

Gravitational waves

a short introduction

A.N. Baushev

Gravitational field equations

1) The index agreement:

The greek indexes (α, β, γ etc.) take values (0, 1, 2, 3). 0 is the time coordinate, the rest are the space coordinates

The latin indexes (a, b, i, j etc.) take values (1, 2, 3), these are the space coordinates.

Repeating indexes imply summation.

The metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \equiv \sum_{\alpha=0}^3 \sum_{\beta=0}^3 g_{\alpha\beta} dx^\alpha dx^\beta$$

The metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$dl^2 = dx^2 + dy^2 + dz^2$$
$$\eta_{\alpha\beta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$ds^2 = g_{00}dt^2 + 2g_{01}dtdx + 2g_{02}dtdy + 2g_{03}dtdz +$$
$$+g_{11}dx^2 + 2g_{12}dxdy + 2g_{13}dxdz + g_{22}dy^2 + 2g_{23}dydz + g_{33}dz^2$$

$$g_{\alpha\beta} = g_{\beta\alpha}$$
$$g_{\alpha\beta} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

A change of coordinates

$$(x^0, x^1, x^2, x^3) \rightarrow (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$$

Spherical coordinates

$$(t, r, \theta, \phi) \quad g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

Rotating cylindrical coordinates

$$(t, z, r, \phi) \quad g_{\alpha\beta} = \begin{pmatrix} -1 + \Omega^2 r^2 / c^2 & -\Omega r^2 & 0 & 0 \\ -\Omega r^2 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Общая теория относительности в общем виде

Символы Кристофеля (в некотором смысле, силы)

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\mu}} \right)$$

Тензор Римана (тензор кривизны)

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta} \Gamma^{\mu}_{\beta\gamma}$$

Тензор Риччи

Уравнения Эйнштейна для пустоты

$$R_{\alpha\beta} \equiv R^{\gamma}_{\alpha\gamma\beta}$$

$$R_{\alpha\beta} = 0$$

Weak gravitational field

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1$$
$$\eta_{\alpha\beta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$h_{\alpha\beta} = \begin{pmatrix} -2\Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Newtonian limit

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

$$ds^2 = -(1 + 2\Phi)dt^2 + (dx^2 + dy^2 + dz^2)$$

The linearized theory in the weak field case

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1$$

The Christoffel symbols (forces, in some sense)

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\mu}} \right)$$

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} \eta^{\alpha\mu} \left(\frac{\partial h_{\mu\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\mu\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\mu}} \right)$$

The Einstein equation for empty space

$$R_{\alpha\beta} = \frac{1}{2} \left(\eta^{\gamma\delta} \left[\frac{\partial^2 h_{\alpha\gamma}}{\partial x^{\beta} \partial x^{\delta}} + \frac{\partial^2 h_{\beta\gamma}}{\partial x^{\alpha} \partial x^{\delta}} - \frac{\partial^2 h_{\alpha\beta}}{\partial x^{\gamma} \partial x^{\delta}} \right] - \frac{\partial^2 h}{\partial x^{\alpha} \partial x^{\beta}} - \eta^{\alpha\beta} \left[\eta^{\gamma\mu} \eta^{\delta\nu} \frac{\partial^2 h_{\mu\nu}}{\partial x^{\gamma} \partial x^{\delta}} - \eta^{\mu\nu} \frac{\partial^2 h}{\partial x^{\mu} \partial x^{\nu}} \right] \right) = 0$$

$$R_{\alpha\beta} = \frac{1}{2} \left(\eta^{\gamma\delta} \left[\frac{\partial^2 h_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} + \frac{\partial^2 h_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 h_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} \right] - \frac{\partial^2 h}{\partial x^\alpha \partial x^\beta} - \eta^{\alpha\beta} \left[\eta^{\gamma\mu} \eta^{\delta\nu} \frac{\partial^2 h_{\mu\nu}}{\partial x^\gamma \partial x^\delta} - \eta^{\mu\nu} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} \right] \right) = 0$$

We make a change

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h, \quad \bar{h} \equiv \eta^{\alpha\beta} \bar{h}_{\alpha\beta} = -h$$

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \bar{h}$$

$$\eta^{\gamma\delta} \left[\frac{\partial^2 \bar{h}_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} + \frac{\partial^2 \bar{h}_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 \bar{h}_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} \right] - \eta^{\alpha\beta} \eta^{\gamma\mu} \eta^{\delta\nu} \frac{\partial^2 \bar{h}_{\mu\nu}}{\partial x^\gamma \partial x^\delta} = 0$$

Infinitesimal coordinate changes

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, |h_{\alpha\beta}| \ll 1$$

$$y^\alpha = x^\alpha + \xi^\alpha, |\xi^\alpha| \ll 1 \quad T(y^\alpha = a^\alpha) = T(x^\alpha = a^\alpha) - \frac{\partial T}{\partial x^\alpha} \xi^\alpha$$

$$\tilde{h}_{\alpha\beta} = h_{\mu\nu} \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \simeq h_{\alpha\beta} - \frac{\partial \xi^\alpha}{\partial x^\beta} - \frac{\partial \xi^\beta}{\partial x^\alpha}$$

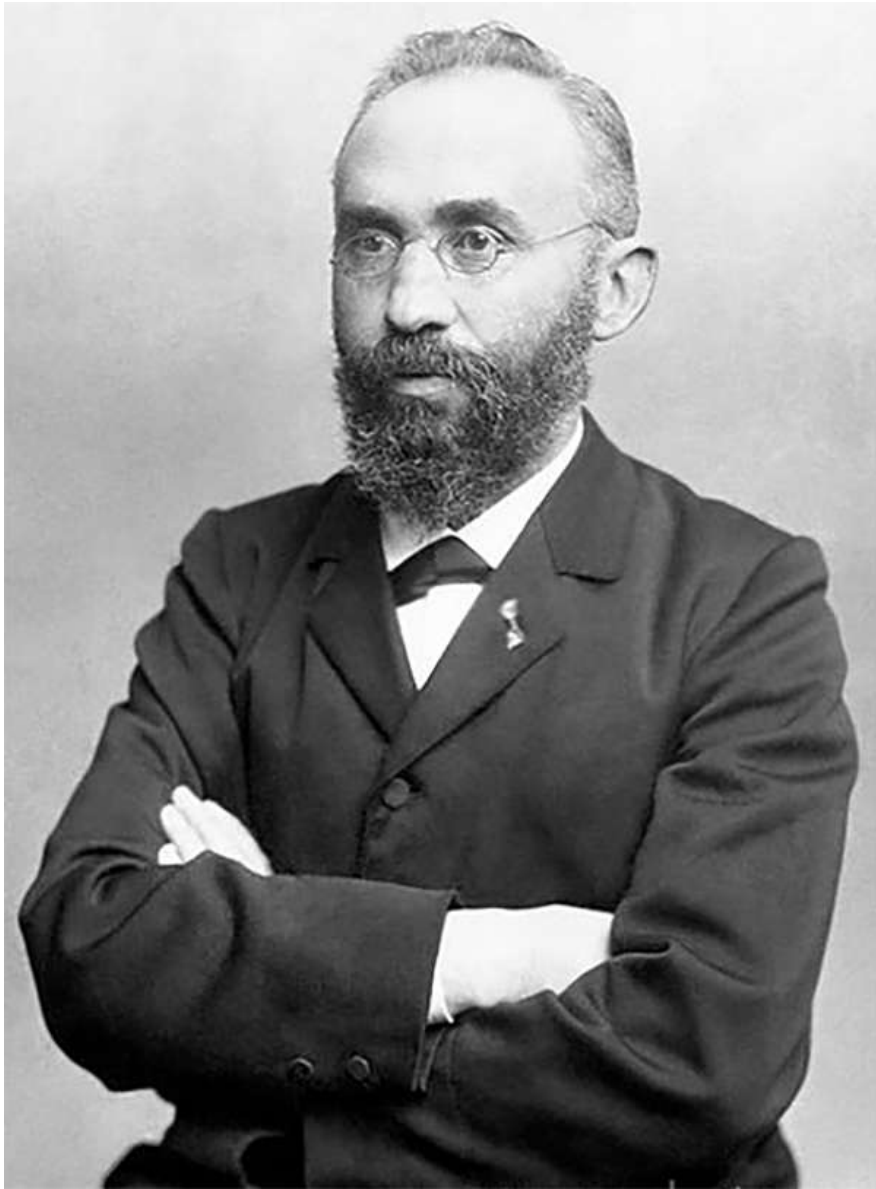
Lorenz gauge

$$\frac{\partial \tilde{h}^{\alpha\beta}}{\partial x^\beta} = 0$$

$$\eta^{\gamma\delta} \frac{\partial^2 \tilde{h}_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} = 0 \quad -\frac{\partial^2 \tilde{h}_{\alpha\beta}}{\partial t^2} + \frac{\partial^2 \tilde{h}_{\alpha\beta}}{\partial x^2} + \frac{\partial^2 \tilde{h}_{\alpha\beta}}{\partial y^2} + \frac{\partial^2 \tilde{h}_{\alpha\beta}}{\partial z^2} = 0$$

$$\square \tilde{h}^{\alpha\beta} = 0 \quad \textit{The last dirty trick} \quad \square \xi_2^\alpha = 0$$

Lorenz or Lorentz?



Hendrik Antoon Lorentz

invented the Lorenz transformation



L. Lorenz

Ludvig Valentin Lorenz

invented the Lorenz gauge condition

$$\hbar_{\alpha\beta} = \Re[A_{\alpha\beta} \exp(ik_\mu x^\mu)] \quad \omega \equiv k_0 = \sqrt{k_1^2 + k_2^2 + k_3^2}$$

$$\square \hbar_{\alpha\beta} = 0, \quad \frac{\partial \hbar_{\alpha\beta}}{\partial x^\beta} = 0 \rightarrow k^\alpha k_\alpha = 0, \quad A_{\alpha\beta} k^\beta = 0$$

$$\square \xi_2^\alpha = 0 \quad y^\alpha = x^\alpha + \xi^\alpha \quad \xi^\alpha = -iC^\alpha \exp(ik_\alpha x^\alpha)$$

So we may put four more conditions!

$$A_{\alpha 0} = 0$$

But these are three conditions, not four, since:

$$A_{\alpha\beta} k^\beta = 0 \rightarrow A_{0\alpha} k^\alpha = 0$$

Then we put the fourth

$$\eta^{\alpha\beta} A_{\alpha\beta} \equiv A_\alpha{}^\alpha = 0$$

Итого

$$A_{\alpha\beta} k^\beta = A_{\alpha 0} = A_\alpha{}^\alpha = 0$$

Thus, the TT-calibration

$$\bar{h}_\alpha{}^\alpha \equiv \bar{h} = \bar{h}_{\alpha\beta} k^\beta = \bar{h}_{\alpha 0} = 0$$

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \bar{h} \quad \rightarrow \quad h_{\alpha\beta} = \bar{h}_{\alpha\beta}$$

$$h_\alpha{}^\alpha \equiv h = h_{\alpha\beta} k^\beta = h_{\alpha 0} = 0$$

Let us consider a flat wave propagating along OZ axis:

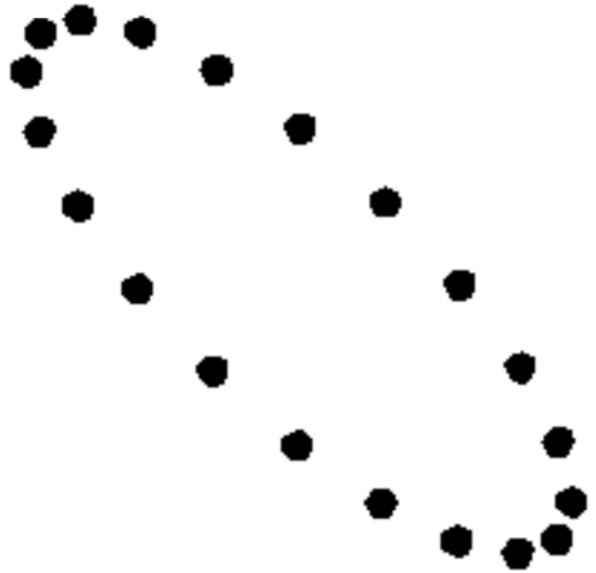
$$k_\alpha = (\omega, 0, 0, \omega)$$

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{h_{xx} dx^2 - h_{yy} dy^2}_{h_+} + \underbrace{2h_{xy} dx dy}_{h_\times}$$

Two polarizations of gravitational waves

$$h_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & 0 & 0 \\ 0 & 0 & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_{\times} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_{xy} & 0 \\ 0 & h_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



A bit of history...

- 1900 — Lorentz supposed that the gravitation may propagate with a speed, not exceeding the speed of light;
- 1905 — Henri Poincaré introduced the term «gravitational wave» (fr. onde gravitique). Laplace objected against the existence of the gravitational waves showing that they would make corrections of the order of v/c to the Newtonian law of gravitation, which contradicted to observations. Poincaré showed that the corrections mutually cancel each other;
- 1916 — Einstein showed that a mechanical system should emit gravitational waves within the framework of general relativity, although under normal conditions energy losses are of the order of $1/c^4$, being negligible and practically unmeasurable (in this work, he still mistakenly believed that a mechanical system that constantly retains spherical symmetry can emit gravitational waves).

Einstein's changing attitude to gravitational waves

- **19 Feb 1916, letter to Schwarzschild:** *“Es gibt also keine Gravitationswellen, welche Lichtwellen analog wären”*
- **22 Jun 1916, article:** *“...so sieht man, daß A (die Ausstrahlung des Systems durch Gravitationswellen pro Zeiteinheit) in allen nur denkbaren Fällen einen praktisch verschwindenden Wert haben muß.”* Nährungsweise Integration der Feldgleichungen, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 1916 688
- **31 Jan 1918, article:** *“Da aber meine damalige Darstellung des Gegenstandes nicht genügend durchsichtig und außerdem durch einen bedauerlichen Rechenfehler verunstaltet ist, muß ich hier nochmals auf die Angelegenheit zurückkommen.”* Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 1916 154
- **1936 undated letter to Max Born:** *“Together with a young collaborator, I arrived at the interesting result that gravitational waves do not exist, though they have been assumed a certainty to the first approximation.”*
- **1936 Princeton lecture:** *“If you ask me whether there are gravitational waves or not, I must answer that I do not know. But it is a highly interesting problem.”*

Herrn John T. Tate
Editor The Physical Review
University of Minnesota
Minneapolis, Minn.

