Introduction to the Standard Model

Lake Baikal Summer School | Supplementary exercises | A. Trautner

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1 Z boson decay*

A Z boson of mass $M_Z = 91.188\,\text{GeV}$ decays at rest to a pair of $\tau^+\tau^-$. A τ lepton has a mass of $m_\tau = 1.777\,\text{GeV}$.

a) Compute energy and momentum of the decay products in GeV.

b) The mean lifetime of τ's at rest is $2.90 \times 10^{-13}\,\text{s}$. How far do the τ's make it on average?

2 Neutron decay and conservation laws*

Name at least one conservation law that is violated by the following hypothetical decays

$$n \to p + e^- \,, \qquad n \to \pi^+ + e^- \,, \qquad n \to p + \pi^- \,, \qquad n \to p + \gamma \,.$$

3 Four-vectors*

The center of mass energy E_cm of a system of two particles is given by $E_\text{cm}^2 = s = (\hat{p}_1 + \hat{p}_2)^2$, where $\hat{p}_1$ and $\hat{p}_2$ are the four-momenta $\hat{p} = (E, \vec{p})$ of the respective particle.

a) Show that s is Lorentz invariant.

b) At HERA protons of $E_p = 920\,\text{GeV}$ were collided with electrons of $E_e = 27.5\,\text{GeV}$. Compute the center of mass energy of an ep-system.

c) What energy for the electrons would be required to achieve the same center of mass energy if the protons are at rest.

4 Finger exercises about the Standard Model*

a) Derive the classical form of Maxwell's equations from the covariant equations of motion of QED stated in the lecture.

b) Create a table of all Standard Model (SM) fields and their charges under all symmetry groups.

c) Write down the full SM Lagrangian (you may use the lecture to collect the pieces and add things that have not been shown).

d) Perform a full count of the number of real parameters by rotating to a "physical" basis. Why does the CKM matrix only have four physical parameters although a general unitary matrix would be expected to have nine?

5 The Lorentz group and $\mathrm{SL}(2,\mathbb{C})$**

In this exercise we will construct an isomorphism

$$\mathrm{SO}(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \,.$$

$\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ is the group of complex 2×2 matrices λ, with $\det(\lambda) = 1$, where all elements are identified that differ only by the action of $\mathbb{Z}_2 \equiv \{1, -1\}$. Consequently every matrix λ is identified with its (additive) inverse $-\lambda$.

a) Show that

$$V_\mu \;\longmapsto\; v \equiv \sigma^\mu\, V_\mu$$

defines an isomorphism between Minkowski space $\mathbb{M}^4$ and the space of two-dimensional Hermitian matrices $\mathbb{H} := \{h \in \mathbb{C}^{2\times 2} | h^\dagger = h\}$ where σ^μ are the Pauli matrices

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) Prove the equation

$$V_\mu \, V^\mu = \det(v) \, .$$

c) Show that each $\lambda \in \mathrm{SL}(2, \mathbb{C})$ corresponds to a Lorentz transformation $\Lambda(\lambda)$ which acts on elements in $\mathbb{H}$ by

$$\Lambda(\lambda) : v \longmapsto v' = \Lambda(\lambda)\, v := \lambda\, v\, \lambda^\dagger \, .$$

Is this a one–to–one correspondence?

d) Now check if this action on $\mathbb{H}$ preserves the group structure of $\mathrm{SO}(1, 3)$, that is

i) $\exists\, \lambda_0 \ : \ \Lambda(\lambda_0) = \mathbb{1}\,,$

ii) $\Lambda(\lambda_1 \lambda_2) = \Lambda(\lambda_1)\Lambda(\lambda_2)\,,$

iii) $\forall\, \Lambda(\lambda)\ \exists \Lambda^{-1}(\lambda) \ : \ \Lambda(\lambda)\, \Lambda^{-1}(\lambda) = \mathbb{1}\,.$

e) Compare the number of real parameters of $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ and $\mathrm{SO}(1, 3)$.

