Aspects of QCD

Alessandro Papa

Università della Calabria & INFN - Cosenza

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UNIVERSITÀ DELLA CALABRIA
DIPARTIMENTO DI FISICA

INFN
Istituto Nazionale di Fisica Nucleare
GRUPPO COLLEGATO DI COSENZA

Outline

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2. Building the theory
   - Classical Lagrangian and its symmetries
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   - Renormalization
3. Using the theory (perturbatively)
   - Infrared safety
   - QCD-improved parton model
4. QCD in the high-energy limit
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   - LHC phenomenology - an example
5. Using the theory (non-perturbatively)
   - General setup
   - Some selected results
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What is QCD?

Quantum Chromodynamics (QCD) is the theory of strong interactions of hadrons

- It is a relativistic quantum field theory based on a non-Abelian gauge group
- At a given distance, the strong interaction is stronger than the electromagnetic, weak and gravitational ones, which can be taken as perturbations;
  it gives account of most of the observed matter in the Universe
- It exhibits two kinds of fields: quarks and gluons;
  however, no isolated states of quarks and gluons have ever been seen;
  they are confined into hadrons
- At short distances ($\lesssim 1\ \mathrm{GeV}^{-1}$) the interaction is weak enough to make perturbation theory amenable ("asymptotic freedom")
- There are many QCD phenomena (mechanism of confinement, nucleon structure, hadron masses, phase diagram at non-zero temperature and baryonic density, etc.) beyond the reach of perturbative QCD, some of them still waiting for an explanation

Why study QCD?

- Very practical (and compelling) reason:

  LHC collides protons, EIC will collide electrons and ions;
  a detailed knowledge of proton/ion structure and of pure QCD background processes is mandatory before claiming discovery of new Physics

- Conceptual reasons: yet many unsolved problems, mostly in the non-perturbative regime

  - theoretical understanding of the “mass gap”
  (CMI Millennium problem, 1 M$ award:
  `https://www.claymath.org/millennium-problems/`)

  - nature of confinement/deconfinement transition at finite temperature

  - QCD phase diagram on the temperature / baryon density plane

  - proton spin

  - connection to the bulk of nuclear physics

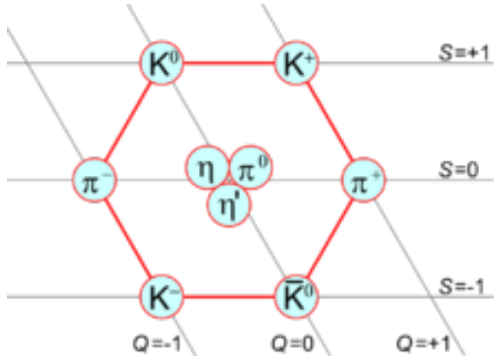
  - ...

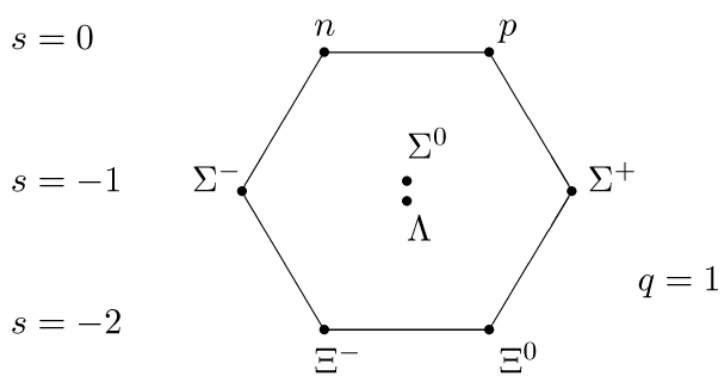
Where was it from?

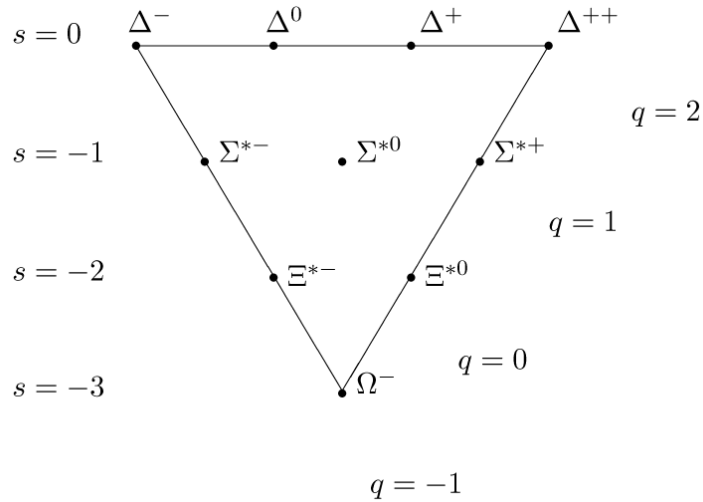
The QCD Lagrangian was proposed in 1972-1973 [Fritsch, Gell-Mann (1972); Fritsch, Gell-Mann, Leutwyler (1973)], not the culmination of a linear development process, but an ingenuous finding at the meeting point of several independent paths:

- Quark model of hadron states
- Current algebra
- Deep inelastic lepton scattering
- Theoretical development of non-Abelian gauge theories

Quark model [Gell-Mann (1964), Zweig (1964)]







Approximate SU(3) flavor symmetry ("Three quarks for Muster Mark", u, d, s):

- mesons ($q\bar{q}$): $3 \otimes \bar{3} = 1 \oplus 8$; baryons ($qqq$): $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

