The international XVI-th Winter School “Actual Cosmology” organized by the Bogoliubov Laboratory of Theoretical Physics, JINR (Dubna)

Some questions of quantum cosmology

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February 1-7, 2021

Content

Introduction

- Quantum cosmology is a description of the universe as a unique quantum object.
- It is important from the point of view of the unification of gravity and quantum field theory.
- Quantum cosmology is connected with the inflationary cosmology.
- Mathematical structure of quantum cosmology is similar to that of theories of strings and superstrings.
- There is a connection between quantum cosmology and the foundations of quantum theory.

Interpretations of Quantum Mechanics and Quantum Cosmology

- Quantum mechanics and more generally quantum theory (including quantum statistical mechanics and quantum field theory) have had great achievements in the description of the microworld.
- The main feature of quantum mechanics, which distinguishes it from classical Newton mechanics is the fact, that even if one has a complete knowledge of a state of a system under consideration and would like to make a certain experiment, more than one alternative result of such an experiment is possible.
- The knowledge of the state of the system can permit us only to calculate the probabilities of different outcomes of the experiment.

- How can we see only one outcome of an experiment and what happens with all other alternatives?
- How can we reconcile ourselves with the fact that the physics of the microworld is described by quantum mechanics while in the macroworld, which we perceive in our everyday experience we encounter the laws of classical physics?
- Copenhagen interpretation: existence of the so called classical realm, where all the results of experiments and observations were registered.
- Classical physics was considered not only as a limiting case of the quantum physics, but also as a pre-requisite of its very existence.
- In quantum mechanics coexist two processes.
- A unitary deterministic evolution of the wave function, describing the quantum system, according to the Schrödinger equation.

- The second process takes place during quantum measurement and is called the reduction of the wave function.
- Is it possible to restore the classical determinism in quantum mechanics?
- De Broglie-Bohm interpretation: behind the curtain there are classical evolutions and trajectories, but we cannot know which one we follow.
- Hugh Everett's many-worlds interpretation of quantum mechanics: there is only one process: unitary evolution according to the Schrödinger equation. There is no reduction of the wave function.
- There is universal wave function or the wave function of the universe.
- It could be very well combined with the spirit of quantum cosmology.

- All the outcomes of the experiment co-exist and the objective result of the measurement under consideration is the establishment of correlations between the measured and measuring subsystems, which are treated on equal footing.
- The Everett interpretation of quantum mechanics is quite logical and economical, however, this economy was achieved by means of the acceptance of parallel existence of different outcomes of a quantum measurement.
- The many-worlds interpretation of quantum mechanics has become more popular due to the successes of cosmology and quantum informatics.

The problem of the preferred basis in the many-worlds interpretation of quantum mechanics

The very essence of the many-worlds interpretation can be expressed by one simple formula.

$$\hat{U}\,|\Phi\rangle_0|\Psi\rangle_i = |\Phi\rangle_i|\Psi\rangle_i.$$

The state $|\Psi\rangle_i$ is a quantum state of the object corresponding to a definite outcome of the experiment, $|\Phi\rangle_0$ is an initial state of the measuring device. The process of interaction between these two subsystems can be described by a unitary operator $\hat{U}$.

Let the initial state of the object be described by a superposition of quantum states:

$$|\Psi\rangle = \sum_i c_i |\Psi\rangle_i.$$

Then the superposition principle leads to

$$\hat{U}|\Phi\rangle_0|\Psi\rangle = \hat{U}|\Phi\rangle_0 \sum_i c_i|\Psi\rangle_i = \sum_i c_i|\Phi\rangle_i \Psi\rangle_i.$$

$|\Phi\rangle_i$ describes the state of the measuring device, which has found the quantum object in the state $|\Psi\rangle_i$.
All the terms of the superposition are realized but in different universes.

Decomposing the wave function of the universe one should choose a certain basis. The result of the decomposition essentially depends on it. Thus, the so called problem of the choice of the preferred basis arises.

Schmidt or bi-orthogonal basis

The essence of the problem can be formulated considering a quantum system consisting of two subsystems. The only essential characteristic of the branching process is the defactorization of the wave function.
Initially,

$$|\Psi\rangle = |\phi\rangle|\chi\rangle.$$

after an interaction between the subsystems

$$\sum_i c_i|\phi\rangle_i|\chi\rangle_i,$$

where more than one coefficient c_i is different from zero.