$\boxed{6}$ Chiral and Dirac basis*

The two Weyl spinors ξ and $\bar{\eta}$ (the bar is part of the name here) can be combined to the Dirac 4-Spinor (in **chiral** basis),

$$\psi_{\text{chiral}} \equiv \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix}.$$

Combining the two Weyl equations of its components, we find that it fulfills the Dirac equation

$$(\gamma_{\mathrm{c}}^\mu p_\mu - m)\, \psi_{\mathrm{c}}(\vec{p}) = 0\,,$$

with the γ-matrices

$$\gamma_{\mathrm{c}}^0 \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_{\mathrm{c}}^i \equiv \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \gamma_{\mathrm{c}}^5 \equiv \mathrm{i}\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

in the so-called **chiral basis**. To change to another basis, we make a unitary transformation

$$\psi_{\text{new}} = \mathbf{U}\, \psi_{\text{chiral}} \qquad \text{with } \mathbf{U}^\dagger\, \mathbf{U} = \mathbb{1}\,.$$

a) Show that ψ_{new} fulfills the Dirac equation with γ–matrices

$$\gamma_{\text{new}}^\mu = \mathbf{U}\, \gamma_{\mathrm{c}}^\mu\, \mathbf{U}^\dagger\,.$$

Another common basis for the γ-matrices is the so-called **Dirac basis**, which can be obtained through

$$\mathbf{U}_{\text{Dirac}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}.$$

b) Calculate explicit expressions for $\gamma_{\text{Dirac}}^\mu$ and γ_{Dirac}^5.

c) The action of the Lorentz group on spinors in chiral basis is

$$\psi_{\mathrm{c}} \longmapsto \begin{pmatrix} \exp\left(\frac{\mathrm{i}}{2}\,\vec{\sigma}\cdot\vec{\theta} + \frac{1}{2}\,\vec{\sigma}\cdot\vec{\beta}\right) & 0 \\ 0 & \exp\left(\frac{\mathrm{i}}{2}\,\vec{\sigma}\cdot\vec{\theta} - \frac{1}{2}\,\vec{\sigma}\cdot\vec{\beta}\right) \end{pmatrix} \psi_{\mathrm{c}}\,.$$

where $\vec{\theta}$ is the vector of rotation angles and $\vec{\beta}$ the vector of rapidities with respect to the three spatial axes. Calculate the action of the Lorentz group on spinors in Dirac basis.

$\boxed{7}$ The gauge covariant derivative**

Under local gauge transformations (Abelian or non–Abelian), a field transforms according to

$$\psi(x) \longmapsto U(x)\,\psi(x)\;,$$

with some position dependent function $U(x)$ that assigns each point of space–time an element of the gauge group G. This represents the freedom to select a different gauge at every point in space–time.

a) Show that the derivative of a field does not transform like the field itself

$$\partial_\mu\psi(x) \not\longmapsto U(x)\,\partial_\mu\psi(x)\;.$$

To maintain the gauge symmetry of the Lagrangian, one has to find a gauge invariant derivative D_μ also called covariant derivative such that the covariant derivative of a field transforms as

$$D_\mu\psi(x) \;\longmapsto\; U(x)D_\mu\psi(x)\,.$$

To this end, one introduces a gauge field A_μ and defines an auxiliary field $\tilde{\psi}$ by

$$\tilde{\psi}(x+\mathrm{d}x) := \psi(x+\mathrm{d}x) + \mathrm{i}gA_\mu(x)\psi(x)\mathrm{d}x^\mu\;,$$

where the gauge field A_μ is chosen such that $\tilde{\psi}$ transforms as

$$\tilde{\psi}(x+\mathrm{d}x) \longmapsto U(x)\,\tilde{\psi}(x+\mathrm{d}x)\;.$$

The covariant derivative can then be defined by

$$\left(D_\mu\psi(x)\right)\mathrm{d}x^\mu := \tilde{\psi}(x+\mathrm{d}x) - \psi(x)\;,$$

b) Show that one can write

$$D_\mu\psi = (\partial_\mu + \mathrm{i}gA_\mu)\psi\,.$$

c) The gauge field A_μ itself transforms non–trivially under the action of the gauge group. Using the transformation properties of ψ and $\tilde{\psi}$, show that A_μ has to transform as

$$A_\mu \longmapsto UA_\mu U^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger\;.$$