- mass splittings explained by an SU(3) octet perturbation

Fermi statistics: $\Delta^{++}(S = 3/2) = |u\uparrow, u\uparrow, u\uparrow\rangle \longrightarrow \epsilon_{ijk}|u_i\uparrow, u_j\uparrow, u_k\uparrow\rangle$

Current algebra

Phenomenology: $H_{\text{weak}-\text{strong}} = \int d^3x \ \sum_A j_A^\mu(x) X_{A,\mu}(x) \ , \qquad X = Z, W^\pm, \gamma$

$j_A^\mu(x)$="(conserved) hadronic current"

E.g., amplitude for $n \to p + e^- + \bar{\nu}_e$: $\propto \langle p|j_-^\mu(0)|n\rangle \frac{-ig_{\mu\rho}}{q^2 - m_W^2} \bar{u}_e \gamma^\rho (1-\gamma_5) v_\nu \ ,$

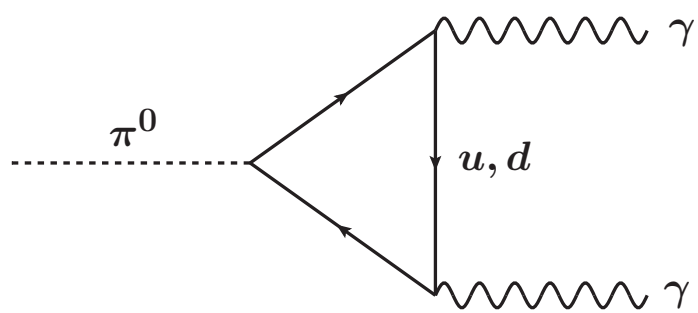
j_-^μ = "current which couples to W^+", $\quad q = p_n - p_p$

Hadronic currents: (approximately) conserved currents related to chiral symmetries of strong interactions

Symmetry group: $SU(3)_L \otimes SU(3)_R$ spontaneously broken to $SU(3)_V$
(or $SU(2)_L \otimes SU(2)_R$ spontaneously broken to $SU(2)_V$ with pions as Goldstone bosons)

In this context, π^0 decay to $\gamma\gamma$ should be suppressed, if not for the anomaly of the $U(1)_A$ symmetry [Adler (1969); Bell, Jackiw (1969)]:

$$\Gamma(\pi^0 \to \gamma\gamma) = N_c^2 (e_u^2 - e_d^2)^2 \frac{\alpha_{\text{em}}^2 m_\pi^3}{64\pi^3 f_\pi}$$



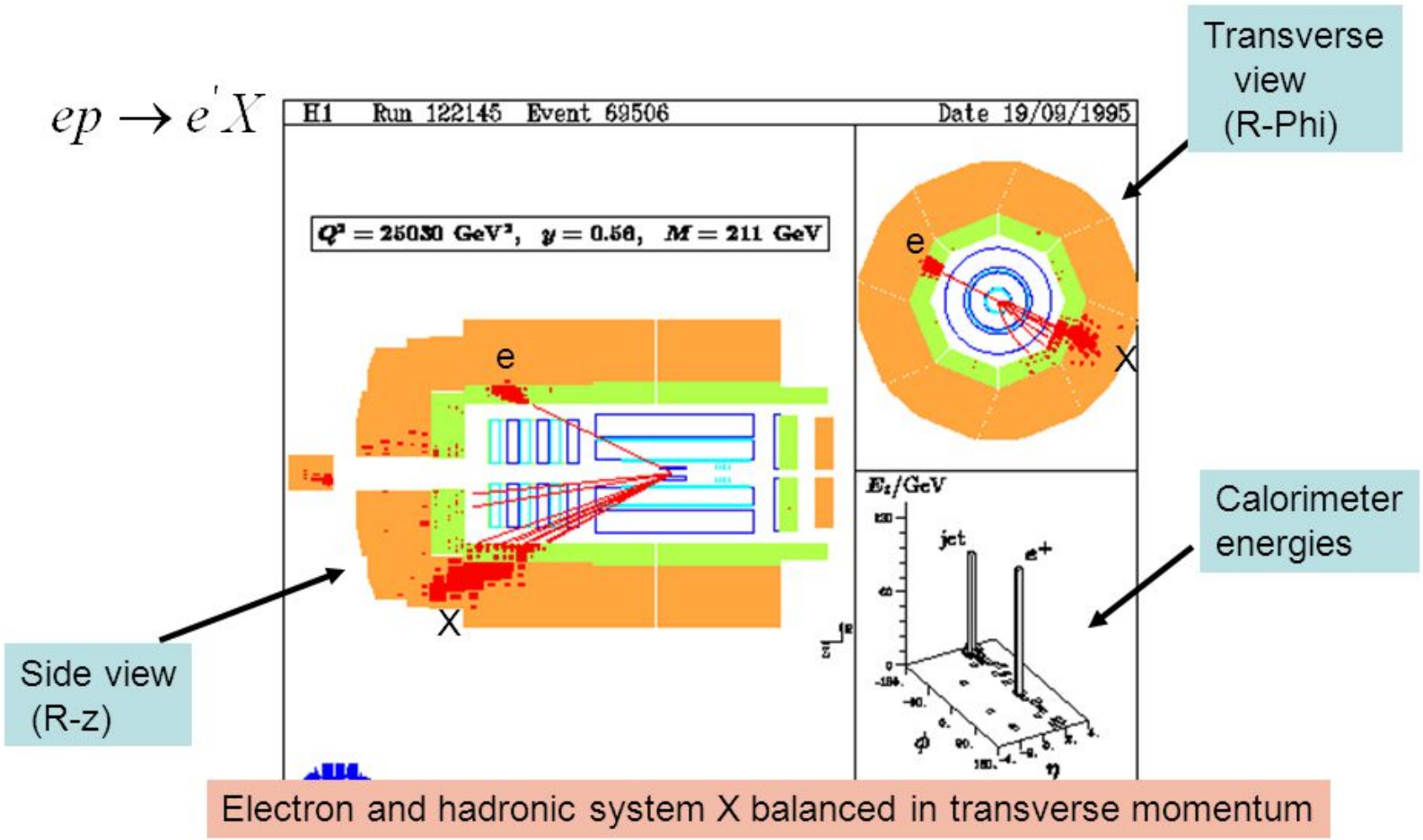
Deep inelastic scattering (or the reality of quarks)

