but as the gravitational redshift calculated in a synchronous frame.

Now we express distances and velocities in terms of z instead of χ. We see the light now with the redshift z if is have been emitted at

$$\chi = \int \frac{cdt}{a} = \frac{c}{a_0} \int \frac{dz}{H(z)}$$

If we consider the dynamics as $a \sim t^{1/\alpha}$, so that $\alpha = 0$ corresponds to de Sitter Universe, $\alpha = 3/2$ is the dust Universe and $\alpha = 2$ is the radiation Universe, then

$$\chi = \frac{c}{a_0 H_0} \frac{1}{1-\alpha} [(1+z)^{1-\alpha} - 1].$$

So that, the proper distance to an object observed with the redshift z is now

$$l_{now} = \frac{c}{(1-\alpha) H_0} [(1+z)^{1-\alpha} - 1]$$

Correspondingly, at the time of emission the proper distance was equal to

$$l_{em} = l_{now}/(1+z)$$

For the velocity of an object within the Hubble flow we have now

$$v_{now} = \frac{c}{1-\alpha}[(1+z)^{1-\alpha} - 1]$$

and at the time of emission

$$v_{em} = \frac{c}{1-\alpha}[1 - (1+z)^{\alpha-1}]$$

If an emitter has non-zero radial peculiar velocity v_r, then the combined redshift is

$$1 + z = (1 + z_c)(1 + z_D)$$

where z_c is the cosmological redshift, and z_D is the standard Doppler shift $(1 + z_D) = \sqrt{\frac{1+v/c}{1-v/c}}$. As z_c and z_D depend upon velocities in a different manners, the redshift of an object with constant proper distance to it is, in general, nonzero.
Moreover, there is no *general* formula for cosmological redshift as a function of the recession velocity, since the redshift depends upon the whole dynamical history of the Universe between the points of emission and observation.

Apart from these definitions of distance and velocity, entering in the Hubble law, there are other definitions. From an empirical point of view, it is useful to define an angular d_A an photometric d_{ph} distances as distances to a source which would have, correspondingly, the same angular size or the same observed energy flux when measured in the Minkowski space. If the physical size and luminosity of the object are known, these distances can be measured directly.

In the Friedmann Universe the angular distance coinsides with the proper distance at emission, $d_A = l_{em}$, as for the photometric distance, it is connected with the angular distance as $d_{ph} = (1+z)^2 d_A$ and this formula is a universal one.

Using the above formula for proper distance at emission we see that in the Friedmann Universe with $\alpha > 0$ the angular distance depends upon redshift non-monotonically, having a maximal value. For example, in the dust Universe ($\alpha = 3/2$) angular distance has a maximum at $z = 5/4$. Since $d_A = l_{em}$ this means that for $z > 5/4$ objects with bigger z were closer to us at the time of emission.

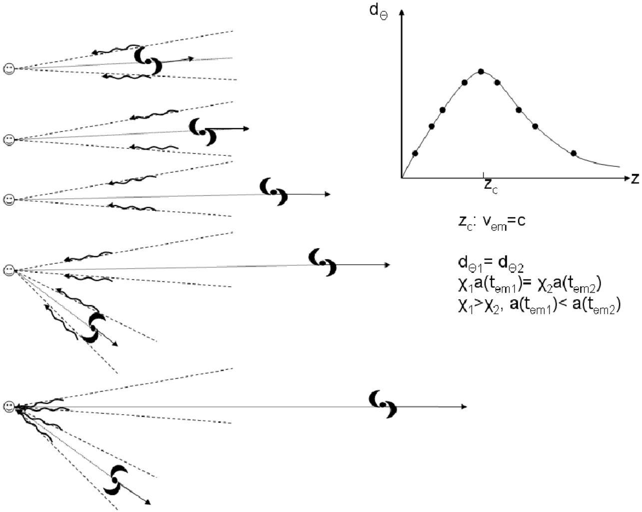

Figure:

We see that:
superluminal recession $\Leftrightarrow$ non-monotonic $d_A(z)$.

It should be noted that $d_A = l_{em}$ does not mean that $\dot{d}_A = v_{em}$ since *apparent* velocity differs from the physical one because the speed of light is finite. The correct formula is

$$\dot{d}_A = v_{em}/(1+z)$$

An interesting effect which could be observable in a near future: the redshift drift.

$$z(t_o) = \frac{a(t_o)}{a(t_e)} - 1$$

and

$$z(t_o + \Delta t_o) = \frac{a(t_o + \Delta t_o)}{a(t_e + \Delta t_e)} - 1$$

which gives us after the expansion at first order in $\Delta t/t$

$$\Delta z = \Delta t_o \left(\frac{\dot{a}(t_o) - \dot{a}(t_e)}{a(t_e)}\right).$$