Note that the gauge field is an element of the Lie algebra $\mathfrak{g}$ corresponding to G. For simplicity, one can regard it as a matrix acting on the vector (with respect to the gauge group) ψ. As every Lie algebra element, A_μ can be written as linear combination of the Lie algebra generators T_a, which fulfill the commutation relations

$$[\mathrm{T}_a,\mathrm{T}_b] = \mathrm{i}\,f_{ab}^{c}\,\mathrm{T}_c\;.$$

Thus one defines

$$A_\mu(x) := A_\mu^a\mathrm{T}_a\;.$$

Note that that gauge degrees of freedom are conventionally labeled with Latin indices. The field-strength tensor $F_{\mu\nu}$ is defined by

$$\mathrm{i}g\,F_{\mu\nu} := \mathrm{i}g\,F_{\mu\nu}^a\mathrm{T}_a := [D_\mu, D_\nu]\;.$$

d) Show that this definition of the field–strength tensor coincides with its definition in component form, i.e. that

$$F_{\mu\nu} = \left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\,f^{\,a}_{\;bc}A_\mu^bA_\nu^c\right]\mathrm{T}_a\;.$$

e) What is the simplest, i.e. lowest order, Lorentz– and gauge–invariant expression that can be formed from $F_{\mu\nu}$?

$\boxed{8}$ Abelian Higgs Mechanism**

Consider a theory with a complex scalar field $\phi(x)$ and a U(1) gauge symmetry with a gauge boson $A_\mu(x)$. The Lagrangian density is given by

$$\mathscr{L} \;=\; (D_\mu\phi)\,(D^\mu\phi)^* + \mu^2\,\phi^*\phi - \frac{\lambda}{2}\,(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\;.$$

Here $F_{\mu\nu}$ is the Abelian field strength tensor and D_μ is the gauge covariant derivative given by

$$F_{\mu\nu} \;:=\; \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad D_\mu \;:=\; \partial_\mu + \mathrm{i}\, e\, A_\mu \;,$$

respectively. This Lagrangian is invariant under the gauge transformation

$$\phi(x) \to \mathrm{e}^{\mathrm{i}\,\alpha(x)}\,\phi(x), \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\,\partial_\mu \alpha(x)\;.$$

Assume that $\mu^2 > 0$ and $\lambda > 0$.

a) Show that the field ϕ develops a vacuum expectation value $\langle\phi\rangle$ and determine $v := |\langle\phi\rangle|$.

b) Is there any freedom in choosing $\langle\phi\rangle$? Why?

c) To quantize the theory we have to expand the field $\phi(x)$ around the true minimum of the potential. Therefore, parametrize

$$\phi(x) \;=\; \mathrm{e}^{\mathrm{i}\,\xi(x)/v}\,(v + \sigma(x))\;,$$

and rewrite the initial Lagrangian in terms of the real scalar fields $\xi(x)$ and $\sigma(x)$.
Hint: Do not expand terms which are non–quadratic in the fields. Factor out $(1 + \frac{\sigma}{v})$ whenever possible.

d) What happens to the gauge symmetry of the theory? What is the symmetry of the ground state? What is special about $\xi(x)$?

e) Summarize the total number of real degrees of freedom of this theory. Which of the fields are physical and why can you tell that they are (un)physical?

f) Perform an adequate gauge transformation to show explicitly that all unphysical fields can be made to disappear from the Lagrangian. Determine the masses of all remaining scalar fields and gauge bosons. What is the total number of real degrees of freedom now?