$$e^-(l) + p(P) \to e^-(l') + X$$

- *inclusive* process
- point-like electromagnetic probe of the proton
- pioneered at SLAC 1967-1973 ($\simeq$ 21 GeV electrons to fixed target)
- extensively studied at HERA 1992-2007 (27.5 GeV electrons, 920 GeV protons, $\sqrt{s} = 318$ GeV)

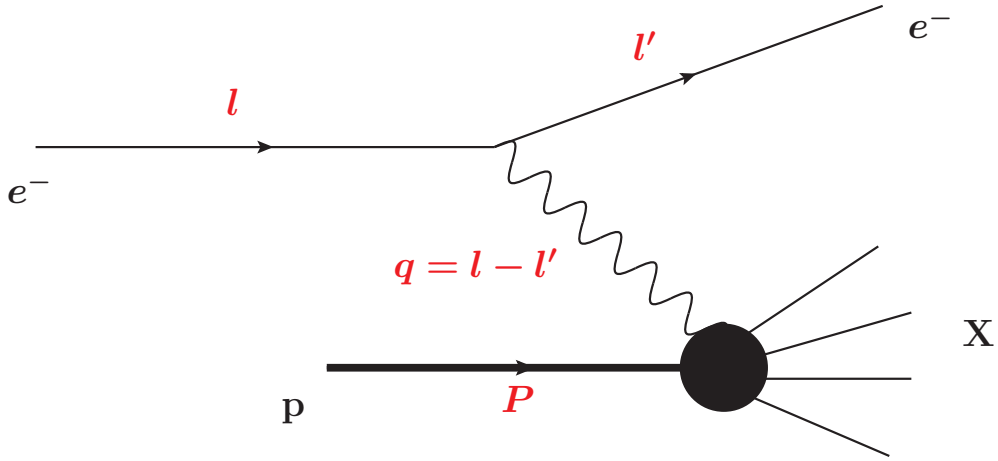
An H1 event display

Event : Combined view (R-z, R-Phi , calorimeter energies)



Electron and hadronic system X balanced in transverse momentum

https://www-h1.desy.de/pictures/H1-event_tutorial.pdf



$$q^2 = (l - l')^2 \equiv -Q^2$$

$$s = (l + P)^2$$

$$\begin{aligned} m_X^2 &\equiv W^2 = (P+q)^2 \\ &= M^2 + 2P\cdot q - Q^2 \end{aligned}$$

$$x_B \equiv \frac{Q^2}{2P\cdot q}\,, \qquad \frac{Q^2}{s+Q^2} \le x_B \le 1\,, \qquad x_B = 1 \text{ elastic limit}$$

$$y \equiv \frac{q\cdot P}{l\cdot P} \simeq \frac{Q^2}{x_B s}, \quad \text{proton rest frame: } y = \frac{E-E'}{E}$$

$$\frac{m_e^2}{s}, \ \frac{M^2}{s}, \ \frac{M^2}{Q^2} \text{ neglected}$$

Deep inelastic regime: m_X and Q large, x_B fixed

$$H_{\rm em} = \int d^3x\; e\, j^\mu(x) A_\mu(x)\,, \qquad j_\mu = j_\mu^{\rm lept} + j_\mu^{\rm h}$$

Lorentz-invariant differential cross section:

$$E'\frac{d\sigma}{d^3\vec{l}'} \simeq \frac{\pi e^4}{2s}\sum_X \delta^{(4)}(p_X - P - q)\left|\langle l'|\, j_\mu^{\rm lept}\,|l\rangle \frac{1}{q^2}\langle X|\, j^{{\rm h},\mu}\,|P\rangle\right|^2 = \frac{2\alpha_{\rm em}^2}{sQ^4} L_{\mu\nu} W^{\mu\nu}$$

$$j_\mu^{\rm lept} = \bar\psi_e \gamma_\mu \psi_e + \ldots\ldots\,, \qquad L_{\mu\nu} = \tfrac{1}{2}{\rm Tr}\left[\gamma_\nu \not{l} \gamma_\mu \not{l}'\right] = 2\left(l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu} l\cdot l'\right)$$

$$\begin{aligned} W_{\mu\nu} &\equiv 4\pi^3 \sum_X \delta^{(4)}(p_X - P - q) \langle P|\, j_\mu^{\rm h}(0)\, |X\rangle\langle X|\, j_\nu^{\rm h}(0)\, |P\rangle \\ &= \frac{1}{4\pi}\int d^4 z\; e^{iq\cdot z} \langle P|\, j_\mu^{\rm h}(z)\, j_\nu^{\rm h}(0)\, |P\rangle \end{aligned}$$