The decomposition can be done in various manners.
The Schmidt or bi-orthogonal basis is formed by eigenvectors of both the density matrices of the subsystems of the quantum system.

$$\hat{\rho}_I = \mathrm{Tr}_{II} |\Psi\rangle\langle\Psi|,$$

$$\hat{\rho}_{II} = \mathrm{Tr}_{I} |\Psi\rangle\langle\Psi|.$$

The eigenvalues of the density matrices coincide and hence the number of non-zero eigenvalues is the same.

$$\hat{\rho}_I |\phi_n\rangle = \lambda_n |\phi_n\rangle,$$

$$\hat{\rho}_{II} |\chi_n\rangle = \lambda_n |\chi_n\rangle.$$

$$|\Psi\rangle = \sum_n \sqrt{\lambda_n} |\phi_n\rangle |\chi_n\rangle.$$

The bi-orthogonal basis is defined by the fixing of the decomposition of the system into subsystems.

The decomposition into the subsystems should be such that the corresponding preferred basis would be rather stable.

For example, when one treats a quantum mechanical experiment of the Stern-Gerlach type, it is natural to consider the measuring device and the atom as subsystems.

In the case when we consider a system with some kind of internal symmetry, like in quantum chromodynamics, the division of the system into subsystems which belong to singlet representations of the internal symmetry group looks also reasonable from the point of view of stability of the bi-orthogonal preferred basis of the many-worlds interpretation.

Dynamics of the preferred basis

The wave function

$$|\Psi\rangle = \sum_n c_n |n\rangle_I |n\rangle_{II}.$$

satisfies the Schrödinger equation

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle,$$

$$H = H_I + H_{II} + V,$$

where V is the interaction Hamiltonian between the subsystems.

One can show that the evolution of the preferred basis vectors is unitary and is governed by some effective Hamiltonians:

$$i\frac{\partial}{\partial t}|m\rangle_I = \mathcal{H}_I|m\rangle_I,$$

$$\mathcal{H}_I = i\sum_m \left(\frac{\partial|m\rangle_I}{\partial t}\ {}_I\langle m|\right).$$

Non-diagonal elements of $\mathcal{H}_I$ are given by

$$(\mathcal{H}_\mathcal{I})_{mn} = (H_I)_{mn} - \frac{{}_I\langle m|\mathrm{Tr}_{II}[V, |\Psi\rangle\langle\Psi|]|n\rangle_I}{p_m - p_n},$$

and, similarly, for $\mathcal{H}_{II}$. The diagonal elements can be fixed as

$$(\mathcal{H}_I)_{mm} = (H_I)_{mm};\ (\mathcal{H}_{II})_{mm} = (H_{II})_{mm},$$

whence it follows that

$$i\frac{\partial c_n}{\partial t} = {}_I\langle n|_{II}\langle n|V|\Psi\rangle.$$

The effective Hamiltonians turn out to be complicated functionals not only of the Hamiltonian H, but also of the quantum state $|\Psi\rangle$ of the system.

The most unexpected conclusion from the unitary dynamics of the proposed basis: Since the observer (identified, for example, with the subsystem I) observes and measures only one relative state of the second subsystem II in his many-worlds branch, he finds that this state undergoes a unitary evolution of the above type. This is in spite of the impure nature of this open subsystem II described by a non-factorizable density matrix.

The second conclusion: the observer studying the dynamics of his relative state measures the effective Hamiltonian $\mathcal{H}_{II}$ and not the fundamental Hamiltonian H of the total system. This means that research into nature at the most fundamental levels requires additional efforts in reconstructing the fundamental dynamical laws on the grounds of the observable reality.

Decoherence

Let us consider the Stern-Gerlach experiment, where the initial state of the atom and of the device is described by the vector

$$|\Psi\rangle = |\Phi\rangle(c_1|\psi_\uparrow\rangle + c_2|\psi_\downarrow\rangle).$$

As a result of the measurement, we have

$$|\Psi\rangle_{\text{final}} = c_1|\Phi_\uparrow\rangle|\psi_\uparrow\rangle + c_2|\Phi_\downarrow\rangle|\psi_\downarrow\rangle.$$

In every measurement process a third system participates - it is the environment. Hence, the initial state of the general system is

$$|\Psi\rangle = |\chi\rangle|\Phi\rangle(c_1|\psi_\uparrow\rangle + c_2|\psi_\downarrow\rangle),$$

while its final state is

$$|\Psi\rangle_{\rm final} = c_1|\chi_\uparrow\rangle|\Phi_\uparrow\rangle|\psi_\uparrow\rangle + c_2|\chi_\downarrow\rangle|\Phi_\downarrow\rangle|\psi_\downarrow\rangle.$$

If we calculate the reduced density matrix tracing out the environmental degrees of freedom, we obtain

$$\begin{aligned}\rho_{\rm reduced} &= {\rm Tr}_{\{\chi\}}|\Psi\rangle_{\rm final\ final}\langle\Psi| \\ &= |c_1|^2|\Phi_\uparrow\rangle\langle\Phi_\uparrow|\psi_\uparrow\rangle\langle\psi_\uparrow| + |c_2|^2|\Phi_\downarrow\rangle\langle\Phi_\downarrow|\psi_\downarrow\rangle\langle\psi_\downarrow|,\end{aligned}$$

where we have used the fact that

$$\langle\chi_\uparrow|\chi_\downarrow\rangle = 0.$$

This expression represents a classical statistical mixture, which substitutes the quantum state due to tracing out the environmental degrees of freedom.
This is the essence of the decoherence approach.
In many cases this reduced density matrix becomes quickly practically diagonal in a certain "good" basis, whose states are sometimes called "pointer states" and behaves more or less classically.
From our point of view the decoherence approach to the problem of quantum measurement and to the problem of classical-quantum relations is less fundamental than the many-worlds approach.