In terms of Hubble parameter it can be rewritten as

$$\frac{\Delta z}{\Delta t_o} = H_0\left(1 + z - \frac{H_e}{H_0}\right)$$

If we go back to the formula for the proper distance at emission as a function of H and z, take a derivative with respect to time, use the formulae for $H(t)$ and $\dot{z}$, we get exactly $v_{em}/(1+z)$.
There is one more distance measure, the light travel distance, which is equal to a distance which the light would travel in Minkowski space-time for a given time interval: $d_l = c\Delta t$. In a popular literature this measure is usually expressed in light-years. Obviously, in an expanding Universe $l_{em} < d_l < l_{now}$. Simple geometric considerations show that $\dot{d}_l$ is directly connected with the redshift:

$$\dot{d}_l = \frac{cz}{z+1}$$

We now turn to other proposal for the velocity which uses completely different ideas. Intuitively, let us "transfer" the velocity from a distant point to the location of an observer. The parallel transport on a Riemannian manifold is a well-defined mathematical object. However, we can not apply it directly to velocities since they are 3-dimensional objects. We can make the parallel transport of 4-vector using appropriate connections (General Relativity in its standard from uses Levi-Civita connections, we use them in the present paper and will comment about other choice later), so to start the procedure we take 4-velocity of a distant object and transport it to the observer point. Then, we restore 3-velocity using transported 4-velocity and 4-velocity of the observer. One property of such definition is clear – any 3-velocity obtained by this procedure is subluminal (a hypothetical superluminal 3-velocity would correspond to imaginary 4-velocity vector which can not be a result of parallel transport of any real 4-velocity vector). However, the procedure is still not fixed completely since for Levi-Civita connection the result of parallel transport depends on the path. What path is better to specify? One proposal is to chose a null geodesics between the emitter and observer. This proposal does not require any additional structures, like particular foliation of space-time. In this sense it can be applied to any space-time. Moreover, *3-velocity defined this way is exactly the velocity which produces in a flat space-time the same redshift as the observer sees in curved space-time*.

Informal prove: let us express the redshift through the energy ratio of emitted and observed photons

$$1 + z = \frac{(k_\mu U^\mu)_e}{(k_\mu U^\mu)_o}$$

Then transport emitter values to the point of observation. The scalar product does not change, as for individual meaning of the variables, note that the wave vector at the point of emission $(k\mu)_e$ is transported along null geodesics and thus gives the wave vector at the point of observation $(k\mu)_o$. As for 4-velocity of emitter, it gives some transported value $\tilde{U}^\mu$. After that, the standard formula expressing z through 3-velocity $\tilde{V}$ can be got exactly the same way as in Special Relativity.

There are however, arguments against this chose. Usually in

physically interesting situations we assume some foliation by hypersurfaces of constant time. The emitter sent the light at some time t_1 which is earlier than the time when the observer received it t_2. This means that the velocity obtained from parallel transport along the light path has a meaning of an average (in some sense) velocity in between t_1 and t_2. To construct a velocity at particular time t we need to transfer 4-velocity along the line $t = const$ – if we consider only radial motion, the line of parallel transport is fully specified. Explicit calculations of such a velocity v_t for FRW Universe have been done in
M. Chodorowski, The kinematic component of the cosmological redshift MNRAS, 413, 585 (2011)
where it was shown that it is connected with the Hubble law velocity by a simple formula

$$v_t = \tanh v_H/c.$$

Later, this formula have been generalised to any sperically symmetric static space-times in
E.Emtsova, A. Toporensky, Velocities of distant objects in General Relativity revisited, Gravitation and Cosmology, 26, 50 (2020).

The solution for Friedmann equation if the Universe is filled by a standard matter with the pressure proportional to the energy density $p = \omega\epsilon$, $\omega = const$:

$$a \sim t^{\frac{2}{3(1+\omega)}}$$

and

$$\epsilon \sim a^{-3(1+\omega)}$$

This means that for a mixture the bigger is ω the rapidly energy density of this component falls. So that, during cosmological expansion the effective ω decreases, and the power index α in $a \sim t^{\alpha}$ increases. This means that the transition *Radiation - Dust - de Sitter* is natural.

However, the transition from an early acceleration to later decceleration needs special efforts. We should abandon the condition $\omega = const$. In the matter content of the Universe is a scalar field ϕ , then

$$\frac{3M_{pl}}{8\pi}H^2 = \frac{\dot{\phi}^2}{2} + V(\phi)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

and for large H we can neglect $\dot{\phi}$ in the first equation and $\ddot{\phi}$ in the second one. So that, the equations in the slow-roll approximation are

$$\dot{\phi} = -\frac{V'}{3H}$$

and

$$H^2 = \frac{8\pi}{3M_{pl}}V$$

The "slow-roll" should not be considered "too literally". For the quadratic scalar field potential we indeed have a slow rolling with the constant $\dot{\phi}$. However, for $V = \lambda\phi^4$ the equations show that $\dot{\phi} \sim \phi$ which results in an exponential decay of the scalar field.