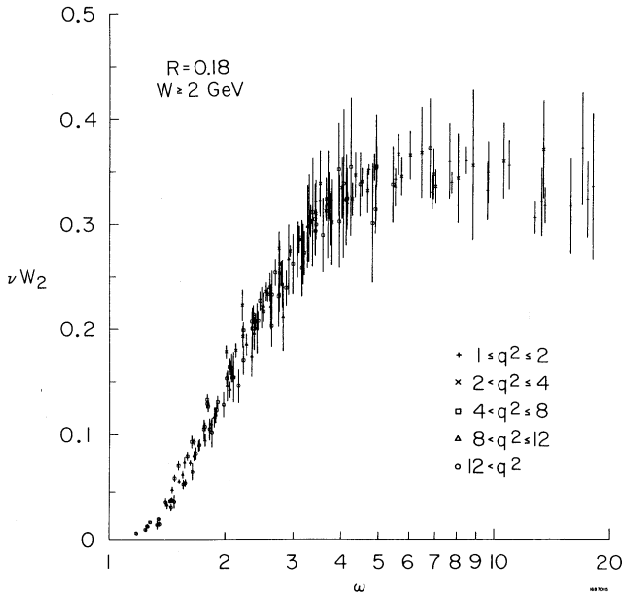
Current conservation, parity, hermiticity, spin average:

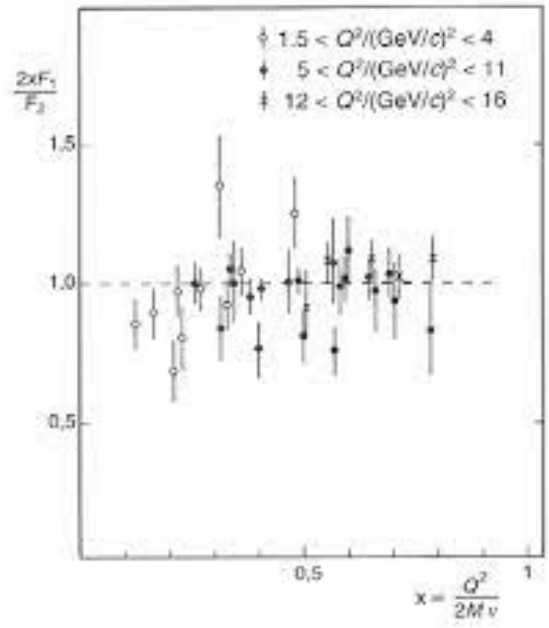
$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x_B, Q^2) + \frac{\left(P^\mu - q^\mu \frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu \frac{P\cdot q}{q^2}\right)}{P\cdot q} F_2(x_B, Q^2)$$

$$\begin{aligned} \frac{d\sigma}{dx_B dy} &\simeq \frac{4\pi\alpha_{\rm em}^2}{x_B y Q^2}\left[(1-y) F_2(x_B, Q^2) + y^2 x_B F_1(x_B, Q^2)\right] \\ &= \frac{2\pi\alpha_{\rm em}^2}{x_B y Q^2}\left[(1+(1-y)^2) F_2(x_B, Q^2) - y^2 F_L(x_B, Q^2)\right]\,, \quad F_L \equiv F_2 - 2x_B F_1 \end{aligned}$$

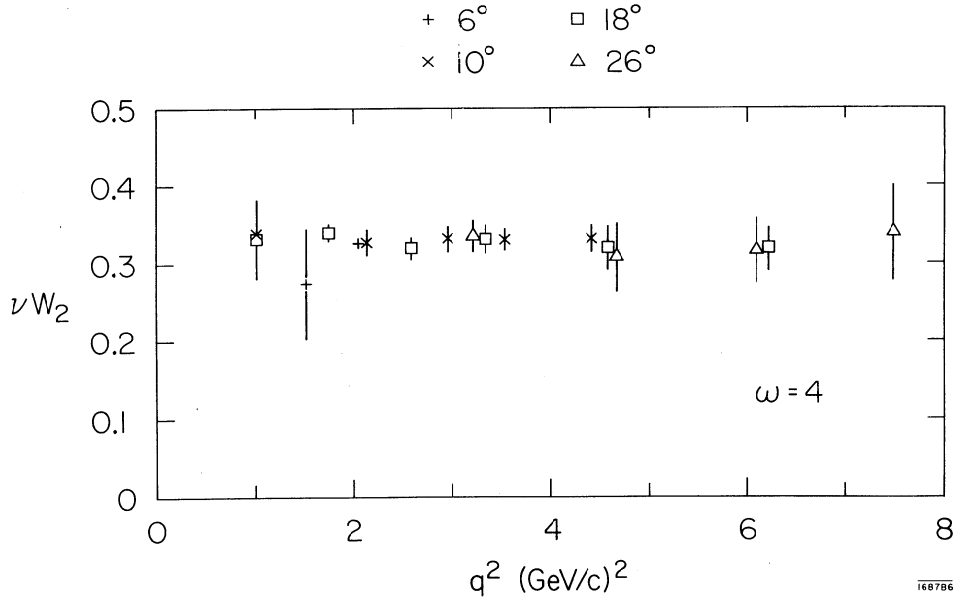
SLAC: F_i depend only on x_B (Bjorken scaling), $F_L \simeq 0$

F_2





F_2

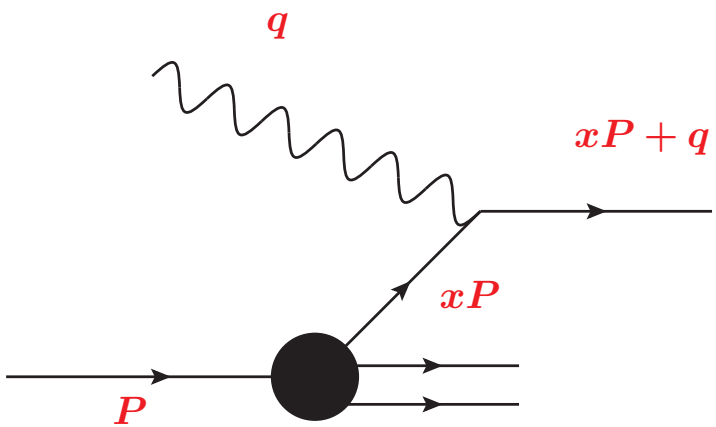


$\omega = 1/x_B$

Parton model [Bjorken, Paschos (1969); Feynman (1972)]

- The hard probe hits the proton on a time scale $\sim 1/Q$, whereas the typical interaction time within the proton is $\sim 1/M \gg 1/Q$ in the deep inelastic regime.