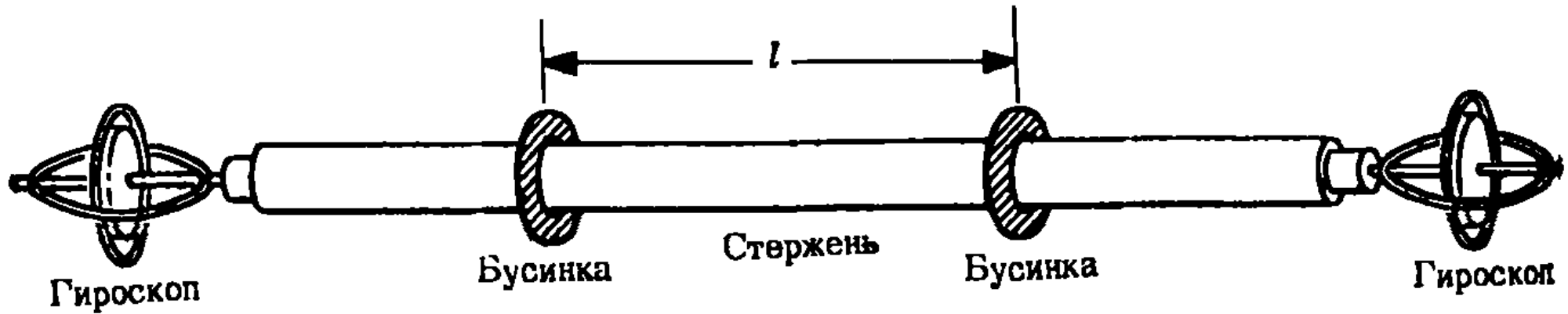
Sehr geehrter Herr:

Wir (Herr Rosen und ich) hatten Ihnen
unser Manuskript zur Publikation gesandt und Sie nicht
autorisiert, dasselbe Fachleuten zu zeigen, bevor es
gedruckt ist. Auf die - übrigens irrtümlichen - Ausführun-
gen Ihres anonymen Gewährsmannes einzugehen sehe ich
keine Veranlassung. Auf Grund des Vorkommnisses ziehe ich
es vor, die Arbeit anderweitig zu publizieren.

Mit vorzüglicher Hochachtung

P.S. Herr Rosen, der nach Sowjet-Russland abgereist ist,
hat mich autorisiert, ihn in dieser Sache zu vertreten.

Bondi, Feinman, 1957



Consider a plane wave along the OZ axis. Then a unit vector along the bar is

$$\vec{n} = (\sin \theta \cos \phi; \sin \theta \sin \phi; \cos \theta)$$

Then the vibration amplitude of the beads is

$$l = l_0 \left(1 + \frac{1}{2} h_{xx} \sin^2 \theta \cos 2\phi + \frac{1}{2} h_{xy} \sin^2 \theta \sin 2\phi \right)$$

Misner, Thorne, Wheeler,
Gravitation, Vol 2

$$a^i = \sum_{j=1}^3 \frac{\ddot{h}_{ij}^{TT} l^j}{2}$$

The Feynman's opinion about the conference

“I am not getting anything out of the meeting. I am learning nothing. Because there are no experiments this field is not an active one, so few of the best men are doing work in it. The result is that there are hosts of dopes here and it is not good for my blood pressure...

...Remind me not to come to any more gravity Conferences!”

R. Feynman, What Do You Care What Other People Think? P91, 1988 (Warsaw meeting 1962)

Emission of gravitational waves

$$h \sim m \frac{da}{dt}$$

$$m_1 a_1 = -m_2 a_2 \quad \rightarrow \quad m_1 \frac{da_1}{dt} + m_2 \frac{da_2}{dt} \simeq 0$$

$$\left(\left(\frac{v}{c} \right)^3 \frac{r_g}{r} \right)^2 \quad r_g = \frac{2Gm}{c^2}$$

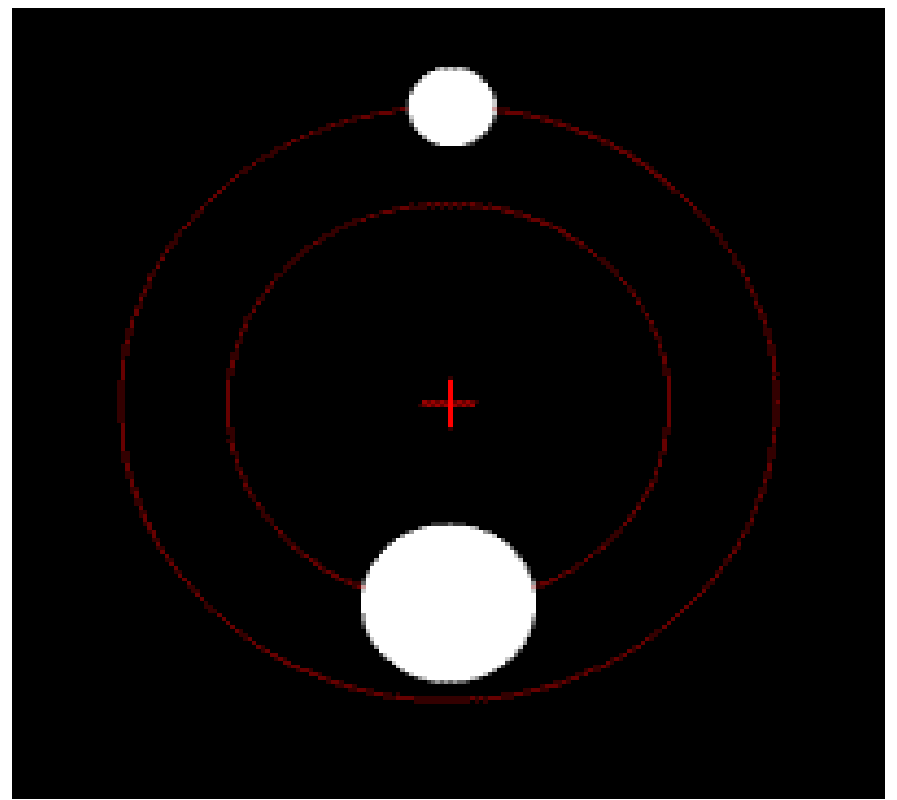
$$D_{ij} = \int \rho (3x_i x_j - r^2 \delta_{ij}) dV$$

$$-\frac{dE}{dt} = \frac{G}{45c^5} \ddot{D}_{ij} \ddot{D}^{ij}$$

Potential sources of gravitational waves

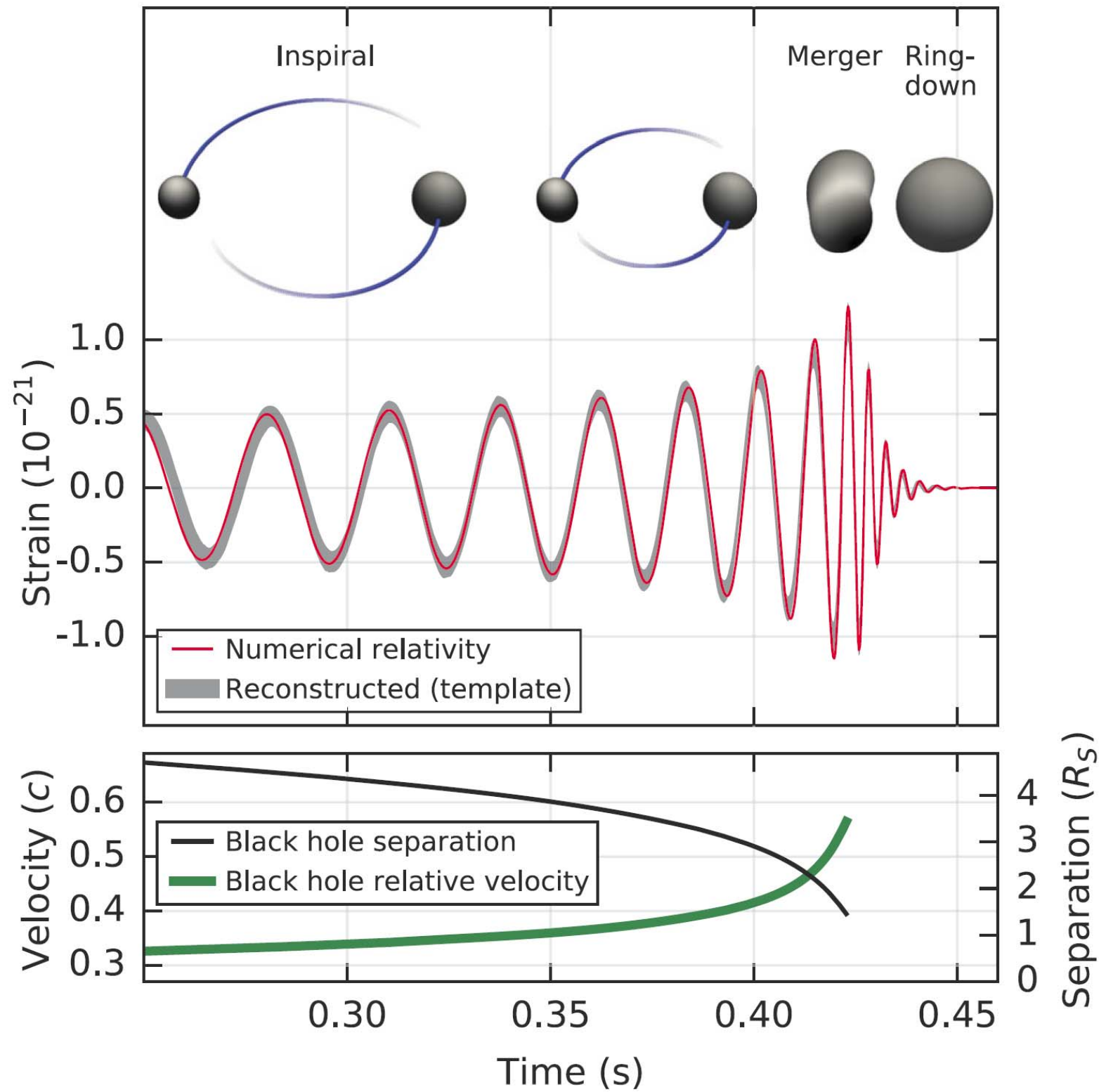
- Binary compact objects (neutron stars, black holes, white dwarfs)
- Collisions of galaxies and their clusters
- Asymmetric collapse of stars
- Cosmological gravitational waves
- In general, almost any accelerated motion of masses (except for a spherically-symmetric one)

A binary system

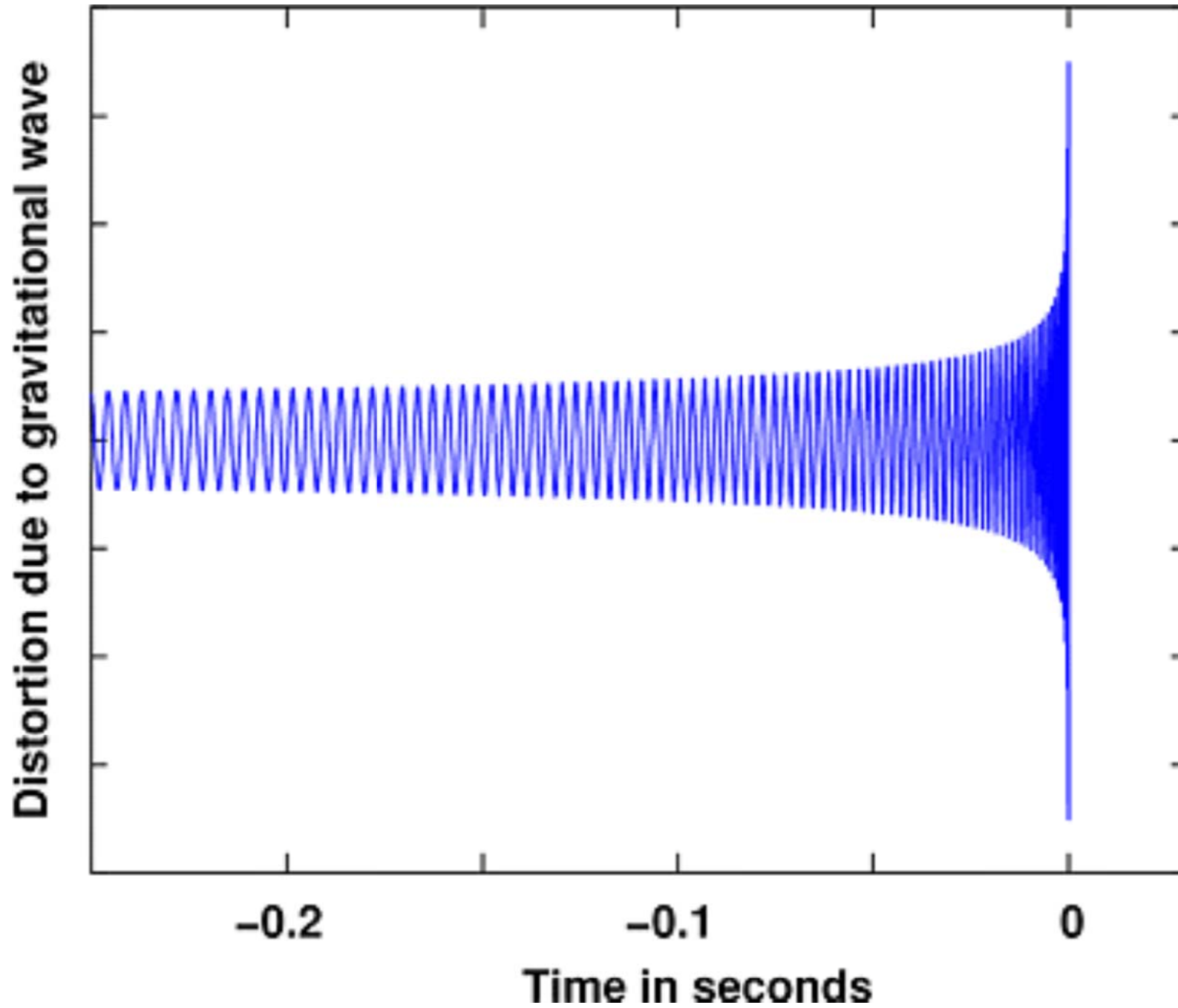


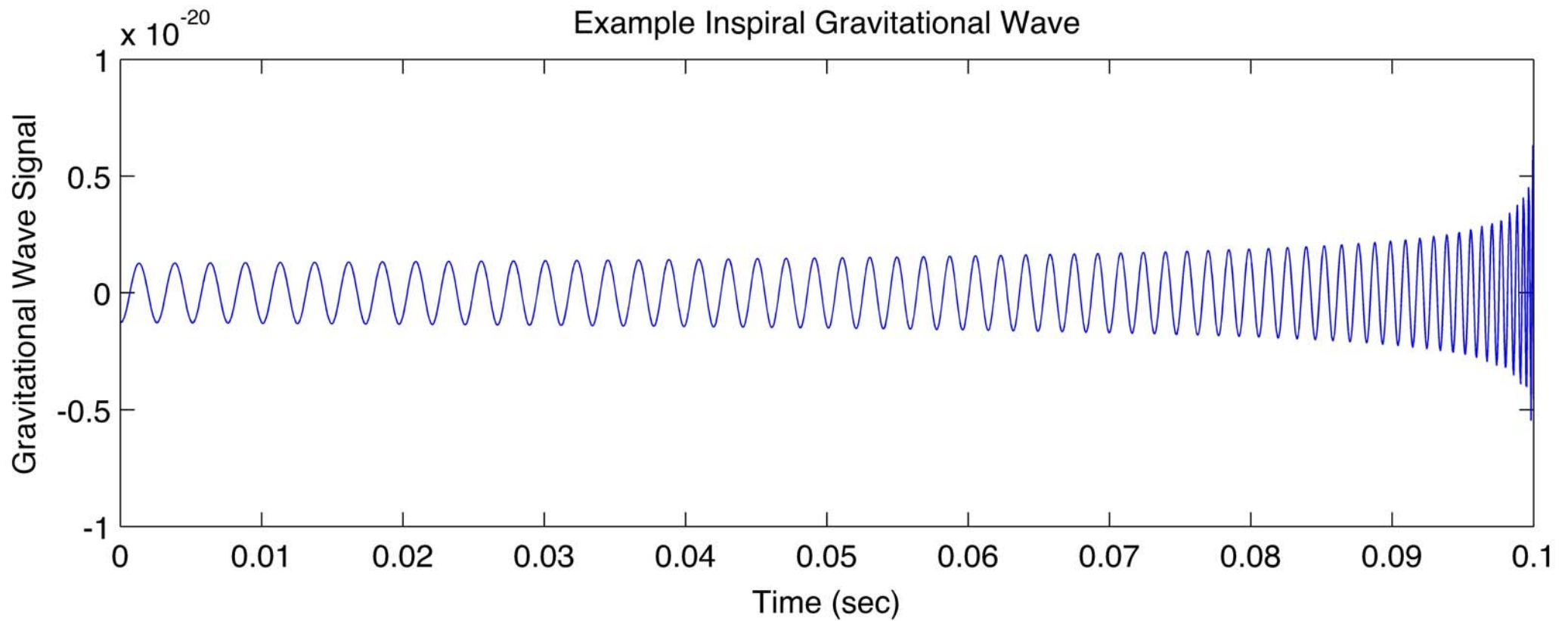
$$-\frac{dE}{dt} = \frac{32G^4 m_1^2 m_2^2 (m_1 + m_2)}{5c^5 r^5}$$

$$-\frac{dr}{dt} = \frac{64G^3 m_1 m_2 (m_1 + m_2)}{5c^5 r^3}$$



A chirp from neutron stars





Schutz (1986);