First, the transition to a classical statistical mixture does not resolve the problem of choice between different alternatives.
Second, the decoherence properties of reduced density matrix depend crucially on the choice of the basis.
In the bi-orthogonal preferred basis approach, the basis is defined by the chosen decomposition of the system under consideration into subsystems.
After that, one can study the dynamics of different elements of the basis and to see if they behave classically.
It appears, that sometimes classicality exists as a stable phenomenon, sometimes as a temporary phenomenon and sometimes it does not exist at all.

Decoherence and ultraviolet divergences in quantum cosmology

What is the environment in quantum cosmology?
There is no external environment, because the object of quantum cosmology is the whole Universe.
We should treat some part of degrees of freedom as essential and observable, while the others could be treated as an environment with subsequent tracing them out in transition to a reduced density matrix.
It is natural to believe that inhomogeneous degrees of freedom play the role of environment while macroscopic variables such as a cosmological radius or initial value of the inflaton scalar field are treated as observables.

What is a quantum state (wave function) of the Universe?
General Relativity is the theory with the constraints (just like electrodynamics or Yang-Mills theory).
Making the Legendre transformation to come from the Lagrange formalism to the Hamilton formalism, one discovers that the Hamiltonian of the General Relativity is proportional a linear combination of constraints.
According to Dirac we should require that the quantum state of the system under consideration is eliminated by the constraint operators.
In quantum cosmology it implies the existence of the Wheeler-DeWitt equation:

$$\hat{H}|\Psi\rangle = 0.$$

The Wheeler-DeWitt equation is very complicated.
There are relatively simple prescriptions for its solutions - the no-boundary prescription and the tunneling prescription.

$$ds^2 = (N^2 - N_a N^a)dt^2 - 2N_i dx^a dt - g_{ab} dx^a dx^b.$$

Arnowitt-Deser-Misner formalism.
The lapse function N and the shift function N^a are Lagrange multipliers. To this multipliers correspond the constraints:

$$H = G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{g}^3 R,$$

where

$$G_{abcd} = \frac{1}{2}\sqrt{g}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}),$$

is called the DeWitt supermetric,

π^{ab} are the conjugate momenta corresponding to the metric components g_{ab}.
Sometimes H is called super-Hamiltonian.
To the shift functions corresponds the constraints

$$H_a = -2g_{ac}\pi^{cd}_{|d}.$$

They describe the invariance of the action with respect to spatial diffeomorphisms and are called supermomenta.
Quantization:

$$\hat{\pi}^{ab} = -i\hbar\frac{\delta}{\delta g_{ab}}.$$

The Hamiltonian is proportional to the linear combination of the constraints and vanishes on the constraint surface.
The time vanishes.
It is called the problem of time.
The formal solution of the Wheeler-DeWitt equation can be represented as a path integral.
One can use there the Euclidean action.
As is known one can describe the quantum tunneling using classical equations of motions in the Euclidean time - instantons.
One can describe the the quantum birth of the Universe as a tunneling from nothing.

It is convenient to write both the no-boundary and tunneling cosmological wave functions in the form:

$$\Psi(t|\varphi, f) = \frac{1}{\sqrt{v_\varphi^*(t)}} \exp\left(\mp I(\varphi)/2 + iS(t,\varphi)\right) \times \prod_n \psi_n(t,\varphi|f_n),$$

$$\psi_n(t,\varphi|f_n) = \frac{1}{\sqrt{v_n^*(t)}} \exp\left(-\frac{1}{2}\Omega_n(t) f_n^2\right),$$

$$\Omega_n(t) = -a^k(t)\frac{\dot{v}_n^*(t)}{v_n^*(t)}.$$

The sign minus or plus in front of Euclidean action $I(\varphi)$ in the exponential corresponds to the no-boundary and to the tunneling wave functions of the Universe, respectively, f_n describe amplitudes of inhomogeneous modes, v_n correspond to solutions of linearized second-order differential equations.