- The photon interacts with the proton via incoherent scattering off point-like and massless partons, carrying a fraction x of the proton momentum and each distributed in the proton according to $f(x)$.



$$(xP+q)^2 \simeq 2x\ P\cdot q$$

$$d\sigma = \sum_i \int dx\ f_i(x) d\sigma_i^{\text{parton}}$$

$$W^{\mu\nu} = \sum_i \int \frac{dx}{x}\ f_i(x)\ W_i^{\text{parton},\mu\nu}$$

Assuming massless partons with spin 1/2 and charge $e_i e$:

$$\begin{aligned} W_i^{\text{parton},\mu\nu} &= e_i^2 \frac{1}{4\pi}\frac{1}{2}\text{Tr}\left[\not{k}\gamma^\nu(\not{k}+\not{q})\gamma_\nu\right] 2\pi\delta((q+k)^2) \\ &= e_i^2(2k^\mu k^\nu + q^\mu k^\nu + k^\mu q^\nu - g^{\mu\nu} q\cdot k)\frac{x}{Q^2}\ \delta(x-x_B)\ , \qquad k \overset{!}{\equiv} xP \end{aligned}$$

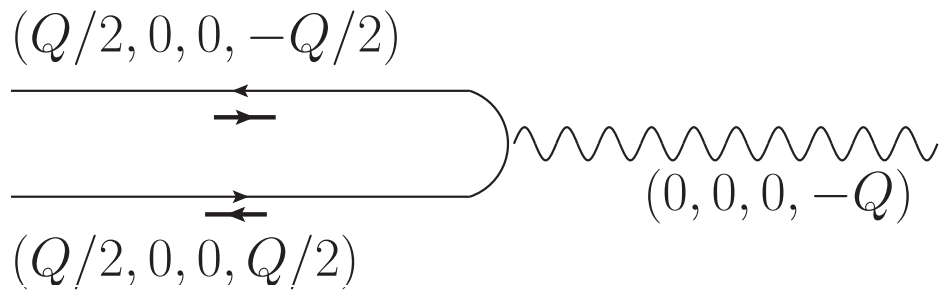
$$\longrightarrow\ F_2 = \sum_i e_i^2 x_B f_i(x_B)\ , \qquad F_L = F_2 - 2x_B F_1 = 0$$

Breit frame

QED conserves parton helicity

$\longrightarrow \Delta S = 1$

transverse photon ($F_L = 0$)



Constraints on partonic distributions:

- Valence: $\int dx\ (f_u(x) - f_{\bar{u}}(x)) = 2\ , \qquad \int dx\ (f_d(x) - f_{\bar{d}}(x)) = 1$
- Strangeness: $\int dx\ (f_s(x) - f_{\bar{s}}(x)) = 0$
- Charge:
  $\frac{2}{3} \int dx\ (f_u(x) - f_{\bar{u}}(x)) - \frac{1}{3} \int dx\ (f_d(x) - f_{\bar{d}}(x)) - \frac{1}{3} \int dx\ (f_s(x) - f_{\bar{s}}(x)) = 1$
- Momentum: $\sum_i \int dx\ x f_i(x) = 1$