Cutler&Flanagan (1994)

arXiv:gr-qc/9402014

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

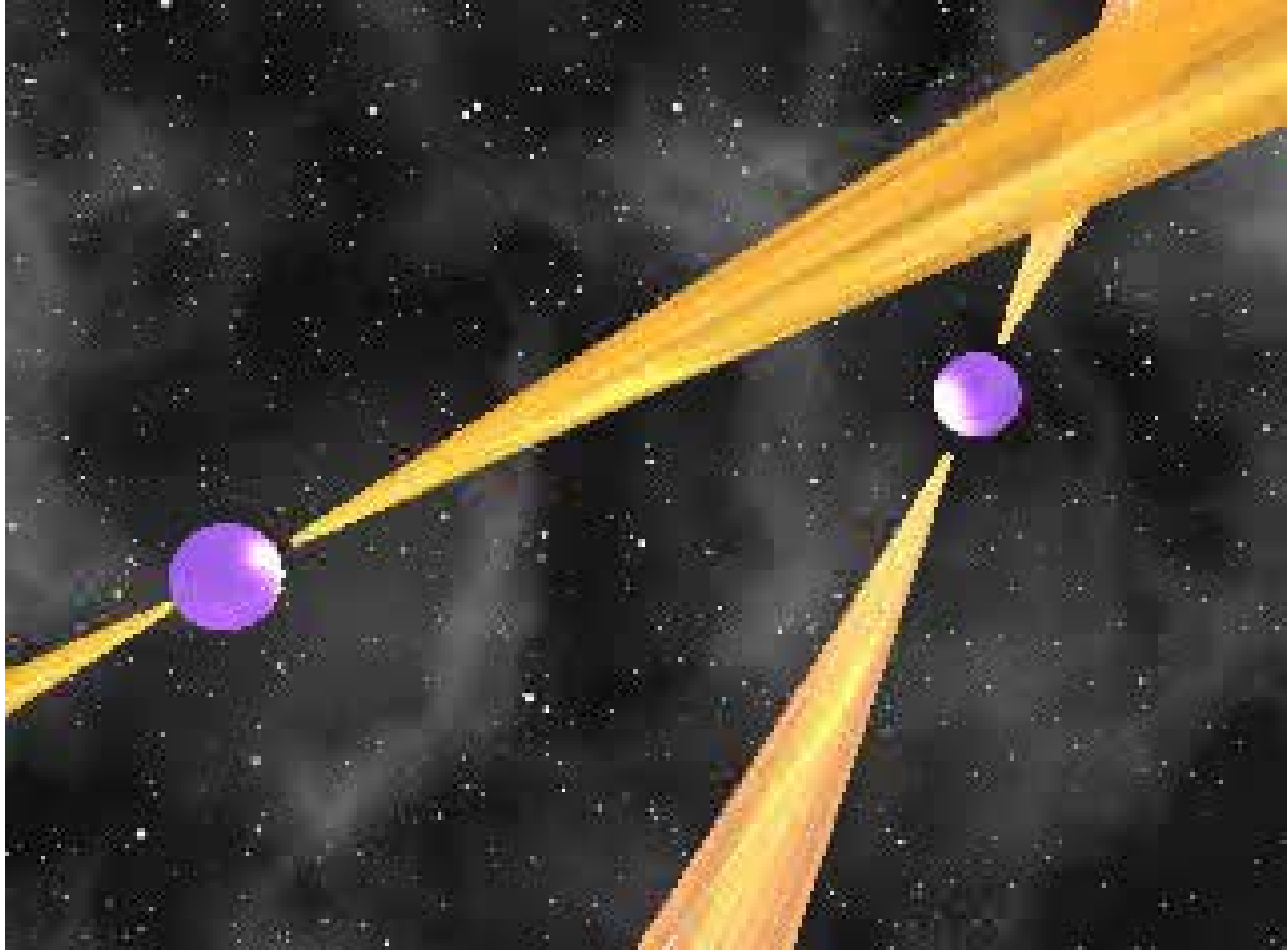
$$M_{chirp} = \frac{c^3}{G} \left(\frac{5\pi^{-8/3}}{96} f^{-11/3} \dot{f} \right)^{3/5}$$

Joseph Weber,
late 60'th

*Resonant
detector*



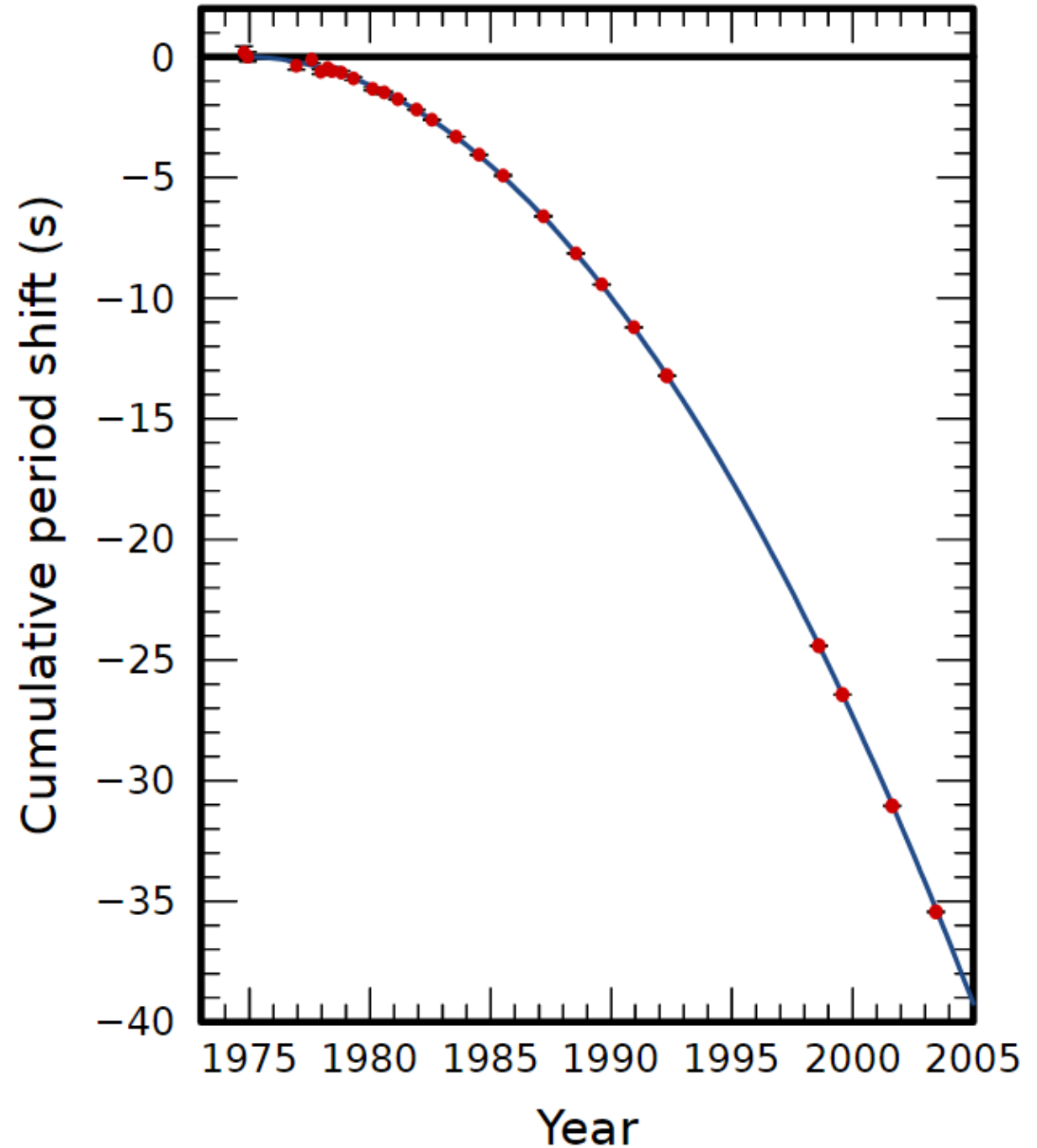
PSR J0737-3039

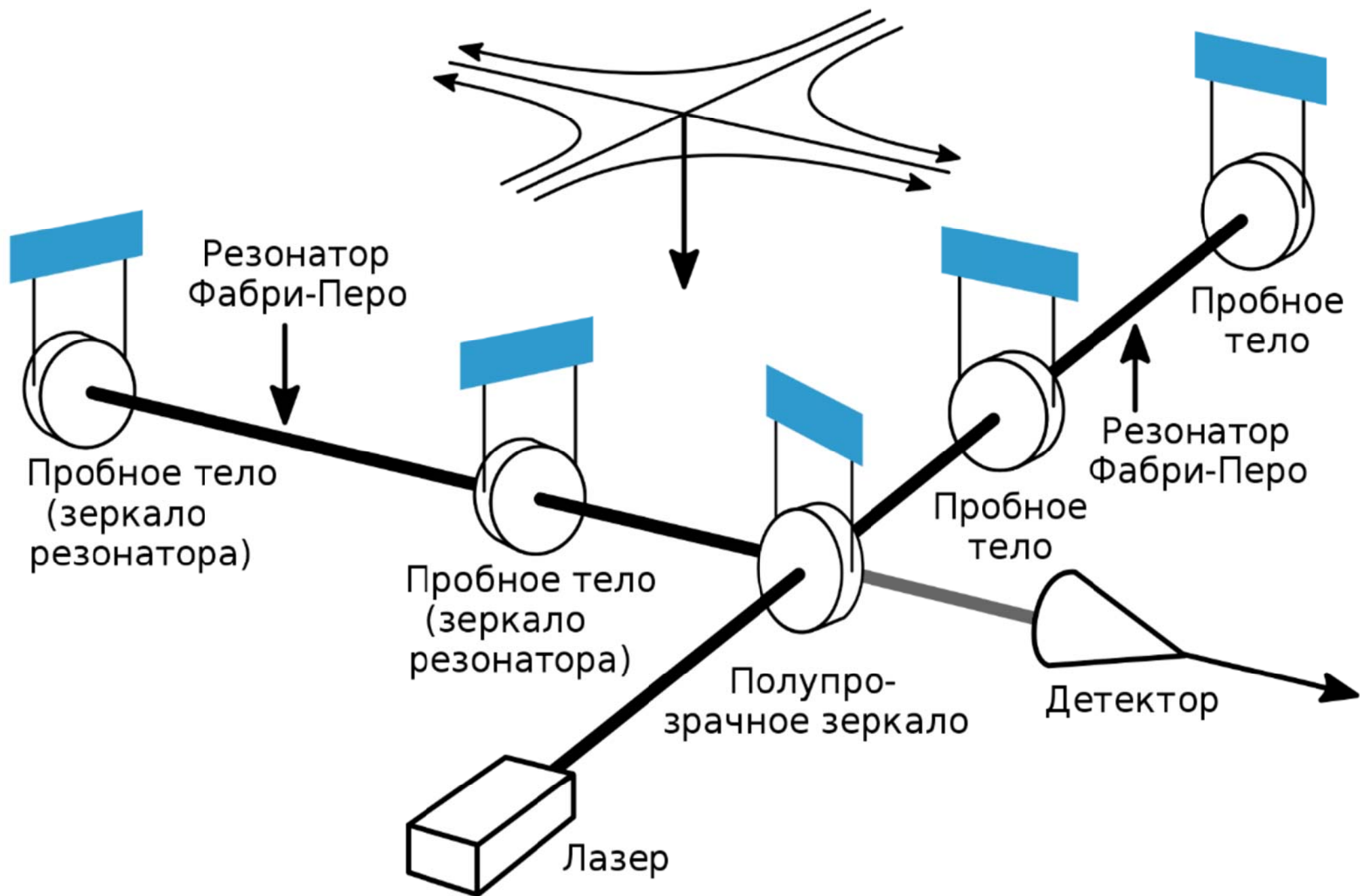


Hulse–Taylor binary pulsar PSR J1915+1606 (1974)

