Describing quantumness of qubits and qutrits by Wigner function's negativity

Vahagn Abgaryan, Arsen Khvedelidze, Astghik Torosyan

Meshcheryakov Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia

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Motivation

Classically, a particle in one dimension with its position q and momentum p is described by a phase space distribution $P_{Cl}(q,p)$. The average of a function of the position and momentum $A(q,p)$ can then be expressed as

$$\langle A \rangle_{Cl} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \, A(q,p) \, P_{Cl}(q,p) \, .$$

A **quantum mechanical** particle is described by a density matrix $\hat{\varrho}$, and the average of a function of the position and momentum operators $\hat{A}(\hat{q},\hat{p})$ is

$$\langle A \rangle_{QM} = \mathrm{tr}\left(\hat{A}\,\hat{\varrho}\right) .$$

A quantum mechanical average can be expressed using a quasiprobability distribution $P_{QM}(q,p)$ as

$$\langle A \rangle_{QM} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \, A(q,p) \, P_{QM}(q,p) \, .$$

Negative probabilities in a physical theory does not exclude that theory. Either the assumed conditions may not be capable of being realized in the physical world, or the situation for which the probability appears to be negative is not one that can be verified directly.

Richard P. Feynman on Negative probability

In classical theory, the probability that a particle has a position q and a momentum p in dq and dp is given by **positive everywhere** distribution function $F(q,p)$.

In quantum mechanics the nearest thing to this is a function (the density matrix in a certain representation) which for a particle in a state with wave function $\psi(q)$ is $F(q,p) = \int \psi^*(q - \frac{y}{2})\, e^{(-\imath p y)}\, \psi(q + \frac{y}{2})\, dy$, and it **may have negative values** for some regions of q and p.

Objective

Wigner quasiprobability distribution (WF) $W(\Omega_N) = \mathrm{tr}\,[\varrho\,\Delta(\Omega_N)]$:

- density matrix $\varrho \in \mathfrak{P}_N$: $\varrho = \varrho^\dagger\,,\ \varrho \geq 0\,,\ \mathrm{tr}\,(\varrho) = 1\,;$
- Stratonovich-Weyl kernel $\Delta(\Omega_N) \in \mathfrak{P}^*_N$: $\Delta = \Delta^\dagger\,,\ \mathrm{tr}\,(\Delta) = 1\,,\ \mathrm{tr}\left(\Delta^2\right) = N\,.$

Global indicator of classicality:

$$\mathcal{Q}_N = \frac{\text{Volume of orbit subspace } \mathcal{O}[\mathfrak{P}^{(+)}_N]}{\text{Volume of orbit space } \mathcal{O}[\mathfrak{P}_N]}\,,$$

where $\mathcal{O}[\mathfrak{P}^{(+)}_N]$ is the unitary orbit space of states with non-negative WF.

KZ-indicator of non-classicality:

$$\delta_N = \int_{\Omega_N} \mathrm{d}\Omega_N \big|W(\Omega_N)\big| - 1\,.$$

Wigner function

Family of the Wigner functions:

$$W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_N) = \frac{1}{N}\left(1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\xi})\right),$$

- $\boldsymbol{\xi}$ is (N^2-1)-dimensional Bloch vector
- vector $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \mu_8 \boldsymbol{n}^{(8)} + \ldots + \mu_{N^2-1} \boldsymbol{n}^{(N^2-1)}$;
- orthonormal vectors $\boldsymbol{n}_\mu^{(s^2-1)} = \frac{1}{2}\,\mathrm{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right)$.

WF is non-negative, $W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_N) \geq 0$, for any state with

$$0 \leq \boldsymbol{\xi}^2 \leq r_*^2(N)\,, \quad \text{where} \quad r_*(N) = \sqrt{N+1}/(N^2-1)\,.$$

Orbit space of $SU(N)$ group adjoint action on state space

The orbit space $\mathcal{O}[\mathfrak{P}_N]$ can be realised as an ordered $(N-1)$-simplex in the space of eigenvalues $\boldsymbol{r} = \{r_1, \dots, r_N\}$ of a density matrix $\varrho = U \varrho_{diag} U^\dagger$:

$$\mathcal{C}^{(N-1)} = \left\{ \boldsymbol{r} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq \cdots \geq r_{N-1} \geq r_N \geq 0 \right\}.$$

The subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}] = \{ \boldsymbol{\pi} \in \textbf{spec}\,(\Delta(\Omega_N)) \mid (\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) \geq 0, \quad \forall\, \boldsymbol{r} \in \mathcal{O}[\mathfrak{P}_N] \}$ is a dual cone $(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) = r_1\pi_N + \cdots + r_N\pi_1$ of a subset $\mathcal{O}[\mathfrak{P}_N] \subset \mathbb{R}^{N-1}$, where $\boldsymbol{\pi} = \{\pi_1, \dots, \pi_N\}$ are the eigenvalues of the SW kernel $\Delta(\Omega_N \,|\, \boldsymbol{\nu})$.