The diagonal of the reduced density matrix corresponding to this wave function

$$\rho(t|\varphi) \equiv \rho(t|\varphi,\varphi) = \int \prod_n df_n |\,\Psi(t|\varphi,f)\,|^2$$

is

$$\rho(t|\varphi) = \frac{\sqrt{\Delta_\varphi}}{|v_\varphi(t)|} \exp\Big(\mp I(\varphi) - \varGamma_{1-\mathrm{loop}}(\varphi)\Big),$$

where

$$\Delta_\varphi \equiv i a^k (v_\varphi^* \dot{v}_\varphi - \dot{v}_\varphi^* v_\varphi).$$

is the Wronskian of the φ-mode functions and $\varGamma_{1-\mathrm{loop}}$ is the one-loop effective action calculated on the DeSitter instanton of the radius $1/H(\varphi)$, where $H(\varphi)$ is the effective Hubble constant.

When $H(\varphi) \to \infty$,

$$\Gamma_{1-\mathrm{loop}} = Z \ln \frac{H(\varphi)}{\mu},$$

where Z is the anomalous scaling of the theory, μ is a renormalization scale.

The condition of the normalizability of the wave function of the Universe is

$$Z > 1.$$

This condition provides us with the selection criterium for particle physics models.

Information about the decoherence behaviour of the system is contained in the off-diagonal elements of the density matrix.

$$\rho(t|\varphi,\varphi') = \left(\frac{\Delta_\varphi \Delta_{\varphi'}}{v_\varphi v^*_{\varphi'}}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}\Gamma - \frac{1}{2}\Gamma' + i(S - S')\right) D(t|\varphi,\varphi')$$

Here $D(t|\varphi,\varphi')$ is the so called decoherence factor:

$$D(t|\varphi,\varphi') = \prod_n \left(\frac{4Re\Omega_n Re\Omega'^*_n}{(\Omega_n + \Omega'^*_n)^2}\right)^{\frac{1}{4}} \left(\frac{v_n v'^*_n}{v^*_n v'_n}\right)^{\frac{1}{4}}.$$

How to cope with ultraviolet divergences appearing in the sum of this type?

One can try to use the dimensional regularization.
The main effect of the dimensional regularization consists in changing the number of degrees of freedom involved in summation.
For example, for a scalar field, the degeneracy number of harmonics in spacetime of dimensionality d changes from the well-known value

$$\dim(n,4) = n^2,$$

to

$$\dim(n,d) = \frac{(2n+d-4)\Gamma(n+d-3)}{\Gamma(n)\Gamma(d-1)}.$$

Making analytical continuation and discarding the poles $1/(d-4)$ one has finite values for $D(t|\varphi, \varphi')$.
However, for scalar, photon and graviton fields one gets an oversubtraction of UV-infinities:

$$|D(t|\varphi, \varphi')| \to \infty, \text{ at } |\varphi - \varphi'| \to \infty.$$

For example, for a massive scalar field

$$\ln |D(t|\varphi, \varphi')| \approx \frac{7}{64} m^3 \bar{a}(a - a')^2,$$

$$a = \frac{1}{H(\varphi)} \cosh H(\varphi)t,$$
$$a' = \frac{1}{H(\varphi')} \cosh H(\varphi')t,$$
$$\bar{a} = \frac{a + a'}{2}.$$

Such a form of a decoherence factor not only does not correspond to decoherence, but also renders the density matrix ill-defined, breaking the condition $\mathrm{Tr}(\rho^2) \leq 1$.
Using the reparametrization of a bosonic scalar field

$$f \to \tilde{f} = a^{\mu} f, \;\; v_n \to \tilde{v}_n = a^{\mu} v_n,$$

one can get the new form of the frequency function

$$\Omega_n(t) = -i a^{3-2\mu}(t) \frac{\dot{\tilde{v}}_n^*(t)}{\tilde{v}_n^*(t)}.$$

In such a way one can suppress ultraviolet divergences.
For the conformal parametrization, $\mu = 1$, for the massive scalar field one has

$$\ln |\tilde{D}(t|\varphi, \varphi')| = -\frac{m^3 \pi \bar{a}(a - a')^2}{64}.$$

For the case of fermions this trick does not work.
The wave function of the Universe filled by fermions has the form

$$\Psi(t,\varphi|x,y) = \Psi_0(t,\varphi)\prod_n \psi_n(t|x_n,y_n),$$

where x, y are Grassmann variables.
Partial wave functions have the form

$$\psi_n(t|x_n,y_n) = v_n - \frac{i\dot{v}_n + \nu v_n}{m} x_n y_n,$$

where the functions v_n satisfy the second-order equation

$$\ddot{v}_n + (-i\dot{\nu} + m^2 + \nu^2)v_n = 0, \quad \nu = \frac{n+\frac{1}{2}}{a}.$$

$$|D(a,\varphi|a',\varphi')| = \exp\left(-\frac{m^2(a-a')^2}{8}\sum_{n=1}\frac{n(n+1)}{\left(n+\frac{1}{2}\right)^2}\right).$$

One can try eliminating ultraviolet divergences by dimensional regularization using the fact that for spinors in spacetime of dimensionality d:

$$\dim(n,d) = \frac{\Gamma(n+2^{(d-2)})\Gamma(n+2^{(d-2)/2}-1)}{[\Gamma(2^{(d-2)/2})]^2\Gamma(n+1)\Gamma(n)}.$$

$$|D(a,\varphi|a',\varphi')| = \exp\left(-\frac{m^2(a-a')^2}{8}I\right),\ \ I<0,$$

and we encounter the same problem as in the case of bosons.