Binary neutron star

They will collide in
~300 mln. years



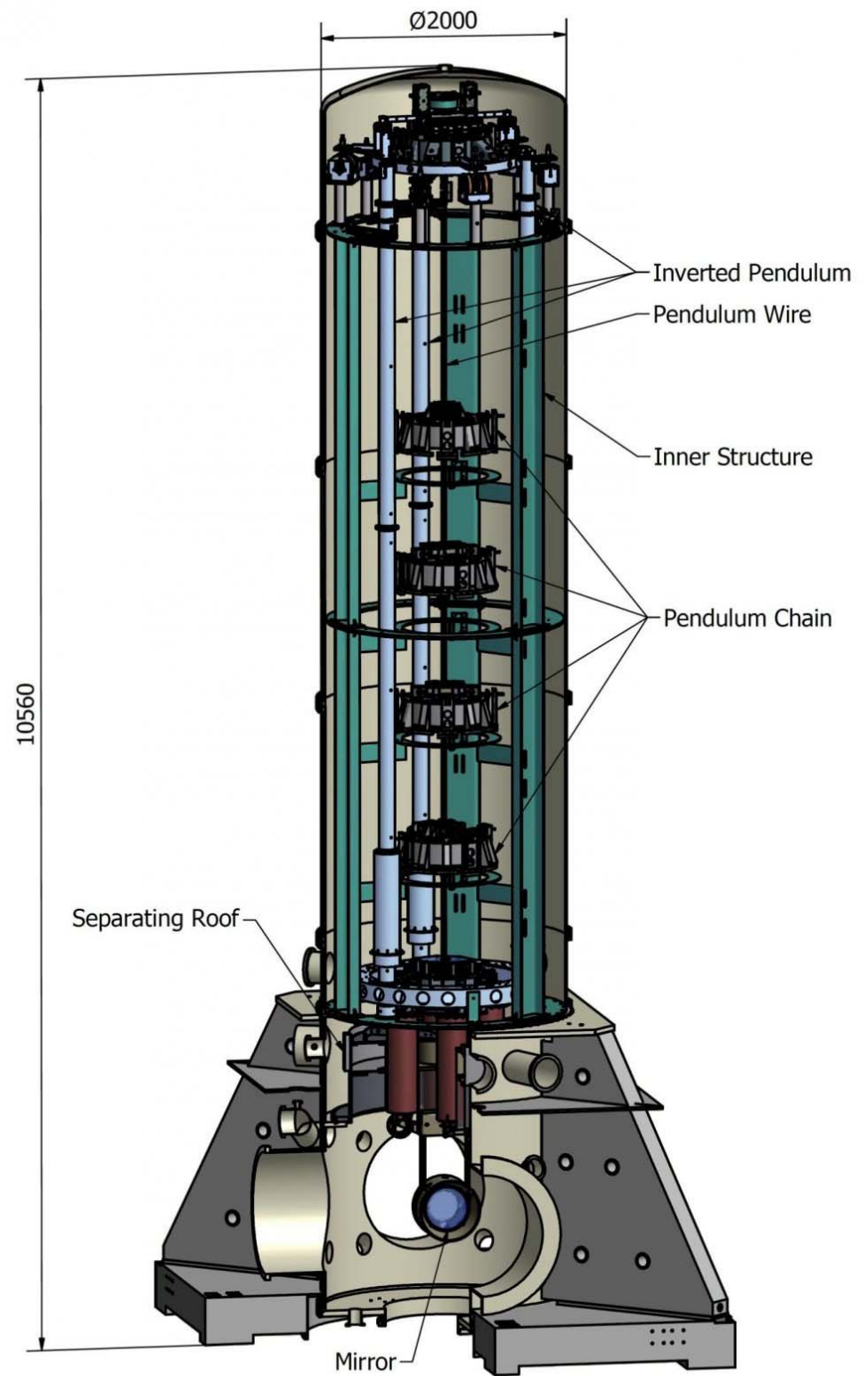


VIRGO



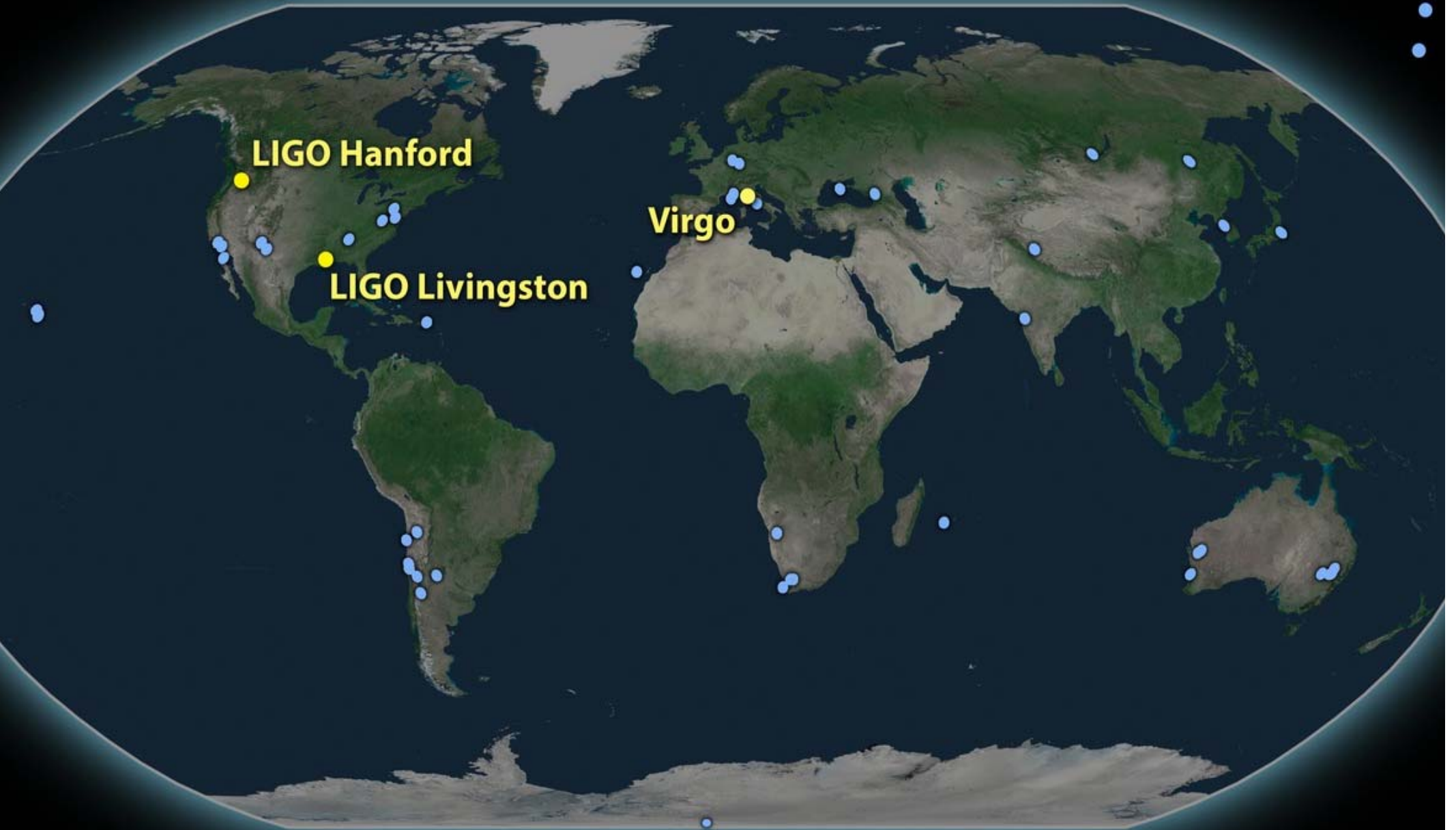
Mirror mount

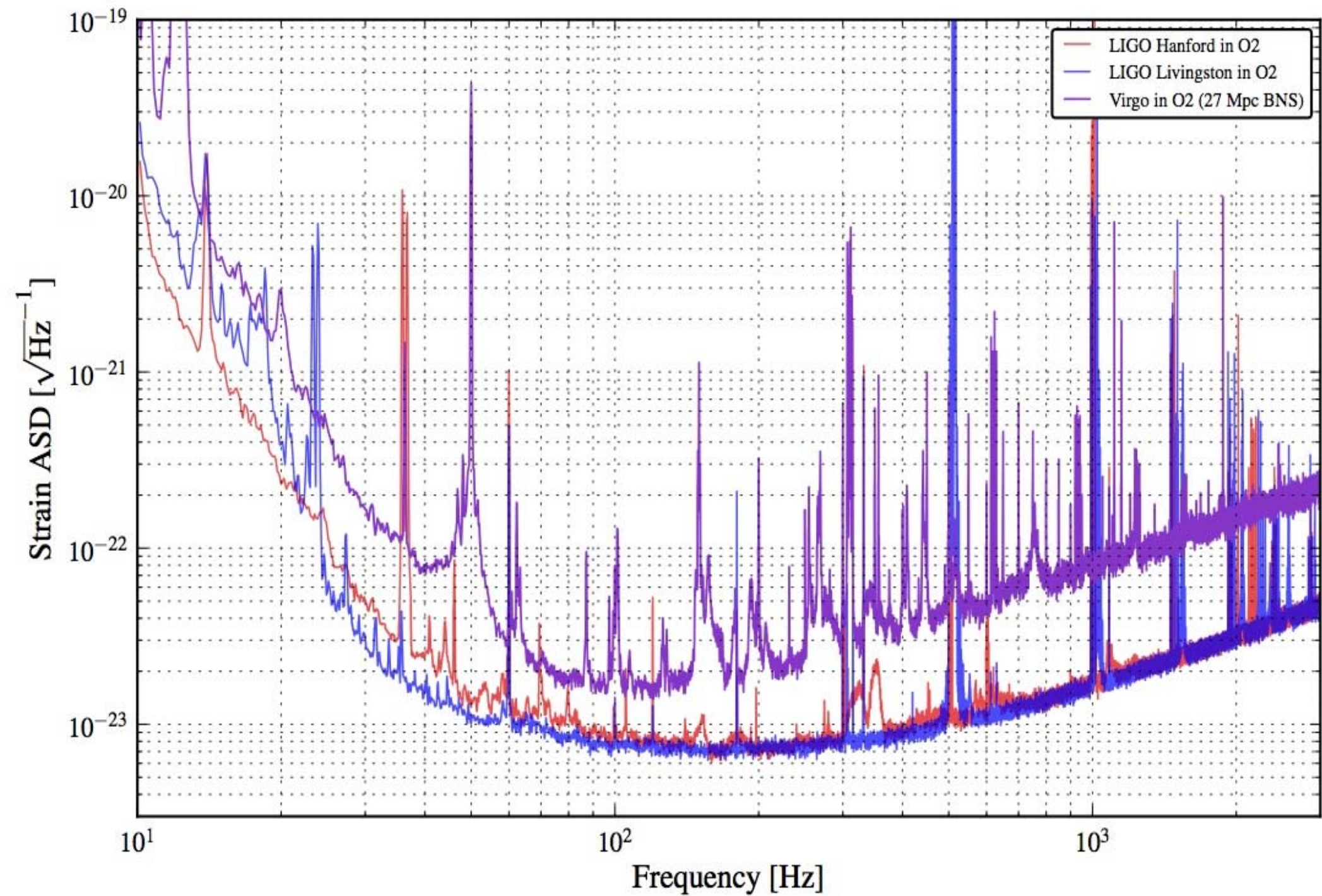
Adalberto Giazotto

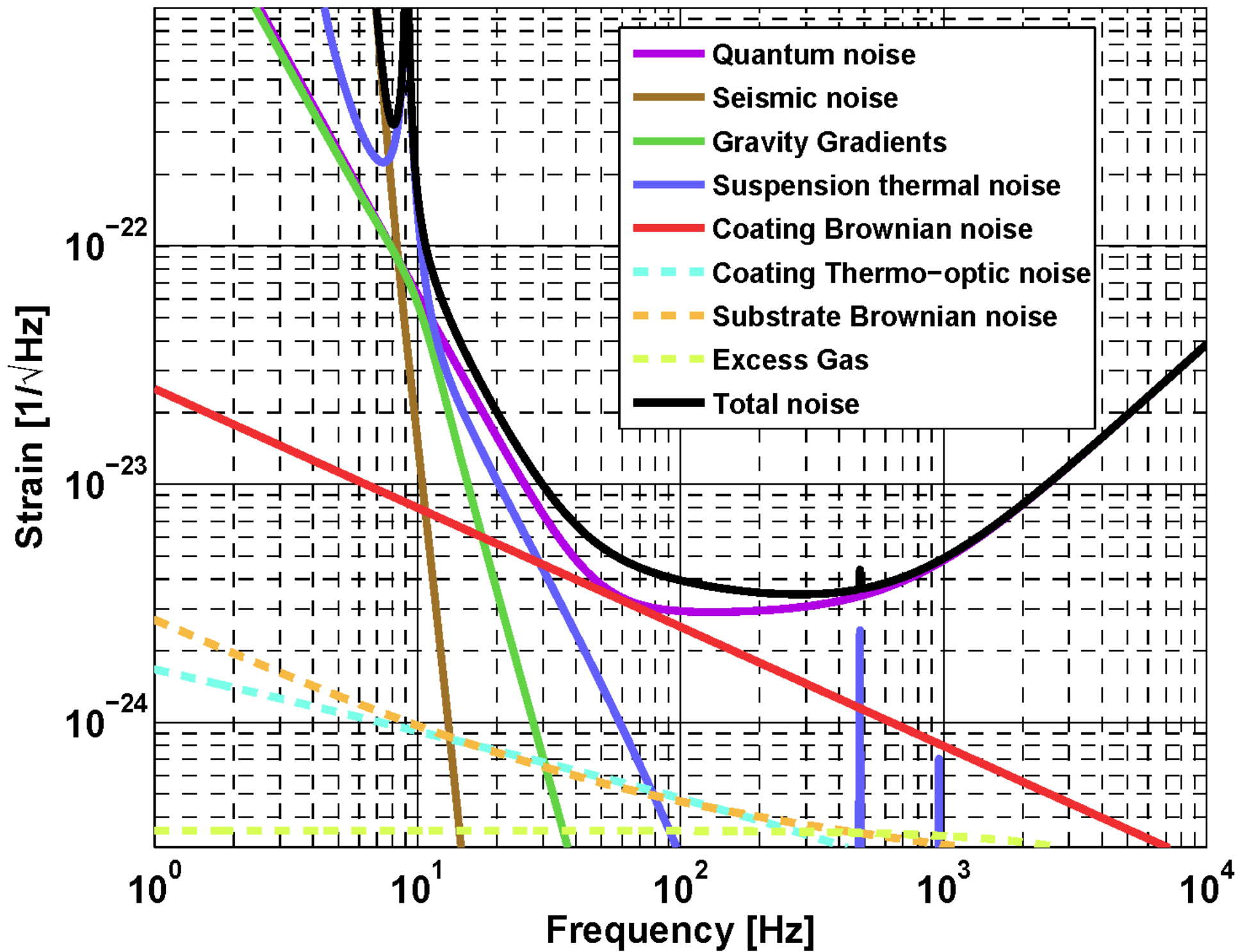


Earth

Sp

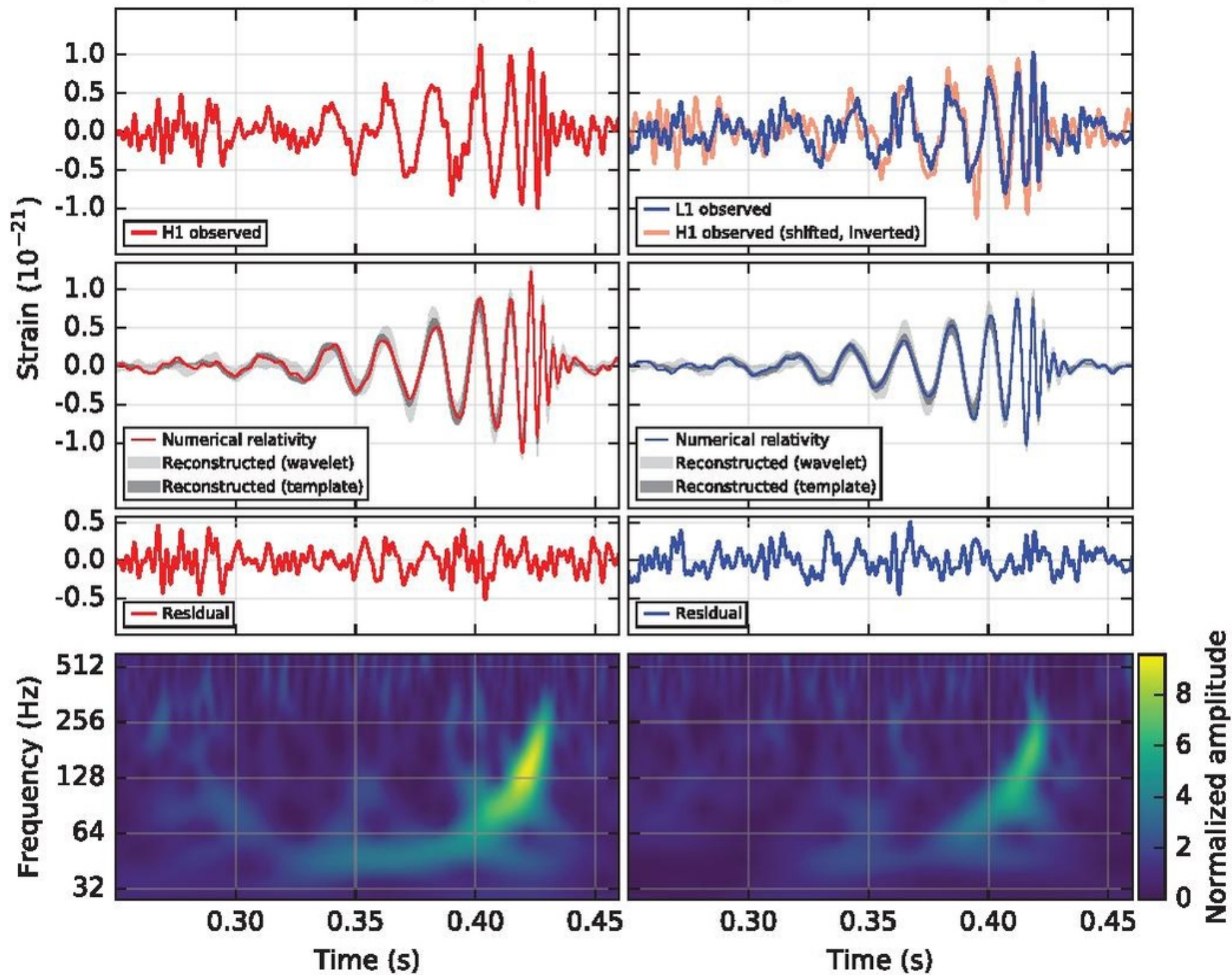




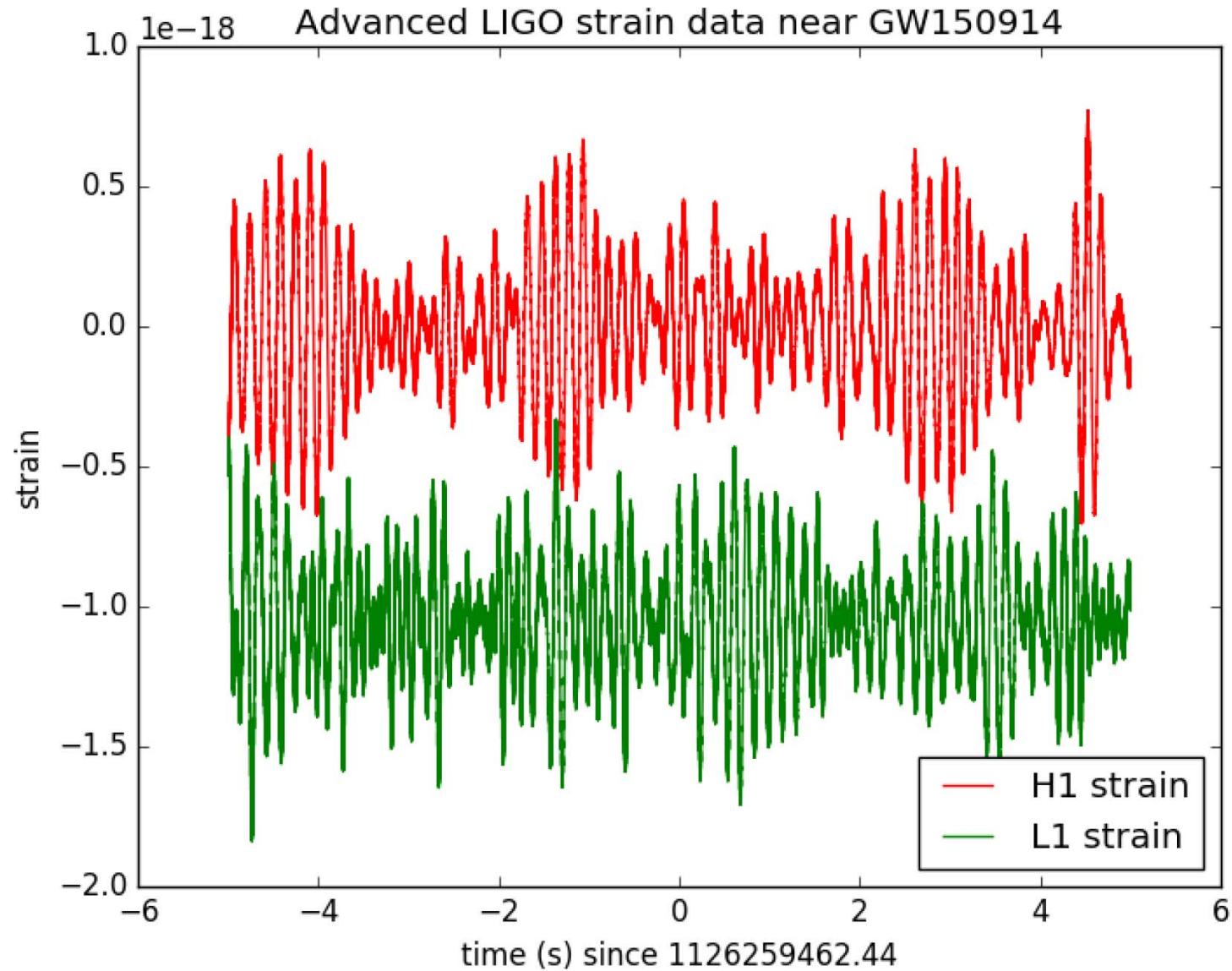


Hanford, Washington (H1)

Livingston, Louisiana (L1)



Data without whitening or band pass



26,768 views | Jun 16, 2017, 04:30pm

Was It All Just Noise? Independent Analysis Casts Doubt On LIGO's Detections



Starts With A Bang Contributor
Starts With A Bang Contributor Group ⓘ
Science

POST WRITTEN BY

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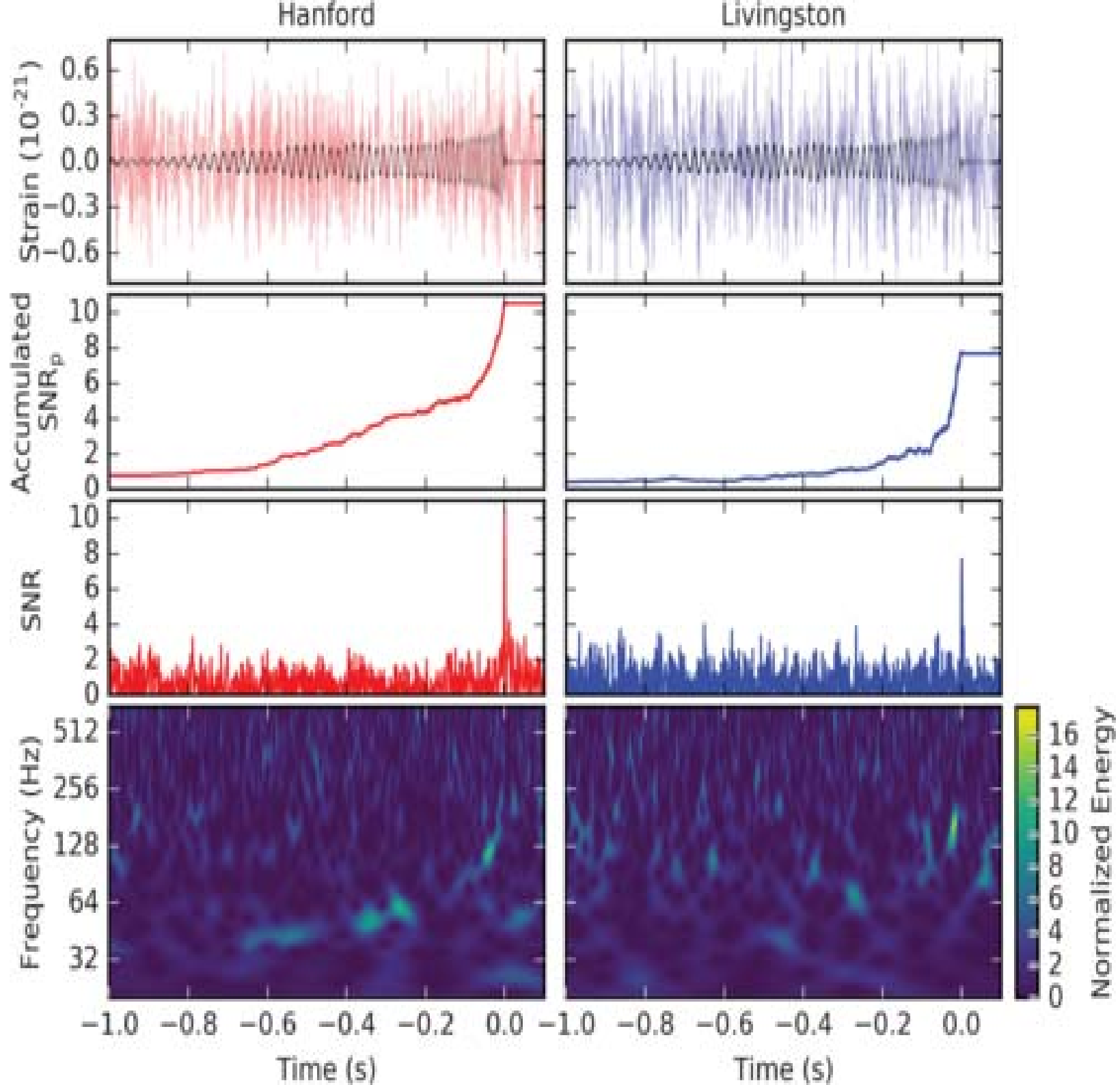


GW150914

- September 14, 2015 at 9:50:45 UTC, detected by the two LIGO detectors, first in Livingston, and 7ms later - in Hanford