For $W_N^{(-)} = \sum_{i=1}^N \pi_i r_{N-i+1}$ and $W_N^{(+)} = \sum_{i=1}^N \pi_i r_i$: $W_N^{(-)} \leq W(\Omega_N) \leq W_N^{(+)}$.

Qutrit orbit space and Wigner function positivity domain

Qutrit density matrix $\varrho_3 = \frac{1}{3}(I + \sqrt{3}\,\sum_{\nu=1}^{8} \xi_\nu \lambda_\nu)$ has the spectrum $r_{1,2} = 1/3\,(1 \pm \sqrt{3}\xi_3 + \xi_8)$ with $1 \geq r_1 \geq r_2 \geq r_3 \geq 0\,.$

Qutrit Stratonovich-Weyl kernel $\Delta = U\,\frac{1}{3}(I + 2\sqrt{3}(\mu_3\lambda_3 + \mu_8\lambda_8))\,U^\dagger\,,$ with $\mu_3 = \sin\zeta\,,\ \mu_8 = \cos\zeta\,,\ \zeta \in [0, \pi/3]\,,$ has the spectrum $\pi_{1,2} = 1/3\,(1 \pm 2\sqrt{3}\sin\zeta + 2\cos\zeta)$ with $\pi_1 \geq \pi_2 \geq \pi_3\,.$

Thus, a qutrit orbit space and its subspace of WF positivity are respectively

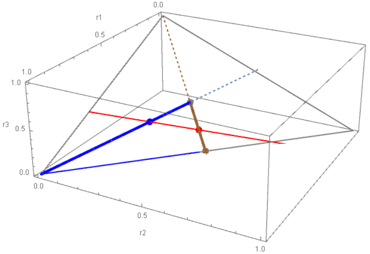
$$\mathcal{O}[\mathfrak{P}_3] : \ \left\{ \boldsymbol{r} \in \mathbb{R}^2 \ \middle|\ \sum_{i=1}^{3} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq r_3 \geq 0 \right\},$$

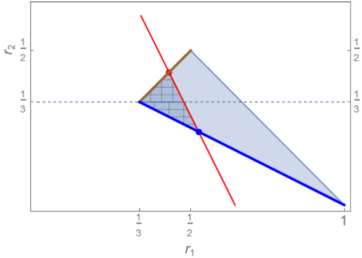
$$\mathcal{O}[\mathfrak{P}_3^{(+)}] : \left\{ \zeta \in [0, \pi/3] \ \middle|\ r_3 \geq \frac{r_1(4\cos\zeta - 1) - r_2\left(1 + 2\cos\zeta - 2\sqrt{3}\sin\zeta\right)}{1 + 2\cos\zeta + 2\sqrt{3}\sin\zeta} \right\}.$$

Strata of a qutrit orbit space:

- stratum $\mathcal{O}_{123}$ of $\dim(\mathcal{O}) = 6$, $r_1 \neq r_2 \neq r_3$, corresponding to regular orbits;
- two strata $\mathcal{O}_{1|23}$ and $\mathcal{O}_{12|3}$, $r_1 \neq r_2 = r_3$ and $r_1 = r_2 \neq r_3$, with $\dim(\mathcal{O}_{1|23}) = \dim(\mathcal{O}_{12|3}) = 4$;
- stratum $\mathcal{O}_0$ of $\dim(\mathcal{O}_0) = 0$, $r_1 = r_2 = r_3$, corresponding to the orbit of maximally mixed state.