One cannot use the conformal reparametrization in this case because standard fermion variables are already presented in the conformal parametrization.
However, there is another way to circumvent this problem.
One can perform a non-local Bogoliubov transformation mixing Grassmann variables x and y.
Choosing it in a certain way one can suppress ultraviolet divergences.
The reasonable idea is to fix this transformation by the requirement that decoherence is absent in a static spacetime.
Then:

$$|D(a,\varphi|a',\varphi')| = \exp\left(-\frac{\pi^2 m^2 (a-a')^2}{192} I\right), \quad I > 0,$$

and is finite.

The main conclusion to be drawn from the above examples is the fact that consistency of the reduced density matrix might determine the very definition of the environment in quantum cosmology.

Classical - quantum duality

Speaking about the problem of time in quantum cosmology and generally, in quantum mechanics, one can remember that some analogue of the classical time can be introduced even in a system with one degree of freedom.
Let us consider a particle with one spatial coordinate and a stable probability distribution for this coordinate.
One can suppose that behind this probability distribution there is a classical motion which we can observe stroboscopically.
We can detect its position many times and obtain a probability distribution for this position.
Classically this measured probability is inversely proportional to the velocity of the particle.
The higher is the velocity of a particle in some region of the space the lesser is the time that it spends there.

In quantum mechanics this probability is given by the squared modulus of its wave function.

$$\psi^*(x)\psi(x) = \frac{1}{|v(x)|T},$$

where T is a normalising time scale, for example, a half period of the motion of the particle.

In this spirit, the probability distributions for the energy eigenstates of the hydrogen atom with a large principal quantum number n were studied.

It was shown that the distributions with the orbital quantum number l having the maximal possible value $l = n - 1$, describe the corresponding classical motion of the electron on the circular orbit.

In contrast, the state with $l = 0$ cannot produce immediately a correct classical limit.
To arrive at such a limit, which represents a classical radial motion of a particle (i.e. on a degenerate ellipse) one should apply a coarse-graining procedure based on the Riemann - Lebesgue theorem.

There is another interesting example: the harmonic oscillator with a large value of the quantum number n. In this case, making a coarse-graining of the probability density one can again reproduce a classical motion of the oscillator.

When one studies the question of the classical-quantum correspondence, one looks for situations where this correspondence is realised.
However, it is reasonable to suppose that such situations are not always realised.
We would like to attract attention to another phenomenon: a particular quantum-classical duality between the systems governed by different Hamiltonians.
Let us suppose that we have a classical motion of the harmonic oscillator, governed by the law

$$x(t) = x_0 \sin \omega t.$$

The velocity is

$$\dot{x}(t) = \omega x_0 \cos \omega t.$$

We can believe that behind this classical motion there is a stationary wave function

$$\psi(x) = \frac{1}{\sqrt{\pi}(x_0^2 - x^2)^{1/4}} e^{if(x)} \theta(x_0^2 - x^2),$$

where θ is the Heaviside theta-function and f is a real function.

Applying the energy conservation law and the stationary Schrödinger equation we find the corresponding potential for the quantum problem:

$$V(x) = \frac{m\omega^2 x_0^2}{2} + \frac{\hbar^2}{2m} \left(\frac{1}{2(x_0^2 - x^2)} + \frac{5x^2}{4(x_0^2 - x^2)^2} + if'' + if' \frac{x}{x_0^2 - x^2} - f'^2 \right),$$

$$\text{if } x^2 < x_0^2.$$

To guarantee the reality of the potential and, hence, the hermiticity of the Hamiltonian, we must choose the phase function f such that

$$f' = C\sqrt{x_0^2 - x^2},$$

where C is a real constant.
Then the potential is equal to

$$V(x) = \frac{m\omega^2 x_0^2}{2} + \frac{\hbar^2}{2m}\left(\frac{1}{2(x_0^2 - x^2)} + \frac{5x^2}{4(x_0^2 - x^2)^2} + C^2(x^2 - x_0^2)\right), \quad \text{if } x^2 < x_0^2.$$

Then for $x^2 > x_0^2$ we can treat the value of the potential as infinite since there the wave function is zero.