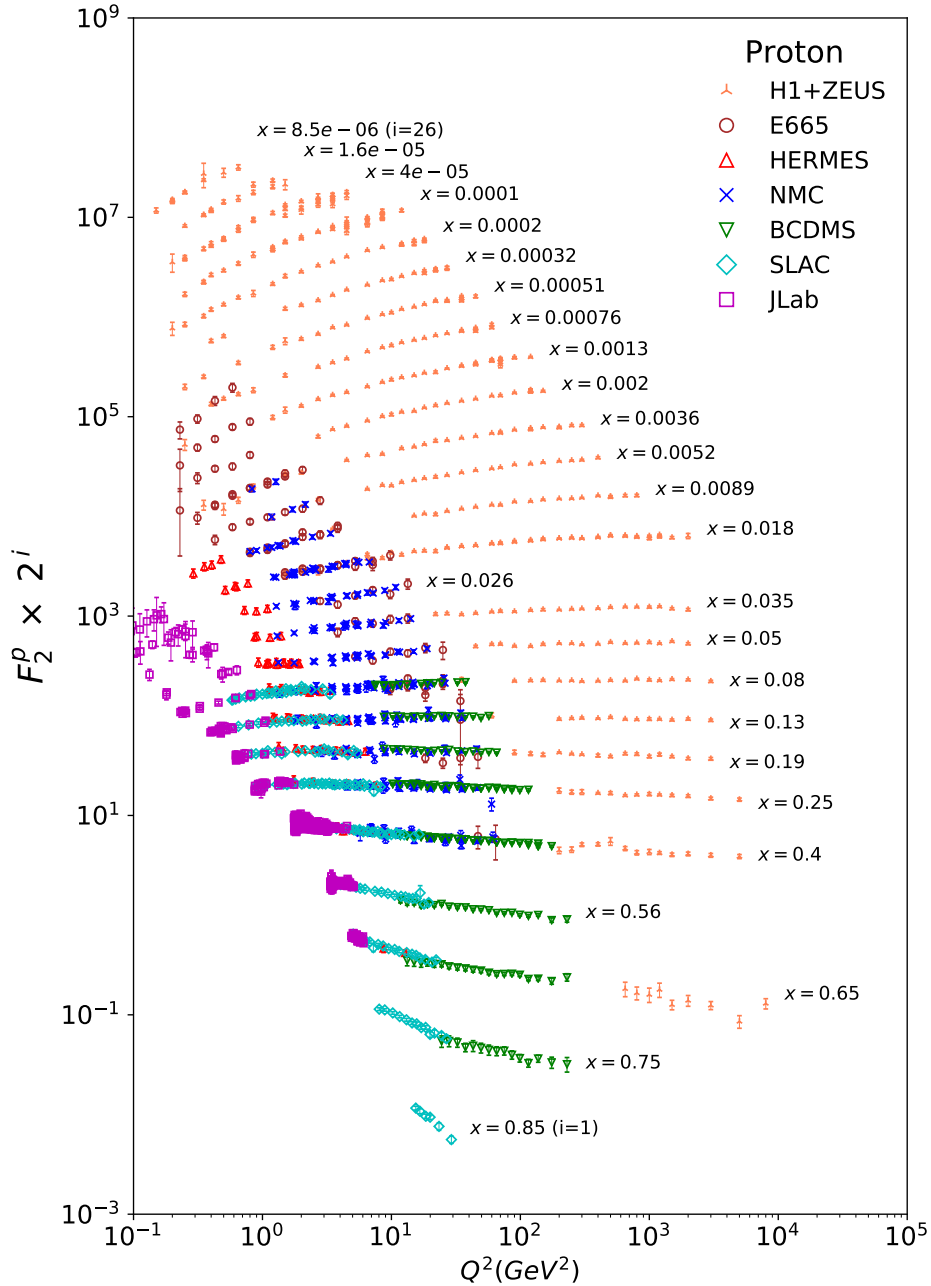
  if only quarks are summed, experimentally violated! Gluons needed!

Bjorken scaling, modern view (PDG 2020)

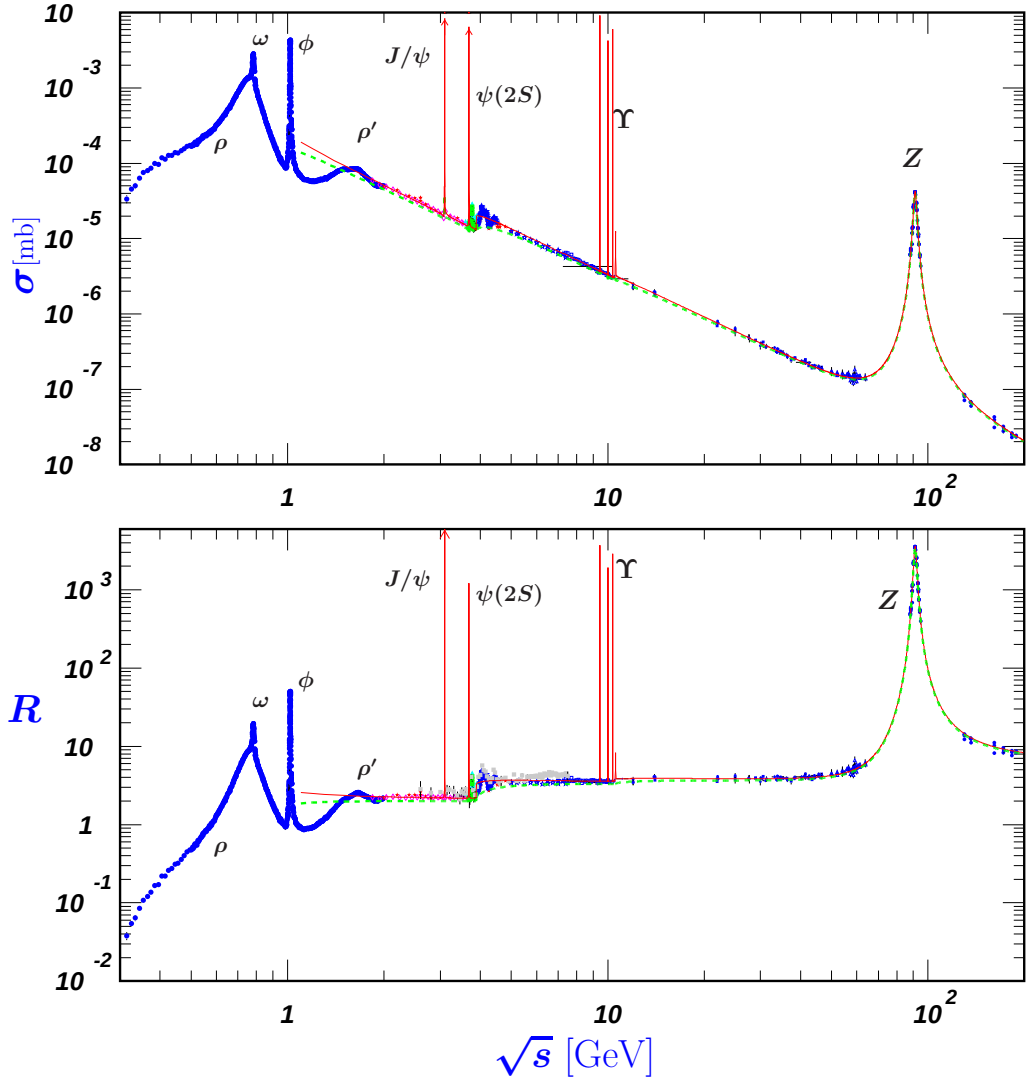
- Scaling violations!