$\boxed{9}$ Fermion masses from spontaneous symmetry breaking**

Consider a theory of a single complex scalar field $\phi(x) \in \mathbb{C}$ and two Dirac fermion fields $\Psi_1(x)$ and $\Psi_2(x)$. The Lagrangian is given by

$$\begin{aligned}\mathscr{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2|\phi|^2 - \lambda|\phi|^4 + \\ + \overline{\Psi}_1 \mathrm{i}\partial_\mu\gamma^\mu\Psi_1 + \overline{\Psi}_2 \mathrm{i}\partial_\mu\gamma^\mu\Psi_2 - m_1\overline{\Psi}_1\Psi_1 - m_2\overline{\Psi}_2\Psi_2 - g(\phi\overline{\Psi}_1\Psi_2 + \phi^\dagger\overline{\Psi}_2\Psi_1)\end{aligned}$$

where m^2, m_1, m_2, λ, g are all real positive coupling constants of the appropriate mass dimension.

a) What is the maximal symmetry of the theory if $g = 0$?

b) What is the maximal symmetry of the theory if $g \neq 0$?

c) For $g \neq 0$ give a possible charge assignment of the different fields under the given symmetry.

d) Suppose that $m^2 < 0$ while the other parameters are still positive. What is the symmetry of the ground state?

e) In the latter case, find the masses of the physical fermions.
Hint: Write the scalar field as $\phi = v + \frac{1}{\sqrt{2}}(\sigma(x) + \mathrm{i}\rho(x))$ where v is the minimum of the potential and expand the Lagrangian in terms of the new variables.

$\boxed{10}$ Higgs decay to Fermions***

A general formula for the rate of of two-body decays $k_i \to k_1 + k_2$ is given by

$$\Gamma = \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3\, 2k_1^0}\, \frac{\mathrm{d}^3 k_2}{(2\pi)^3\, 2k_2^0} |\mathcal{T}|^2 \frac{1}{2k_i^0}\,(2\pi)^4\,\delta^{(4)}\left(k_i - k_1 - k_2\right)\;,$$

where k_i, k_1, k_2 are the four-momenta of the initial and final state particles and $\mathcal{T}$ is the transition matrix element.

a) Simplify this formula for decays in the rest frame of the initial particle. Hint: $\delta(f(x)) = \sum_{x_0} \delta(x - x_0)|f'(x_0)|^{-1}$ where x_0 are the (simple) zeros of $f(x)$.

b) The Lagrangian for Higgs-fermion interaction with Yukawa coupling y_f has been given in the lecture. Use this to compute the matrix element $\mathcal{T}$ and its absolute square.

You may find the following identities helpful:
(here u and v are spinor solutions of the free Dirac equation $\left(\not{p} - m\right) u_s(p) = 0$, $\left(\not{p} + m\right) v_s(p) = 0$ where s denotes the spin, Γ stands for any combination of gamma matrices, and the slash is defined as $\not{p} := \gamma^\mu p_\mu$)

$$(\bar{u}_1 \Gamma u_2)^* = \bar{u}_2 \bar{\Gamma} u_1 \ ,$$

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \ , \qquad\qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \ .$$

c) Combine these results to compute the partial decay with of a Higgs boson to a pair of leptons. What would be different for quarks?

d) What is the branching fraction of $h \to e^+e^-$ as compared to $h \to \mu^+\mu^-$?

e) The production cross section for a Higgs from a proton-proton collision at the LHC at center-of-mass energy of 13 TeV is about $\mathcal{O}(50)$ pb. The total recorded luminosity is about $140\,\mathrm{fb}^{-1}$. The Higgs mass is about $m_h = 125\,\mathrm{GeV}$ and the total decay width in the Standard Model is $\Gamma_{h,\mathrm{tot}} = 4\,\mathrm{MeV}$. How many $h \to e^+e^-$ events do you expect in the data already? How many at the high luminosity LHC (HL-LHC) with $3000\,\mathrm{fb}^{-1}$? Given the constraints on y_μ shown in the lecture, what constraints (very roughly) would you expect on y_e in the future?