This example is rather artificial.

We have elaborated it to hint at the possibility of encountering a similar effect in cosmology.

One can imagine a situation where behind the visible classical evolution of the universe looms a quantum system, whose Hamiltonian is quite different from the classical Hamiltonian governing this visible classical evolution.

Many-worlds interpretation, probabilities and Anthropic Principle

The probability treatment of the predictions of quantum mechanics is quite natural when one speaks about multiple experiments or about multiple identical systems.
What sense can it have when one considers the Universe as a whole?
Can we say that one branch of the wave function of the Universe is more probable than another?
Here arises the idea to combine the many-worlds interpretation of quantum mechanics with the Anthropic Principle.
At first glance, one can think that the many-worlds interpretation is "anti-anthropic" because it deprives a human being of its privileged position in the Universe.

It can be combined quite harmoniously with the anthropic principle and cosmology.

The Universe is described by a unique wave function, which is adequate to a quantum reality where all possible versions of the evolution are realized.

In some of the branches of the wave function of the universe not only the classical properties, but also the Life and Mind arise, while in other there is nothing similar.

Thus, the world described by our branch of the wave function exists not because it had to arise because of some necessity, but because it was possible and all the possibilities are realized.

It is not necessary to require that our world and other similar worlds are the most probable from the point of view of the measure on the Hilbert space, where the wave function of the Universe is defined.

It could be just in the opposite way.

Just like Life is localized in a rather small part of the usual space, it could be localized in a tiny part of the Hilbert space.

However, it does not mean that everything is possible!

The requirements of consistency of theories can impose rather stringent restrictions on the concrete physical laws which govern dynamics in all possible branches of the wave function of the universe.

Density matrix of the universe and the cosmological bootstrap

What if the absence of quantum coherence is fundamentally encoded in the quantum state of the Universe?
In other words, a fundamental quantum state of the Universe is mixed and is described by a cosmological density matrix?

Such a state can arise in the framework of the Euclidean Quantum Gravity path integral.
A mixed state of the universe arises naturally if there exists an instanton with two turning points (surfaces of vanishing external curvature). Such an instanton naturally arises if one considers a closed Friedmann universe with the following two essential ingredients: effective cosmological constant and radiation corresponding to the set of conformally invariant quantum fields.

The Euclidean Friedmann equation in this case is

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^2} - H^2 - \frac{C}{a^4},$$

where $H^2 = \Lambda/3$ is an effective cosmological constant and the constant C characterizes the amount of radiation in the universe.
The turning points for solutions of this equation are

$$a_{\pm} = \frac{1}{\sqrt{2}H}\sqrt{1 \pm (1 - 4CH^2)^{1/2}}, \quad 4CH^2 \leq 1.$$

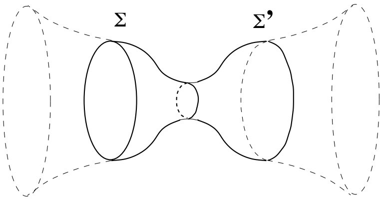


Figure: Picture of instanton representing the density matrix. Dashed lines depict the Lorentzian Universe nucleating from the instanton at the minimal surfaces Σ and Σ'.

For the pure quantum state the instanton bridge between Σ and Σ' breaks down. However, the radiation stress tensor prevents these half instantons from decoupling – the minimal value a_- stays nonzero.

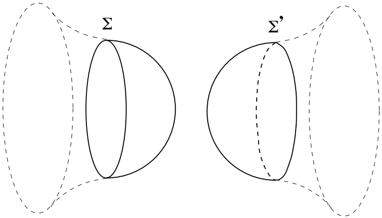


Figure: Density matrix of the pure Hartle-Hawking state represented by the union of two no-boundary vacuum instantons.

The relevant density matrix is the path integral over metric and matter field histories interpolating between their boundary values at Σ and Σ',

$$\rho[\varphi,\varphi'] = e^{\Gamma} \int\limits_{g,\phi\big|_{\Sigma,\Sigma'}=(\varphi,\varphi')} D[g,\phi] \exp\big(-S_{\rm E}[g,\phi]\big).$$

Here $S_{\rm E}[g,\phi]$ is the Euclidean action of the model.
The partition function $e^{-\Gamma}$ for this density matrix follows from integrating out the field φ in the coincidence limit of its two-point kernel at $\varphi'=\varphi$.
This corresponds to the identification of Σ' and Σ, the underlying Euclidean spacetime acquiring the "donut" topology $S^1\times S^3$.
The semiclassical saddle point of the path integral for $e^{-\Gamma}$ is just the instanton of the above type.