  Milder at large x,
  stronger at small x

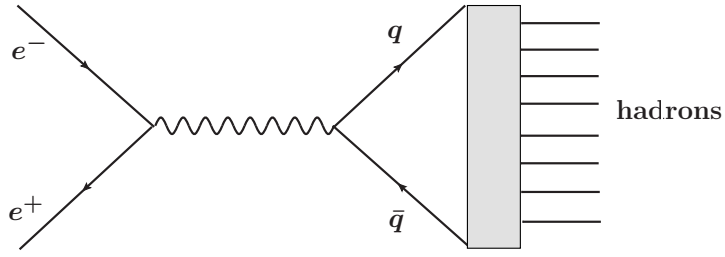
- The parton model must be improved!



R-ratio



$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q e_q^2$$



Theoretical development of non-Abelian gauge theories

- Non-Abelian gauge theories [Yang, Mills (1954)]
- Quantization of Yang-Mills theories [Faddeev, Popov (1967)]
- Renormalization of Yang-Mills theories ['t Hooft (1971)]
- "Asymptotic freedom" of Yang-Mills theories ... ['t Hooft (1972); Gross, Wilczek (1973); Politzer (1973)]
- ... and only of Yang-Mills theories [Coleman, Gross (1973)]
- Which gauge symmetry?

  $SU(N_c = 3)$ ... [Fritsch, Gell-Mann (1972); Fritsch, Gell-Mann, Leutwyler (1973)]

  ... + hypothesis that hadrons are color singlet states.

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The paradigm of gauge theories: QED

- Start from the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

invariant under global U(1) (electric charge conservation)

$$\psi \to e^{i\theta}\psi \ , \qquad \bar{\psi} \to e^{-i\theta}\bar{\psi} \ , \qquad \theta \text{ constant}$$

- Promote the U(1) invariance to local,

$$\psi \to e^{i\theta(x)}\psi \ , \qquad \bar{\psi} \to e^{-i\theta(x)}\bar{\psi} \ ,$$

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \ , \qquad D_\mu \equiv \partial_\mu + ieA_\mu \ ,$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\theta \quad (A_\mu \text{ photon field})$$

- Add a kinetic term for the photon field

$$[D_\mu, D_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv ie\, F_{\mu\nu} \ , \quad F_{\mu\nu} \text{ gauge invariant (photon neutral)}$$

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ ,$$

which gives the correct Maxwell equations

- Start from the free Dirac Lagrangian (just one quark flavor, for the moment)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \ , \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \ , \quad i = 1,2,3 \text{ (three color states)}$$

invariant under global SU(3) in the fundamental representation

$$\psi \to U\psi \ , \qquad U = \exp(i\theta^a t^a) \in \mathrm{SU}(3) \ , \qquad a = 1, \ldots, 8$$

SU(3) generators: $[t^a, t^b] = if^{abc}t^c$, f^{abc} structure constants, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$

- Promote the SU(3) invariance to local (*i.e.* $\theta^a \to \theta^a(x)$),

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \ , \qquad D_\mu \equiv \partial_\mu + igA_\mu \ , \qquad A_\mu \equiv t^a A^a_\mu$$

$$A_\mu \to UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \quad (A^a_\mu, a = 1, \ldots, 8, \text{ gluon fields})$$

- Add a kinetic term for the gluon fields

$$[D_\mu, D_\nu] = igF_{\mu\nu} \ , \quad F_{\mu\nu} = t^a F^a_{\mu\nu} \ , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu \ ,$$

$F^a_{\mu\nu}$ are not gauge invariant: $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$

$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ 3- and 4-gluon interaction terms here

Remarks

- a mass term for the gluon field $A^{a,\mu} A^a_\mu$ would break gauge invariance
- all terms in $\mathcal{L}_{\mathrm{QCD}}$ have dimension four, due to

  $$[\psi] = M^{3/2} \,, \quad [A^a_\mu] = M \,, \quad [\partial_\mu] = M \,, \quad [g] = M^0$$

  Lorentz- and gauge-invariant terms such as $(\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}))^2$ or similar would spoil renormalizability

- a term of the form $\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{a,\mu\nu} F^{a,\rho\sigma}$ cannot be excluded

  $\mathcal{L}_\theta \propto \vec{E}^a \cdot \vec{B}^a$ and therefore breaks parity (P) and time reversal (T)
  (recall: $\vec{E}$ is P-odd and T-even, $\vec{B}$ is P-even and T-odd)

  Current upper bounds on the electric neutron moment impose $\theta \lesssim 10^{-10}$.