- Merging of two black holes, $36_{-4}^{+5} M_{\text{sun}}$ and $29_{-4}^{+5} M_{\text{sun}}$
- A single black hole of mass $62_{-4}^{+4} M_{\text{sun}}$ was formed, and $3_{-0.5}^{+0.5} M_{\text{sun}}$ were gone with gravitational waves.
- The distance to the event is $410_{-140}^{+160} \text{Mpc}$, ~ 1300 mln. light years

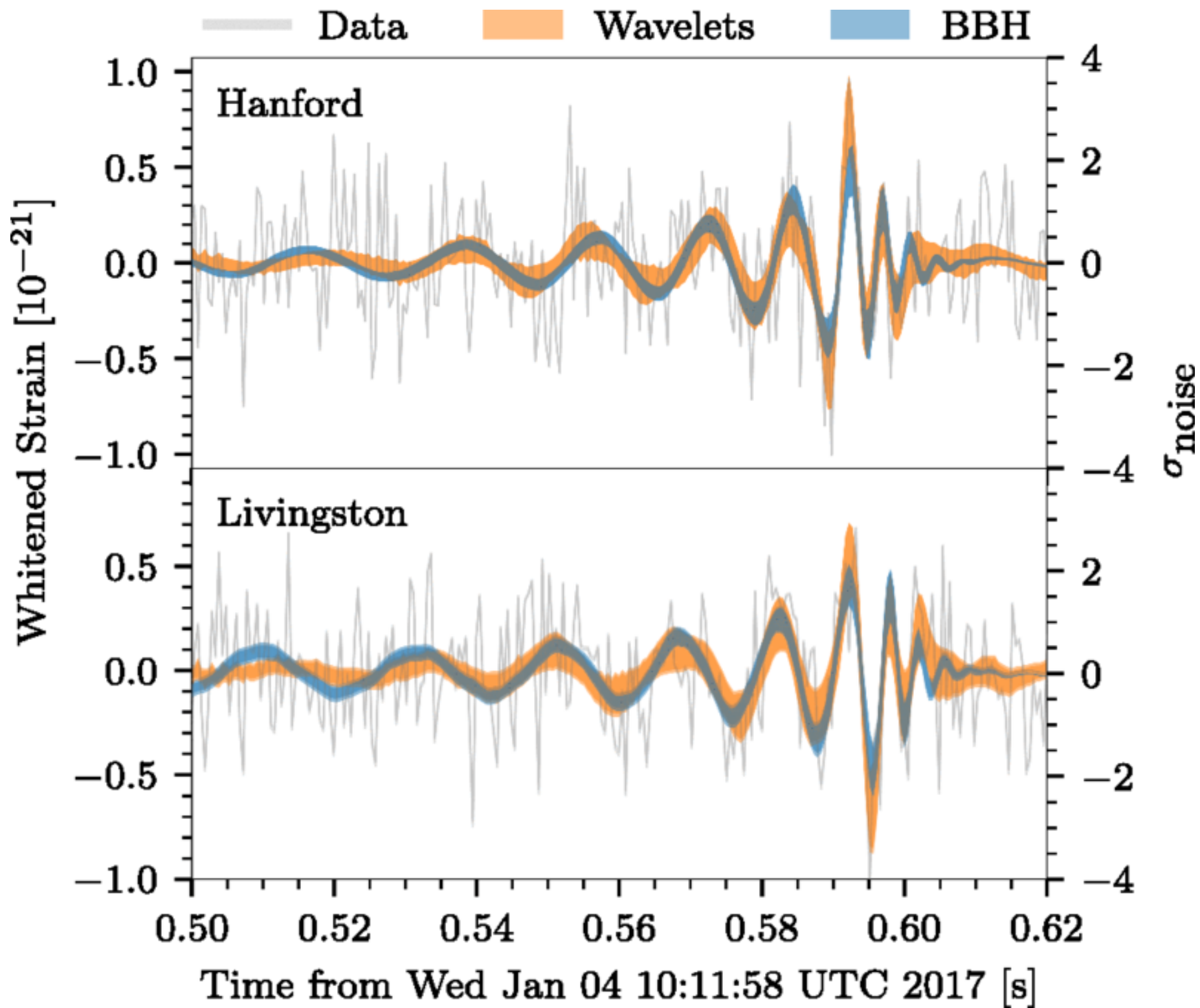
GW151226

- December 26, 2015, detected by the two LIGO detectors, first in Livingston, and 1.1 ms later - in Hanford
- Merging of two black holes, $14.2_{-3.7}^{+8.3} M_{\text{sun}}$ and $7.5_{-2.3}^{+2.3} M_{\text{sun}}$. A single black hole of mass $20.8_{-1.7}^{+6.1} M_{\text{sun}}$ was formed, and $1_{-0.2}^{+0.1} M_{\text{sun}}$ were gone with gravitational waves.
- The distance to the event is 440_{-190}^{+180} Mpc, ~ 1400 mln. light years.