The metric of this instanton is conformally equivalent to the metric of the Einstein static universe:

$$ds^2 = d\eta^2 + d^2\Omega^{(3)},$$

where η is the conformal time related to the cosmic time τ by the relation $d\eta = d\tau/a(\tau)$.
This opens the possibility of exact calculations for conformally invariant quantum fields, because their effective action on this minisuperspace background is exhausted by the contribution of the conformal anomaly, relevant Casimir energy and free energy.
At the quantum level the Friedmann equation gets modified to

$$\frac{\dot{a}^2}{a^2} + B\left(\frac{1}{2}\frac{\dot{a}^4}{a^4} - \frac{\dot{a}^2}{a^4}\right) = \frac{1}{a^2} - H^2 - \frac{C}{a^4},$$

where the amount of radiation constant C is given by the bootstrap equation

$$m_P^2 C = m_P^2 \frac{B}{2} + \frac{dF(\eta)}{d\eta} \equiv \frac{B}{2} m_P^2 + \sum_\omega \frac{\omega}{e^{\omega\eta} - 1}.$$

Here $F(\eta)$ is the free energy which for the conformally coupled scalar field is given by the series of terms contributed by field-theoretical oscillators with frequencies $\omega/a(\tau)$ on a 3-sphere of the radius $a(\tau)$

$$F(\eta) = \sum_\omega \ln\left(1 - e^{-\omega\eta}\right) = \sum_{n=1}^{\infty} n^2 \ln\left(1 - e^{-n\eta}\right).$$

Here η is the period of the cosmological instanton in units of the conformal time – effective inverse temperature of the gas of conformal particles. The constant $B = \beta/8\pi^2 M_P^2$ here describes the contribution associated with the conformal anomaly and Casimir energy of the model, where β is a dimensionless coefficient of the Gauss-Bonnet term of the stress tensor trace anomaly. Similar expressions hold for other conformally invariant fields of higher spins.

We have obtained a highly non-trivial system of equations. While the geometry of the instanton depends on the amount of radiation through the modified Friedmann equation, the amount of radiation, in turn, depends on the parameters of the instanton. We called this phenomenon "cosmological bootstrap".

The Friedmann equation can be rewritten as

$$\dot{a}^2 = \sqrt{\frac{(a^2 - B)^2}{B^2} + \frac{2H^2}{B}(a_+^2 - a^2)(a^2 - a_-^2)} - \frac{(a^2 - B)}{B}$$

and has the same two turning points $a_\pm$ as in the classical case provided

$$a_-^2 \geq B.$$

This requirement is equivalent to

$$C \geq B - B^2H^2, \quad BH^2 \leq \frac{1}{2}.$$

Together with $CH^2 \leq 1/4$ the admissible domain for instantons on a two-dimensional plane of C and H^2 reduces to the curvilinear wedge below the hyperbola and above the straight line to the left of the critical point:

$$C = \frac{B}{2}, \quad H^2 = \frac{1}{2B}.$$

More detailed analysis shows that cosmological instantons form one-parameter families classified by the number of oscillations of the scale factor during the instanton time period $k = 1, 2, ...$. Because of these oscillations they can be called garlands.

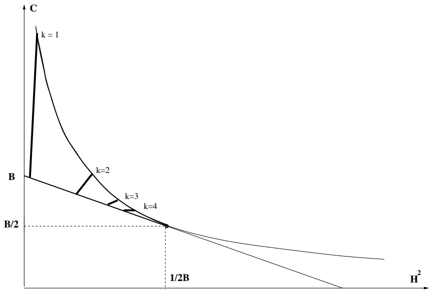


Figure: The instanton domain in the (H^2, C)-plane is located between bold segments of the upper hyperbolic boundary and lower straight line boundary. The first one-parameter family of instantons is labeled by $k = 1$. Families of garlands are qualitatively shown for $k = 2, 3, 4$. $(1/2B, B/2)$ is the critical point of accumulation of the infinite sequence of garland families.

The suggested approach allows one to resolve the problem of the so-called infrared catastrophe for the no-boundary state of the Universe based on the Hartle-Hawking instanton.
The Euclidean action on this instanton is negative and inversely proportional to the value of the effective cosmological constant.
This means that the probability of the universe creation with an infinitely big size is infinitely high.
The effect of conformal anomaly allows one to avoid this counter-intuitive conclusion.
One can construct instantons with one turning point which smoothly closes at $a_- = 0$ with $\dot{a}(\tau_-) = 1$.
Such instantons correspond to the Hartle-Hawking pure quantum state.
The on-shell effective action

$$\varGamma_0 = F(\eta) - \eta \frac{dF(\eta)}{d\eta} + 4m_P^2 \int_{a_-}^{a_+} \frac{da\dot{a}}{a} \left(B - a^2 - \frac{B\dot{a}^2}{3} \right),$$

diverges to plus infinity.