$\boxed{11}$ Charged pion decay, chiral enhancement and parity violation in the SM***

Consider the charged pion decay $\pi^+ \to \ell^+\nu_\ell$ into a lepton $\ell = e, \mu$ and its neutrino ν_ℓ. In terms of quarks $\pi^+ = u\bar{d}$.

a) What prohibits the energetically allowed decays $\pi^+ \to \ell^+ + \gamma$?

b) From naive phase space considerations, what branching ratio $\Gamma(\pi^+ \to e^+\nu_e)/\Gamma(\pi^+ \to \mu^+\nu_\mu)$ would you expect for this decay?

c) Draw the Feynman diagram for this process. Regarding the coupling vertices, do not forget about quark mixing.

d) The leptonic matrix element can easily be computed and is given by

$$\langle \ell^+\nu_\ell|\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell|0\rangle = \bar{u}_{\nu_\ell}\gamma_\mu(1-\gamma_5)v_{\ell^+} \ .$$

By contrast, the hadronic matrix element contains the bound state pion and cannot be straightforwardly computed. However, using Lorentz invariance and dimensional analysis we can parametrize it as

$$\langle 0|\bar{d}\gamma^\mu(1-\gamma^5)u|\pi^+\rangle = -f_\pi p_\pi^\mu \ ,$$

where p_π is the four-momentum of the pion and f_π a constant of dimension $[\mathrm{mass}]^1$ that is called the pion decay constant. This constant contains all of our ignorance of the inner structure of the pion (from the absolute decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$ one can extract $f_\pi \approx 130\,\mathrm{MeV}$ but the numerical value is unimportant for this exercise).

e) Using these matrix elements, compute the total matrix element for the decay and use the Dirac equation on the outer legs to simplify its form.

f) Confirm that the matrix element, and therefore the decay amplitude, vanishes if $m_\ell = 0$. Interpret this in view of the helicities of the outgoing particles and angular momentum conservation.

g) Compute the squared matrix element $|\mathcal{M}|^2$ and simplify it, eventually using that $\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) = 0$.

h) Use the general formula for two body decays stated in exercise 10 to compute the differential decay amplitude and total decay width in the pion rest frame.

i) Compute the branching ratio $\Gamma(\pi^+ \to e^+\nu_e)/\Gamma(\pi^+ \to \mu^+\nu_\mu)$ and compare to the experimentally determined value which is $1.230(4) \times 10^{-4}$, how does that compare to the naive expectation?

$\boxed{12}$ $e^+ + e^- \to \mu^+ + \mu^-$***

A general Lorentz invariant expression for the cross section of a two particle reaction with four-momenta $p_a + p_b \to p_1 + p_2$ is given by

$$\mathrm{d}\sigma \;=\; \frac{1}{4\sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2}}|\mathcal{M}|^2\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2-p_a-p_b)\,\frac{\mathrm{d}p_1^3}{(2\pi)^3\,2p_1^0}\frac{\mathrm{d}p_2^3}{(2\pi)^3\,2p_2^0}\;,$$

where $\mathcal{M}$ is the transition matrix element to be computed by Feynman rules.

a) Find the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ in the center of mass frame for the case of negligible masses $m_a = m_b = m_1 = m_2 = 0$, with the relevant scattering angle θ defined as the angle between $\vec{p_a}$ and $\vec{p_1}$.

b) For the process $e^+ + e^- \to \mu^+ + \mu^-$, determine the matrix element $\mathcal{M}$ from the Feynman rules of QED (if you don't know where to find those, see e.g. the Appendix of Peskin&Schröder).

c) Compute the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ for this process as a function of the scattering angle θ assuming (i) unpolarized electrons in the initial state and (ii) the approximation that the center of mass energy $s := (p_a + p_b)^2 \gg m_{e,\mu}^2$ (effectively setting $m_{e,\mu} = 0$).

You may use the following identities:
(here u and v are spinor solutions of the free Dirac equation $\left(\not{p} - m\right) u_s(p) = 0$, $\left(\not{p} + m\right) v_s(p) = 0$ where s denotes the spin, Γ stands for any combination of gamma matrices, and the slash is defined as $\not{p} := \gamma^\mu p_\mu$)

$$\left(\bar{u}_1 \Gamma u_2\right)^* = \bar{u}_2 \bar{\Gamma} u_1\;,$$

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m\;, \qquad\qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m\;,$$

$$\mathrm{tr}\left(\not{a}\gamma_\mu \not{b}\gamma_\nu\right) = 4\left(a_\mu b_\nu + a_\nu b_\mu - \eta_{\mu\nu}\, a\cdot b\right)\;.$$