GW170104

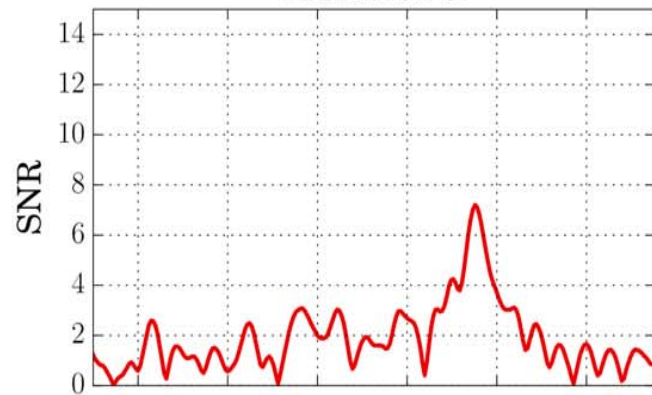
- January 14, 2015 at 10:11:58,6 UTC, detected by the two LIGO detectors, first in Hanford, and 3 ms later - in Livingston
- Merging of two black holes, $31.2_{-6.0}^{+8.4} M_{\text{sun}}$ and $19.4_{5.9}^{+5.3} M_{\text{sun}}$. A single black hole of mass $48.7_{-4.6}^{+5.7} M_{\text{sun}}$ was formed, and $2 M_{\text{sun}}$ were gone with gravitational waves.
- The distance to the event is 880_{-390}^{+450} Mpc, ~ 2900 mln. light years.



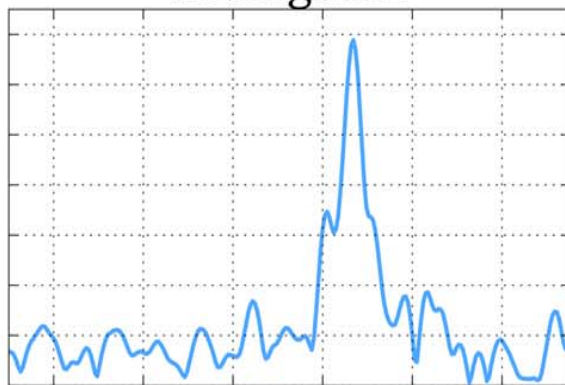
GW170814

- August 14, 2017 at 9:50:45 UTC, detected by all three detectors, first in Livingston, and 8ms later - in Hanford, and 14ms later by VIRGO. The possible area of the event decreased from 1160 to 60 square degrees.
- Merging of two black holes, $31 M_{\text{sun}}$ and $25 M_{\text{sun}}$. A single black hole of mass $53 M_{\text{sun}}$ was formed, and $3 M_{\text{sun}}$ were gone with gravitational waves.
- The distance to the event is 567 Mpc, ~1800 mln. light years.
- The first experimental estimation of the polarization of gravitational waves

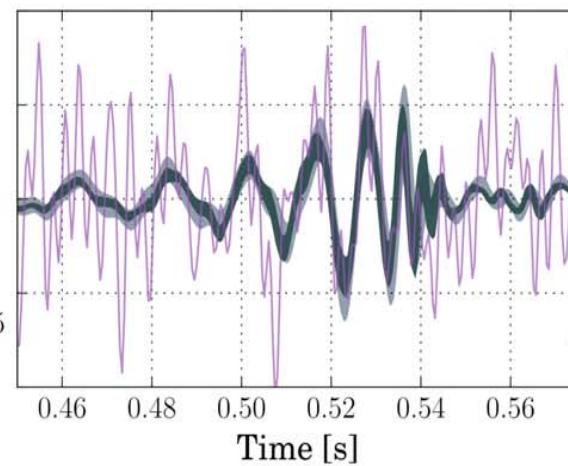
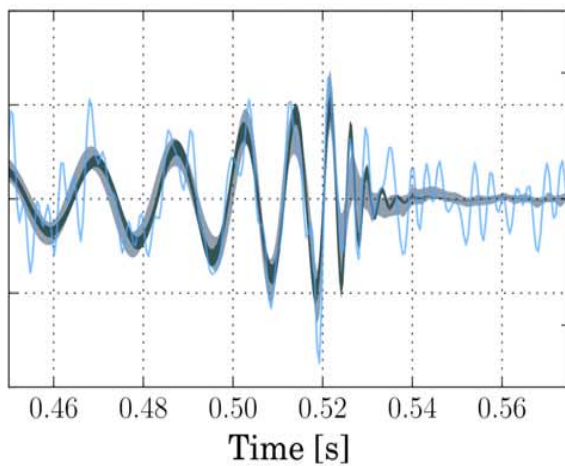
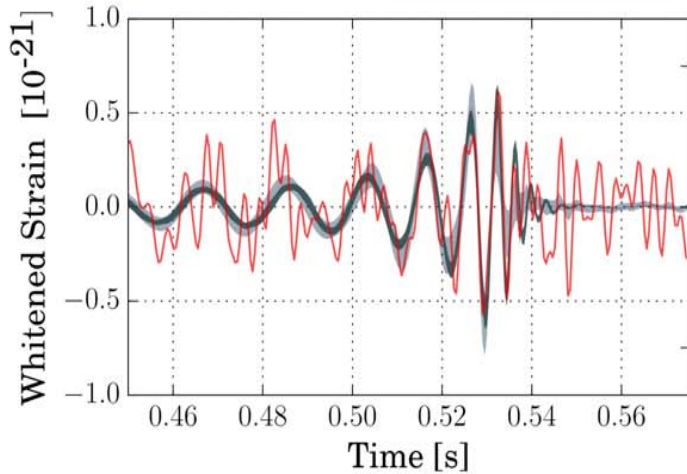
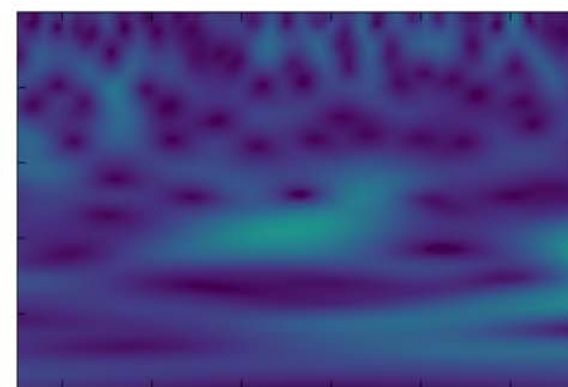
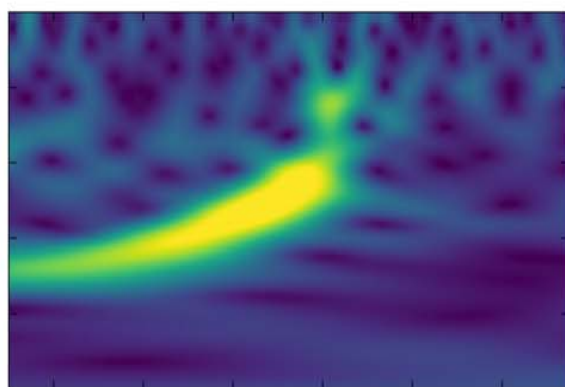
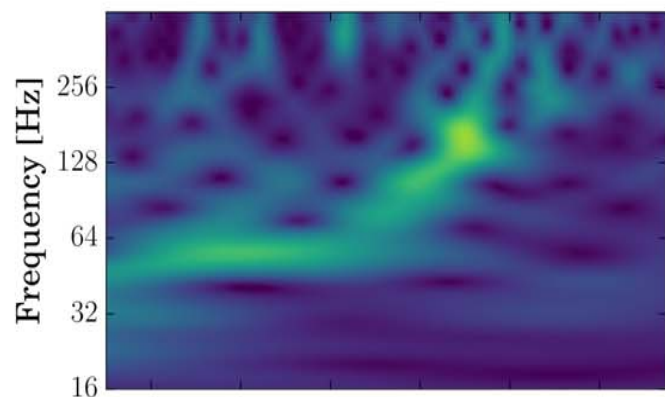
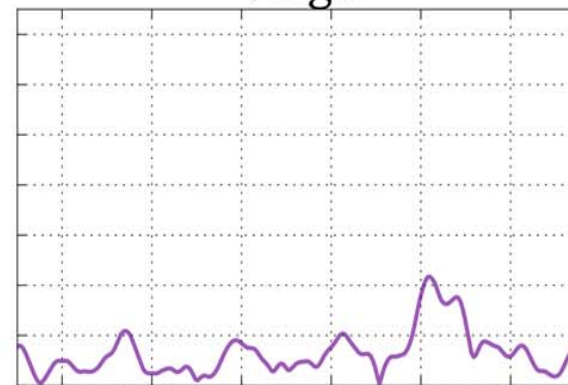
Hanford