Indeed, for $a_- = 0$ and $\dot{a}_- = 1$

$$\eta = \int_0^{a_+} \frac{da}{\dot{a}a} = \infty, \quad F(\infty) = F'(\infty) = 0,$$

and the effective Euclidean action diverges at the lower limit to $+\infty$.

$$\varGamma_0 = +\infty, \quad \exp(-\varGamma_0) = 0,$$

and this fact completely rules out all pure-state instantons, and only mixed quantum states of the universe with finite values of the effective Euclidean action $\varGamma_0$ turn out to be admissible.
This is a dynamical mechanism of selection of mixed states in the cosmological ensemble described by the density matrix.

All the references can be found in the paper by
A. O. Barvinsky and A.Yu. Kamenshchik,
Preferred basis, decoherence and a quantum state of the Universe,
arXiv:2006.16812[gr-qc],
to be published in the Springer series "Fundamental Theories of Physics", in the volume, dedicated to the memory of H. D. Zeh.

Are classical physics and cosmology deterministic?

Let us consider a simple mechanical model: Norton's Dome.
A classical point particle finds itself at some maximum of the potential

$$V = V_0 - V_1 x^{\frac{3}{2}},\ \ V_0 > 0,\ \ V_1 > 0.$$

To such a potential corresponds a surface which is not spherical but has some kind of cusp at the top.
The second Newton law for the particle is

$$\ddot{x} = \frac{3}{2} V_1 x^{\frac{1}{2}}.$$

Obviously, we have a solution:

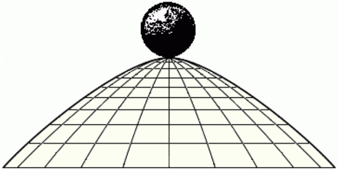
$$x(t) = 0.$$

There is also a whole family of nontrivial solutions:

$$x(t) = \frac{V_1}{64}(t - t_0)^4.$$

Here t_0 is the moment of the beginning of the motion "rolling down" of the particle.
There is no reason to begin falling at $t = t_0$ and there is no probability interpretation here.

John D. Norton (November 2003), "Causation as Folk Science", Philosophers' Imprint. 3 (4): 122.



It was possible also to have the potential

$$V = V_0 - V_1 x^{\alpha}, \; V_0 > 0, \; V_1 > 0, \; 1 < \alpha < 2.$$

Then we have two solutions:

$$x(t) = 0$$

and

$$x(t) = \left(\frac{V_1}{\alpha - 1}\right)^{\frac{1}{2-\alpha}} (t - t_0)^{\frac{2}{2-\alpha}}.$$

Norton's Dome and cosmology

V. Husain and V. Tasic,
"An Indeterminate Universe: Dark Energy and Norton's Dome",
arXiv:2011.01450
The Friedmann equations for a flat universe filled with a perfect fluid with

$$p = w\rho, \ -1 > w > -\frac{2}{3}$$

are written in such a form that they are nonsingular at $a = 0$ and permit both the solutions $a(t) = 0$ and

$$a(t) = a_0(t - t_0)^{\frac{2}{3(1+w)}}.$$

At $w = -\frac{5}{6}$, one has a perfect analogy with the Norton's Dome.

The first Friedmann equation was written in such a form:

$$\dot{a} = Ca^{-\frac{3w+1}{2}}.$$

and not in the form

$$\frac{\dot{a}^2}{a^2} = \frac{C^2}{a^{3(w+1)}}.$$

In the paper
V. Gorini, A. Y. Kamenshchik, U. Moschella and V. Pasquier,
Tachyons, scalar fields and cosmology,
Phys. Rev. D **69**, 123512 (2004)
we have had quite similar differential equation.
In the flat Friedmann universe we have a tachyon field (i.e.
Born-Infeld field) with the Lagrangian

$$L = -V(T)\sqrt{1 - \dot{T}^2}.$$

The energy density is

$$\rho = \frac{V(T)}{\sqrt{1 - \dot{T}^2}}.$$

The pressure is

$$p = -V(T)\sqrt{1-\dot{T}^2}.$$

Let us introduce

$$s \equiv \dot{T}.$$

Then the Klein-Gordon equation is

$$\frac{\dot{s}}{1-s^2} + 3\frac{\dot{a}}{a}s + \frac{V_{,T}}{V(T)} = 0.$$

At $s = 1$, we have a singularity.
In the vicinity of the singularity

$$s = 1 - \tilde{s},$$

where $\tilde{s}$ satisfies the equation

$$\dot{\tilde{s}} = A\tilde{s}^{3/4}, \;\; A = 3 \cdot 2^{3/4} V(T),$$

which has two solutions:

$$\tilde{s} = 0$$

and

$$\tilde{s} = \frac{A^4}{256}(t - t_0)^4.$$

These are exactly the Norton's Dome solutions!