$\boxed{13}$ Group theoretical anomaly coefficient***

For a simple Lie algebra, the anomaly coefficient $A(\boldsymbol{r})$ of a representation $\boldsymbol{r}$ is defined by

$$A(\boldsymbol{r})\,d^{abc} \;:=\; \mathrm{tr}\left[\mathsf{T}_{\boldsymbol{r}}^a\{\mathsf{T}_{\boldsymbol{r}}^b, \mathsf{T}_{\boldsymbol{r}}^c\}\right]\;,$$

where $\mathsf{T}_{\boldsymbol{r}}^a$ are the generators of a representation $\boldsymbol{r}$ and d^{abc} is an invariant totally symmetric tensor.

a) Explain why the anomaly function of a gauge theory based on a simple Lie group with chiral fermions in the representation $\boldsymbol{r}$ is proportional to $A(\boldsymbol{r})$.
Hint: Think about the derivation of the anomaly based on triangle diagrams.

b) Show that $A(\bar{\boldsymbol{r}}) = -A(\boldsymbol{r})$ for the complex conjugate representation $\bar{\boldsymbol{r}}$. What does this imply for (pseudo–)real representations?

In general, one can show that $A(\boldsymbol{r})$ vanishes for all simple Lie algebras except $\mathrm{SU}(N)$ with $N \geq 3$.

$\boxed{14}$ Gauge anomalies of the Standard Model***

Show that the Standard Model (SM) is free of gauge anomalies. Several considerations might be helpful:

- The SM gauge group is $\mathrm{SU}(3)_\mathrm{c} \otimes \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$, you may look up the field content in a book of your choice.
- As in the previous exercise, the relevant part for the vanishing of the anomalies are the traces over the generators of the respective gauge group.
- Be particularly careful with the mixed anomalies such as $[\mathrm{U}(1)_\mathrm{Y}]\,[\mathrm{SU}(3)_\mathrm{c}]^2$, etc.
- Keep in mind that the generators of $\mathrm{SU}(N)$ are traceless.

What can you say about the contributions to the anomalies within one generation of matter fields? How do the anomalies change if you add a right–handed neutrino in the representation $(\mathbf{1},\mathbf{1})_0$? Can you relate the anomaly freedom of $\mathrm{SO}(10)$ to the anomaly freedom of the SM?

15 **Global symmetries, anomalies, and topological vacuum terms of the Standard Model*****
The structure of the SM is such that it gives rise to the accidental global symmetries $\mathrm{U}(1)_\mathrm{B}$ and $\mathrm{U}(1)_\mathrm{L}$ corresponding to the classically conserved quantities of baryon (B) and lepton number (L). The corresponding charges thus are $+1/3$ for all types of quarks under $\mathrm{U}(1)_\mathrm{B}$ and $+1$ for all types of leptons under $\mathrm{U}(1)_\mathrm{L}$.

a) Show that $\mathrm{U}(1)_\mathrm{B}$ and $\mathrm{U}(1)_\mathrm{L}$ both are anomalous. What are the physical implications?

Now add three right–handed neutrinos in the SM representation $(\mathbf{1},\mathbf{1})_0$.

b) Show now that $\mathrm{U}(1)_{\mathrm{B-L}}$ is anomaly free but $\mathrm{U}(1)_{\mathrm{B+L}}$ is still anomalous.

This implies that $\mathrm{U}(1)_{\mathrm{B-L}}$ may actually be gauged in GUT models which come with right–handed neutrinos (such as for example the SO(10) model). On the contrary, the anomalous violation of $\mathrm{U}(1)_{\mathrm{B+L}}$ occurs in instanton and sphaleron processes. The latter might be relevant for Baryogenesis.

c) The QCD gauge group $\mathrm{SU}(3)_\mathrm{c}$ has a physical θ parameter (see lecture). Why do neither $\mathrm{U}(1)_\mathrm{Y}$ nor $\mathrm{SU}(2)_\mathrm{L}$ have physical θ parameters?