Livingston

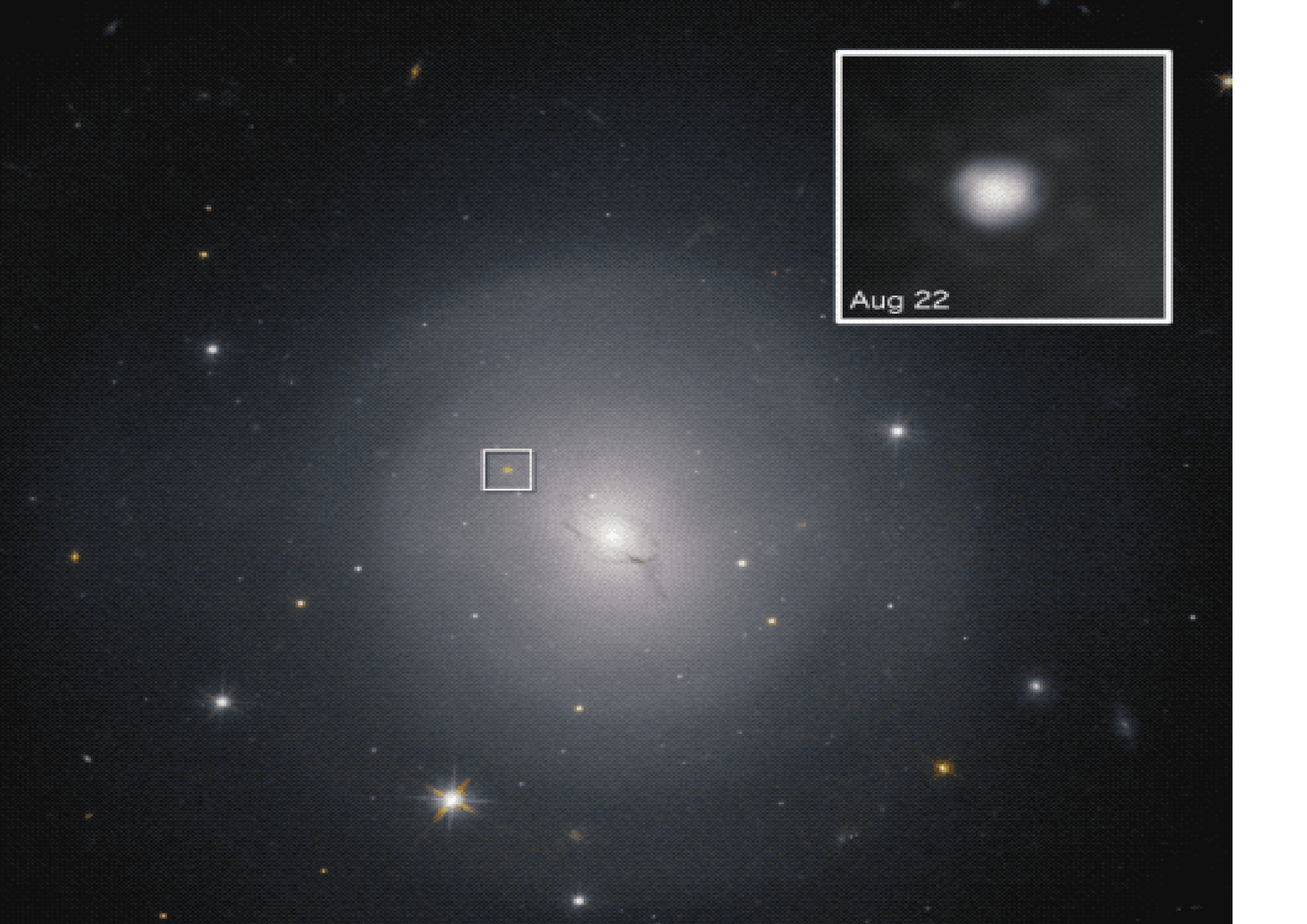


Virgo

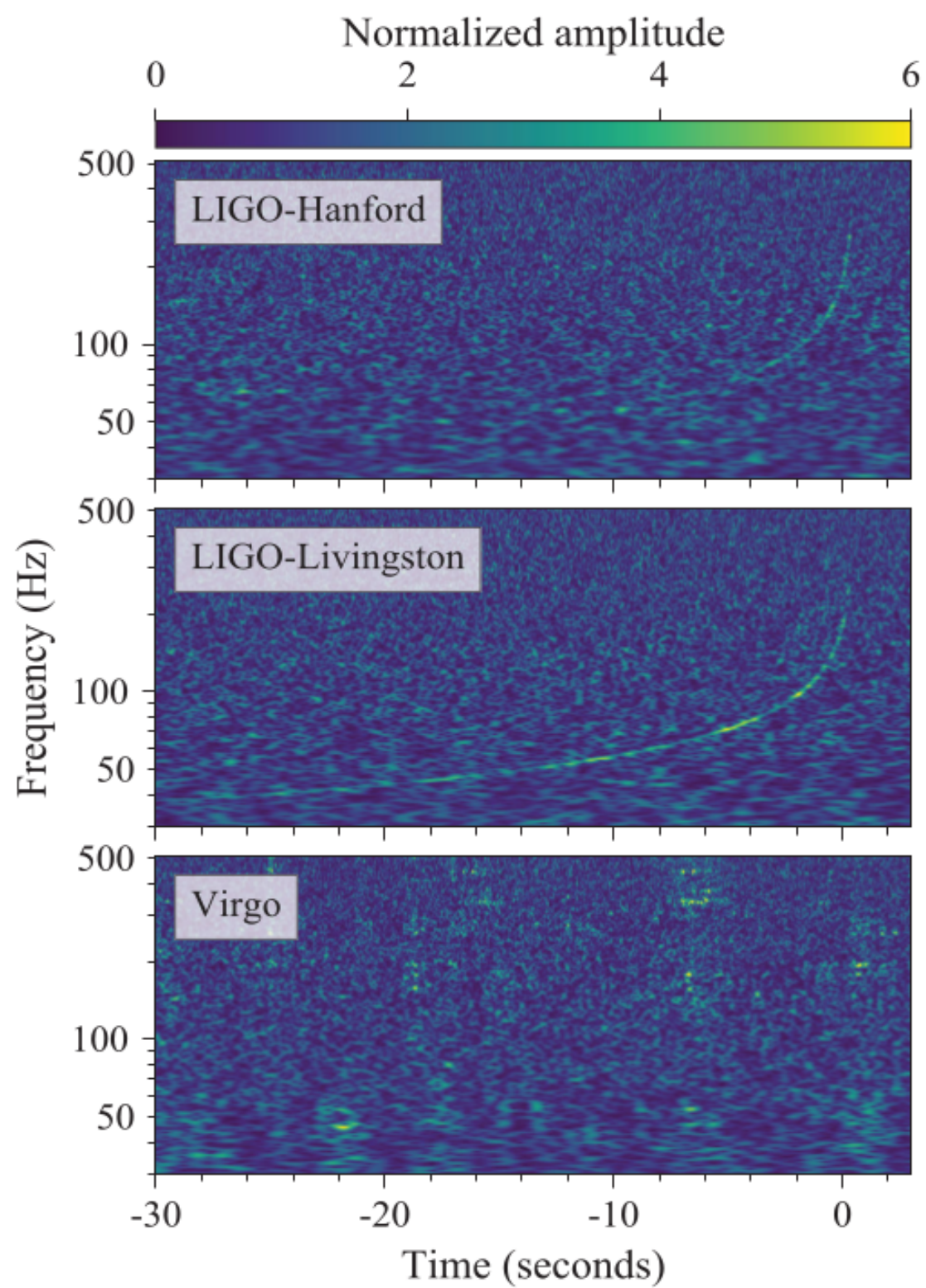

 σ_{noise}

GW170817

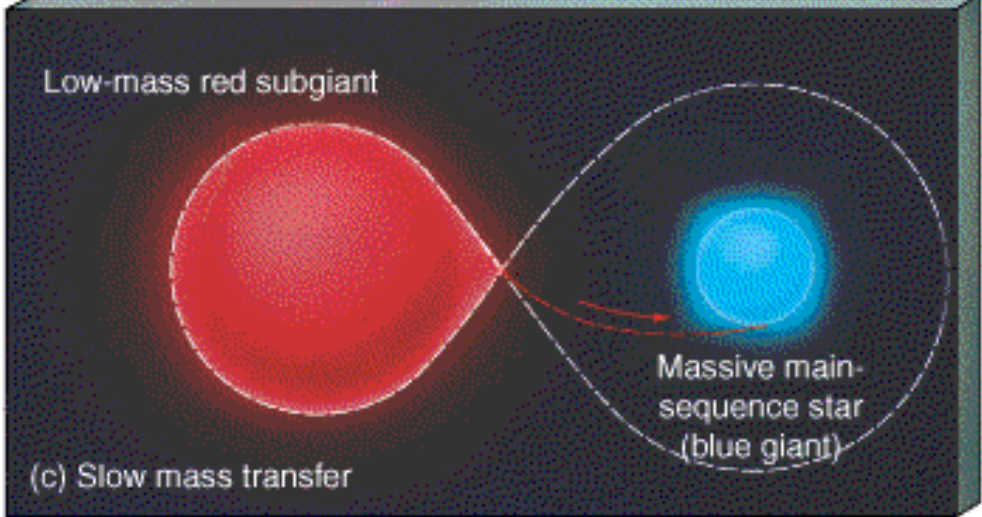
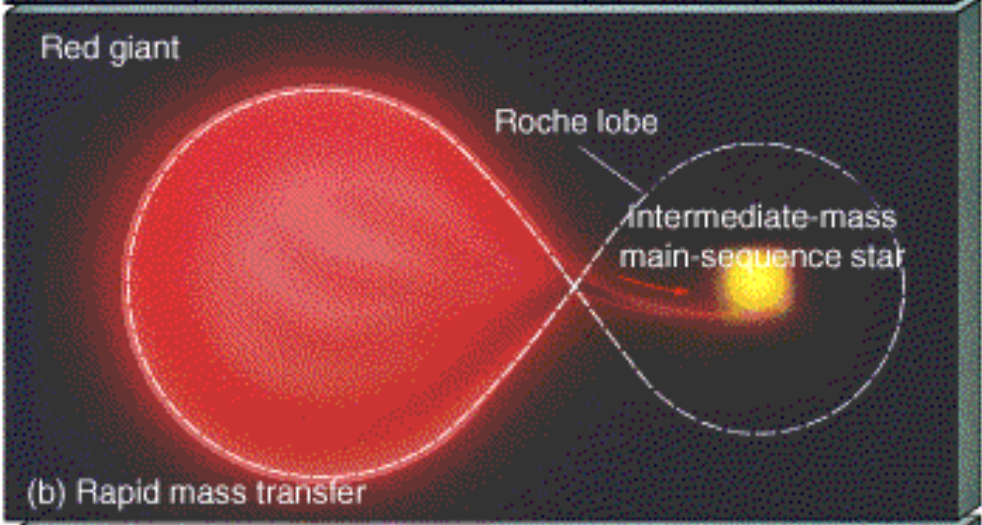
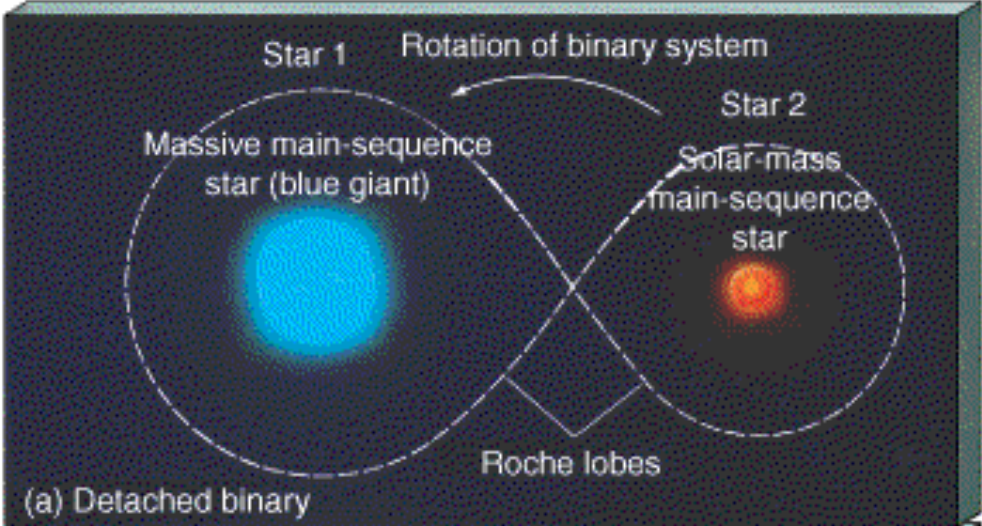
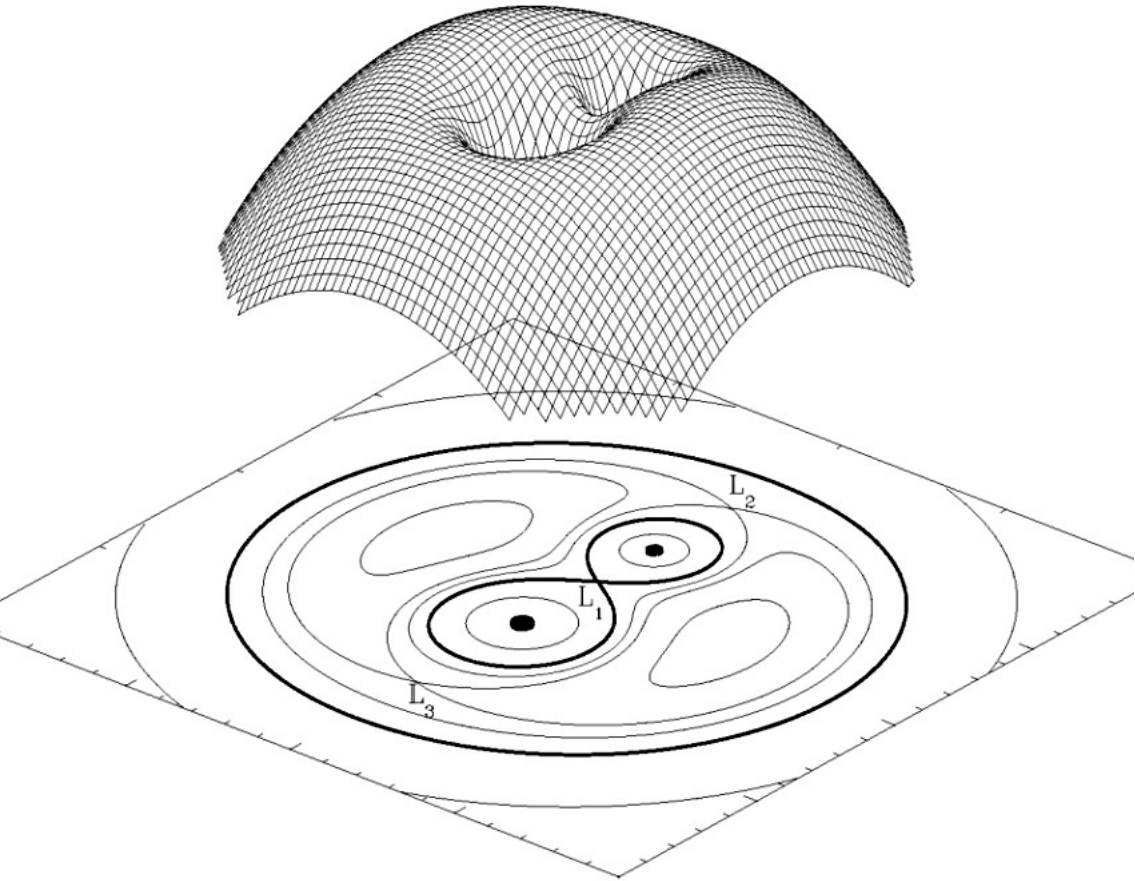
- August 17, 2017 at 12:41:04,4 UTC, by all three detectors.
- **A merging of two neutron stars**, $1.36\text{--}2.26 M_{\text{sun}}$ и $0.86\text{--}1.36 M_{\text{sun}}$, more than $0.025 M_{\text{sun}}$ emitted as grav. waves
- NGC 4993 (Hydra constellation), distance 40_{-14}^{+8} Mpc, ~ 130 mln. light years
- The source was localized inside 28 square degrees
- First electromagnetic component
- Short gamma-burst GRB 170817A in $1,74 \pm 0,05$ s after the maximum of the chirp
- Ultraviolet and infrared signals
- Radio and X-rays appeared only in a couple of days