Wigner function lower bound positivity region:





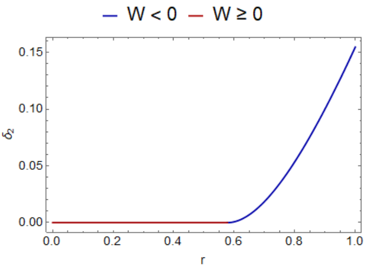
Quantumness of a single qubit

KZ-indicator for a single qubit is zero for the Bloch radius $r \in [0, \frac{1}{\sqrt{3}}]$:

$$\delta_2 = \theta[r - \frac{1}{\sqrt{3}}] \left(\frac{3r^2+1}{2\sqrt{3}r} - 1 \right) .$$

Global $\mathcal{Q}$-indicator for a qubit:

$$\mathcal{Q}_2[\mathrm{g}_{\mathrm{HS}}] = \frac{1}{3\sqrt{3}} .$$



The Wigner function of a qubit is positive definite inside the Bloch ball of radius $r_*(2) = 1/\sqrt{3}$.

Quantumness of a single qutrit

Qutrit KZ-indicator for $\zeta = 0$:

$$\delta_3^{(0)} = \theta[2\sqrt{3}\xi_3 + 2\xi_8 - 1]\,\theta[\sqrt{3}\xi_8 - \xi_3] \times$$

$$\theta[1 - 2\xi_8]\,\theta[1 - 8\xi_8]\,\frac{2\left(\left(\sqrt{3}\xi_3 + \xi_8\right) - \frac{1}{2}\right)^3}{9\xi_3\left(\xi_3 + \sqrt{3}\xi_8\right)} .$$

Qutrit KZ-indicator for $\zeta = \pi/3$:

$$\delta_3^{(\pi/3)} = \theta[4\xi_8 - 1]\,\theta[1 - 2\xi_8]\,\Big[\,\theta[2\sqrt{3}\xi_3 - 2\xi_8 - 1] \times$$

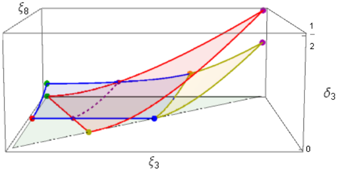
$$\theta[\sqrt{3}\xi_8 - \xi_3]\left(\frac{2\left((\sqrt{3}\xi_3 + \xi_8) + \frac{1}{2}\right)^3}{9\xi_3(\xi_3 + \sqrt{3}\xi_8)} - 2\right) +$$

$$\theta[\xi_3]\,\theta[2\xi_8 - 2\sqrt{3}\xi_3 + 1]\left(\frac{-\,(4\xi_8 - 1)^3}{18\left(\xi_3^2 - 3\xi_8^2\right)}\right)\Big] .$$

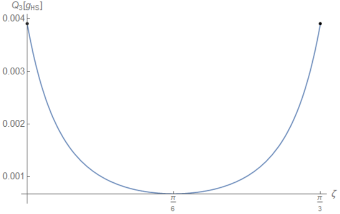
Qutrit $\mathcal{Q}$-indicator of classicality:

$$\mathcal{Q}_3[g_{\mathrm{HS}}] = \frac{20\cos^2(\zeta - \pi/6) + 1}{128\left(4\cos^2(\zeta - \pi/6) - 1\right)^5} ,$$

$$\min_{\zeta\in[0,\frac{\pi}{3}]} \mathcal{Q}_3(\zeta) = \mathcal{Q}_3\left(\frac{\pi}{6}\right) = \frac{7}{2^7\, 3^4} .$$



Qutrit KZ-indicators $\delta_3^{(0)}$ (red surface) and $\delta_3^{(\frac{\pi}{3})}$ (blue and yellow surfaces) as functions of two invariants ξ_3 and ξ_8 .



Qutrit global indicator in Hilbert-Schmidt metric.

Summary

- The global $\mathcal{Q}$-indicator is sensitive to two features of classicality: the negativity of the Wigner function and local quantum uncertainty which is accumulated in the form of geometric measure on the orbit space inherited from a physically motivated quantum information content.

- The KZ-indicator points to the existence of three classes of states:
  - "absolutely classical" states: $\delta = 0$ for all values of moduli parameters ζ;
  - "absolutely quantum" states: δ depends on ζ but is never zero;
  - "relatively quantum-classical" states whose classicality is susceptible to a Wigner function representation.

- A relation between these indicators was established: $\mathcal{Q}_N = 1 - \int_{supp(\delta_N)} \mathrm{dP}_N$,
where dP_N is the normalized probability measure on the orbit space evaluated for a given random ensemble of states.

Thank you!