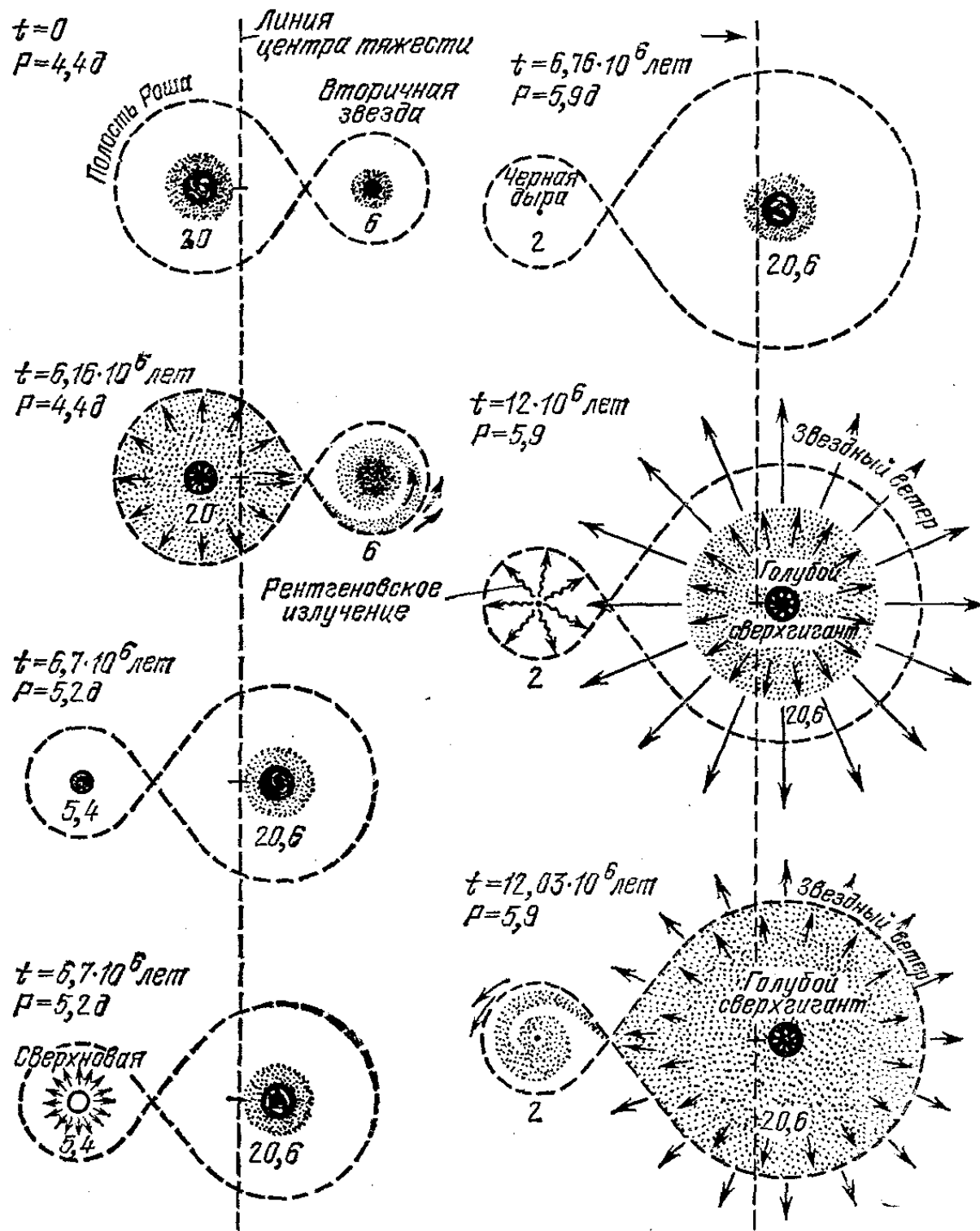


Aug 22

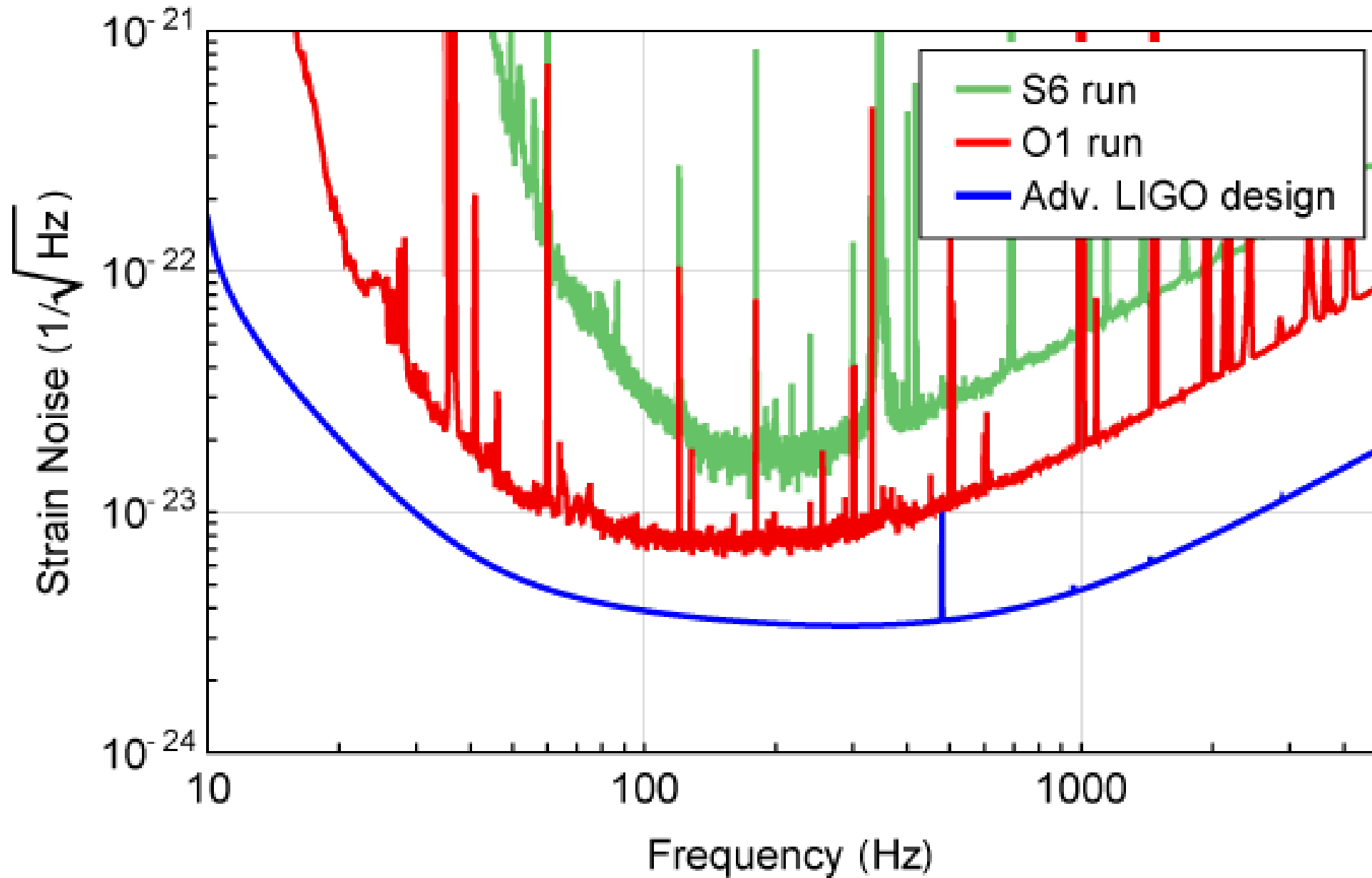


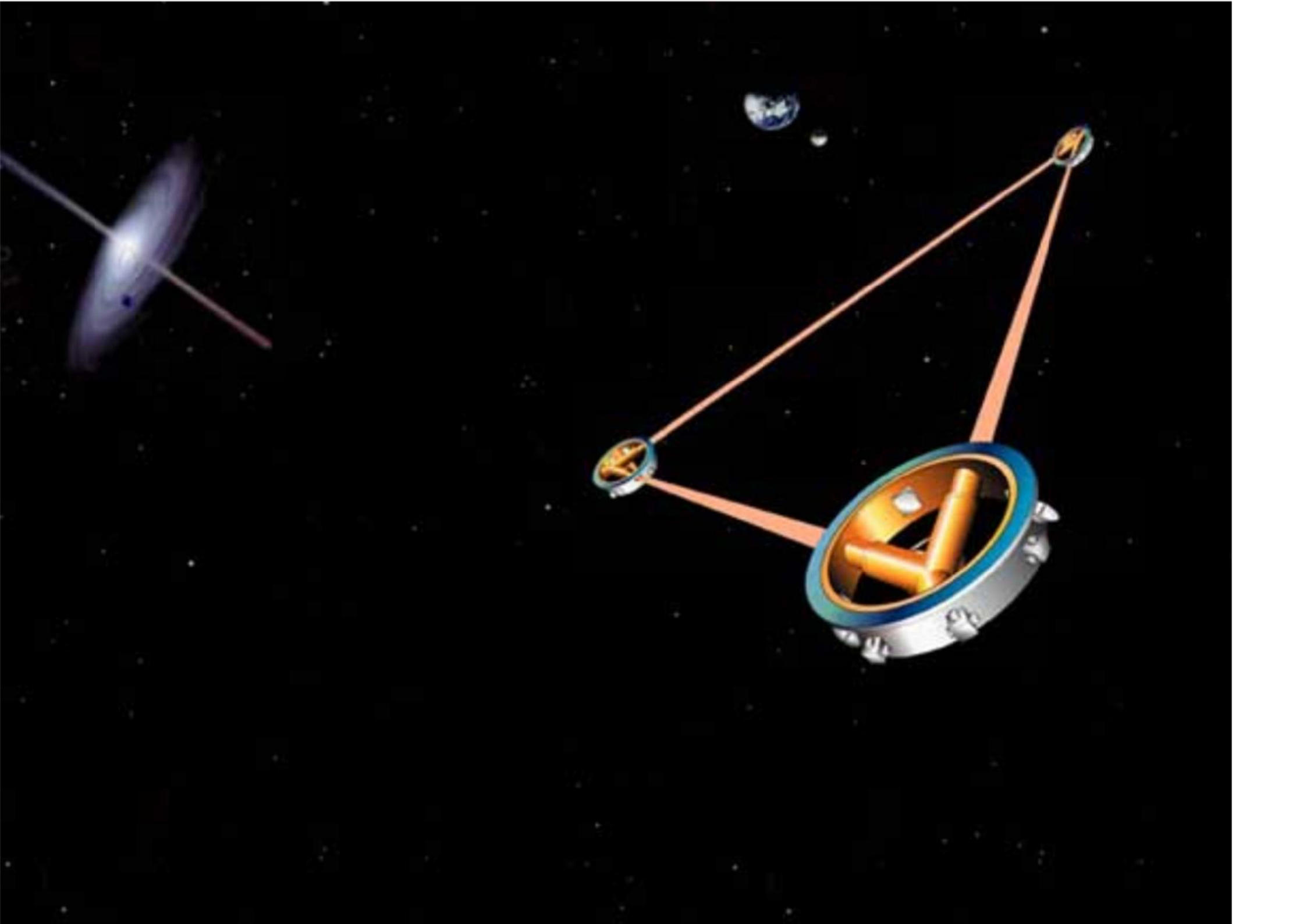
Evolution of a binary system



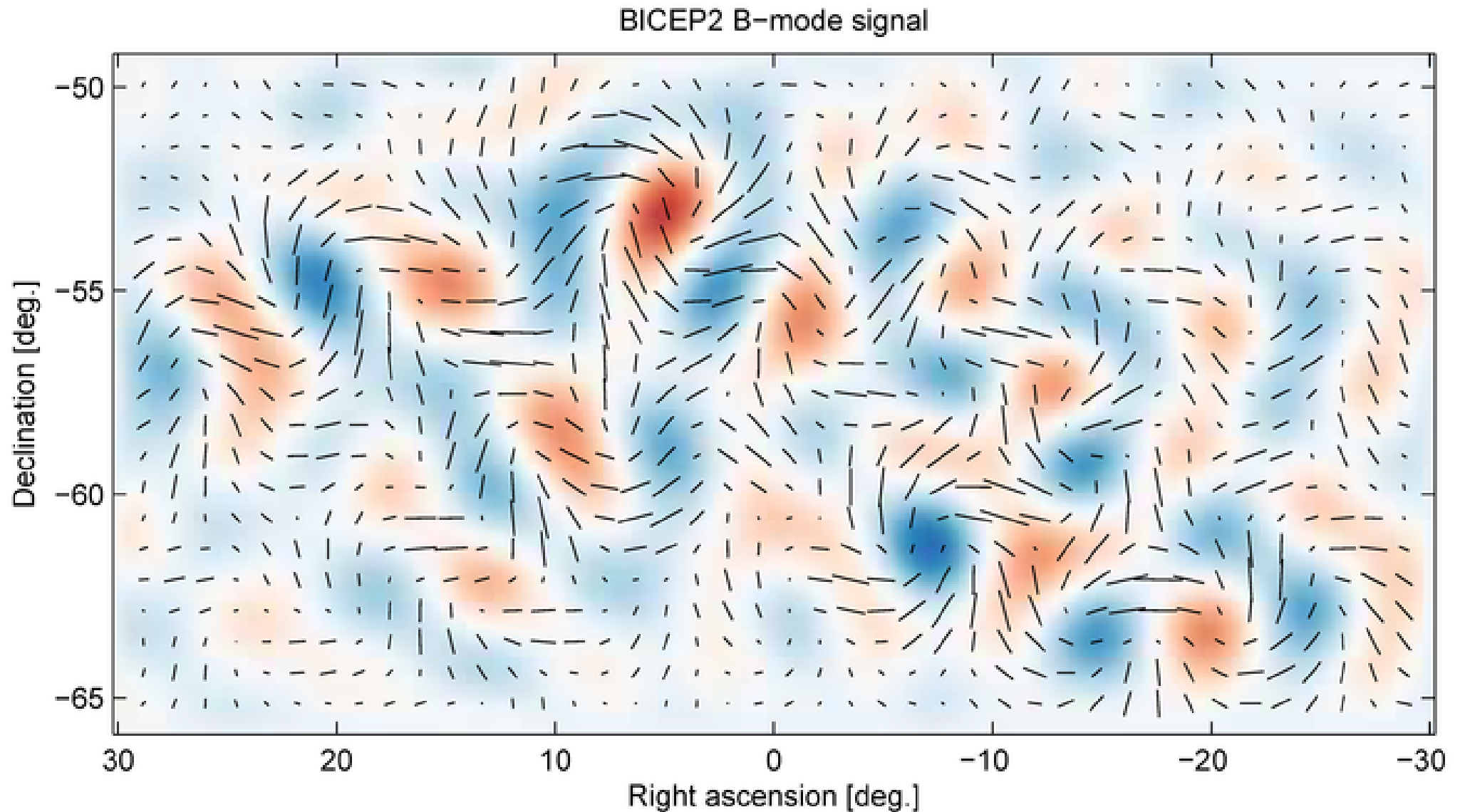


Advanced LIGO



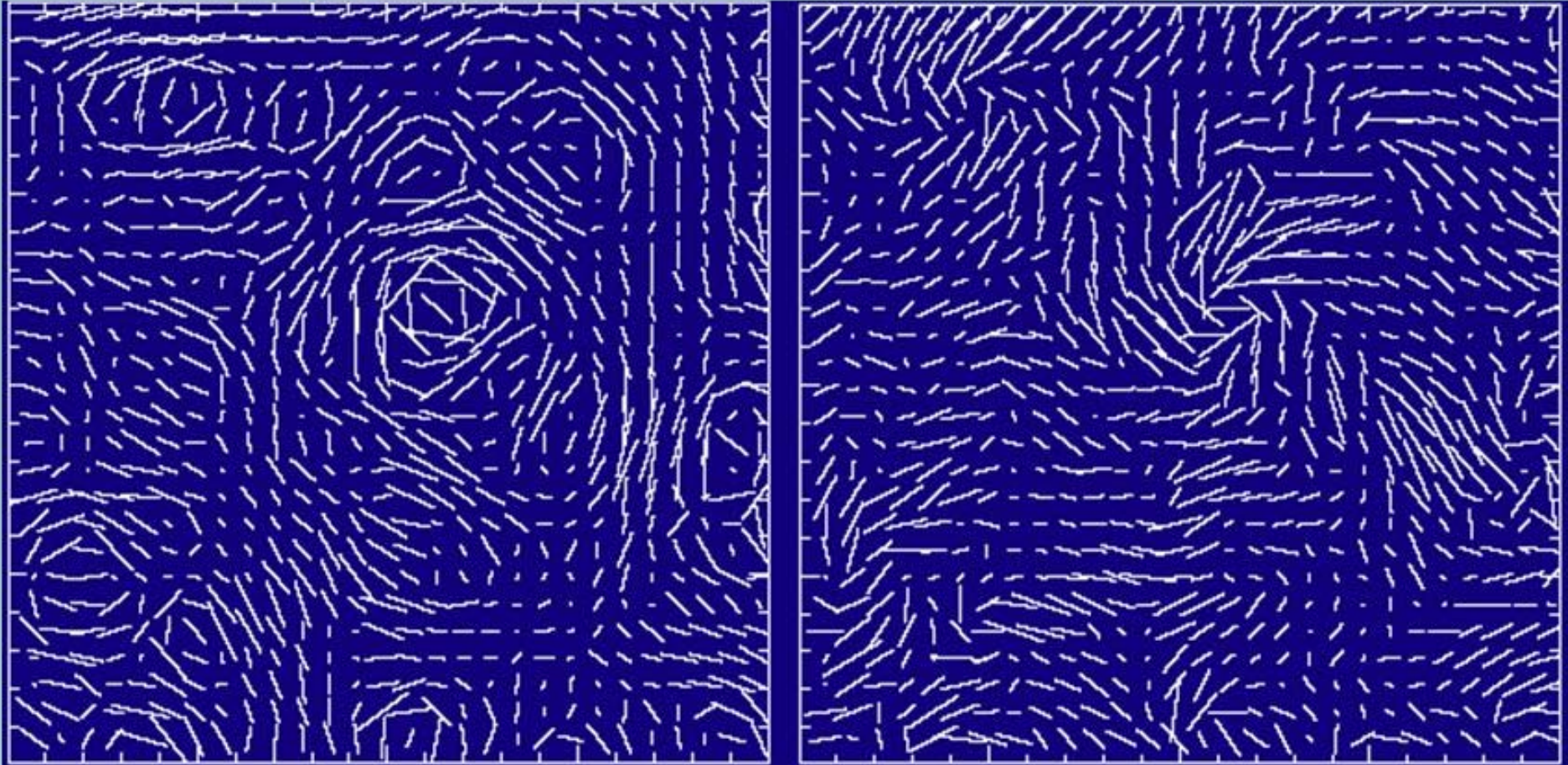


Cosmological gravitational waves



E-mode vs. B-mode

(curl-free) (curl)



Primordial tensors induce curl of CMB photons' polarization (B-mode).

Image: Seljak and Zaldarriaga