QCD - Classical symmetries

Extension to many flavors: $\bar{\psi}(i\not{D} - m)\psi \longrightarrow \sum_{f=1}^{N_f=6} \bar{\psi}_f(i\not{D} - m_f)\psi_f$

- With no information about masses, $\mathcal{L}_{\text{QCD}}$ is invariant under $\psi_f \to e^{i\alpha_f}\psi_f$

  exact symmetry: $U(1)_u \otimes U(1)_d \otimes U(1)_s \otimes U(1)_c \otimes U(1)_b \otimes U(1)_t$

  each flavor is conserved separately

  $0 = \delta\mathcal{L} = \partial_\mu j^\mu, \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_f)}\delta\psi_f \propto \bar{\psi}_f\gamma^\mu\psi_f$

- Light quarks: $m_u \sim 2 - 4$ MeV, $m_d \sim 3 - 5$ MeV, $m_s \sim 100$ MeV [PDG 2020]

  $$\mathcal{L}_F^{(ud)} = \bar{\psi}(i\not{D} - M)\psi\,, \qquad \psi = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}\,, \qquad M = \begin{pmatrix}m_u & 0 \\ 0 & m_d\end{pmatrix}\,,$$

  approximate (good) symmetry: $U(2) = \underbrace{U(1)}_{\text{baryon no.}} \otimes \underbrace{SU(2)}_{\text{isospin}}$

  $$\mathcal{L}_F^{(uds)} = \bar{\psi}(i\not{D} - M)\psi\,, \qquad \psi = \begin{pmatrix}\psi_u \\ \psi_d \\ \psi_s\end{pmatrix}\,, \qquad M = \begin{pmatrix}m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s\end{pmatrix}\,,$$

  approximate symmetry: $U(3) = \underbrace{U(1)}_{\text{baryon no.}} \otimes \underbrace{SU(3)}_{\text{isospin}}$

- $m_{u,d} \ll$ "typical hadron mass" (barely true also for m_s):

  assume $m_l = 0$ for $l = u, d$ ($L_f = 2$) or even for $l = u, d, s$ ($L_f = 3$) (chiral limit)

$$\psi_{R,L} = \frac{1 \pm \gamma_5}{2}\,\psi \equiv P_{R,L}\psi\ , \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\mathcal{L}_F^{(L_f)} = \bar{\psi}_R(i\not{D})\psi_R + \bar{\psi}_L(i\not{D})\psi_L - \bar{\psi}_R M\psi_L - \bar{\psi}_L M\psi_R\ , \quad M \text{ mass matrix } L_f \times L_f$$

for $M \to 0$, symmetry: $U(L_f)_L \otimes U(L_f)_R = \underbrace{U(1)_L \otimes U(1)_R}_{U(1)_V \otimes U(1)_A} \otimes \underbrace{SU(L_f)_L \otimes SU(L_f)_R}_{SU(L_f)_V \otimes SU(L_f)_A}$

$$U(1)_L:\ \psi_L \to e^{i\theta_L}\psi_L\ , \qquad U(1)_R:\ \psi_R \to e^{i\theta_R}\psi_R\ ,$$
$$U(1)_V:\ \psi \to e^{i\theta}\psi\ , \qquad U(1)_A:\ \psi \to e^{i\tilde{\theta}\gamma_5}\psi$$

$$SU(L_f)_L:\ \psi_L \to e^{i\theta_L^a t^a}\psi_L\ , \qquad SU(L_f)_R:\ \psi_R \to e^{i\theta_R^a t^a}\psi_R\ ,$$
$$SU(L_f)_V:\ \psi \to e^{i\theta^a t^a}\psi\ , \qquad SU(L_f)_A:\ \psi \to e^{i\tilde{\theta}^a t^a\gamma_5}\psi$$

- (Classically) conserved currents:

$$\underbrace{U(1)_V}_{\text{baryon no.}} : j_V^\mu = \sum_{i=1}^{L_f} \bar{\psi}_i \gamma^\mu \psi_i \ , \qquad \underbrace{U(1)_A}_{\text{anomalous}} : j_5^\mu = \sum_{i=1}^{L_f} \bar{\psi}_i \gamma^\mu \gamma_5 \psi_i \ ,$$

$$\underbrace{SU(L_f)_V}_{\text{isospin}} : j_V^{a,\mu} = \sum_{i=1}^{L_f} \bar{\psi}_i \gamma^\mu t^a \psi_i \ , \qquad \underbrace{SU(L_f)_A}_{\text{spont. broken}} : j_A^{a,\mu} = \sum_{i=1}^{L_f} \bar{\psi}_i \gamma^\mu \gamma_5 t^a \psi_i$$

$$\partial_\mu j_V^\mu = \delta \mathcal{L}_F^{(L_f)} = 0$$

$$\partial_\mu j_A^\mu = \delta \mathcal{L}_F^{(L_f)} = 2i\bar{\psi}\gamma_5 M\psi + \text{quantum anomaly}$$

$$\partial_\mu j_V^{a,\mu} = \delta \mathcal{L}_F^{(L_f)} = i \sum_{i,j} (m_i - m_j) \bar{\psi}_i t_{ij}^a \psi_j$$

$$\partial_\mu j_A^{a,\mu} = \delta \mathcal{L}_F^{(L_f)} = i \sum_{i,j} (m_i + m_j) \bar{\psi}_i t_{ij}^a \psi_j$$

$$\text{quantum anomaly} = L_f \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{a,\mu\nu} F^{a,\rho\sigma}$$

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   - Renormalization
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Quantization

- Let's start again from QED:

$$\mathcal{L}_{\rm QED} = \bar{\psi}(i\not{D} - m)\psi \; - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \; , \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Not suitable for quantization: $\Pi^\mu = \frac{\partial \mathcal{L}_{\rm QED}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \longrightarrow \Pi^0 = 0$

Solution: gauge fixing! (different gauges lead to the same physics).

Covariant gauges: $\Delta\mathcal{L}_{\rm QED} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ (Maxwell theory + $\partial_\mu A^\mu = 0$).

Photon propagator: $D^{\mu\nu}(k) = \frac{i}{k^2}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$

- Ward identity: process with incoming photon, no matter other external particles

$$\mathcal{M} = \epsilon_\mu(k)\mathcal{M}^\mu = \underbrace{(\epsilon_\mu(k) + \alpha k_\mu)}_{\text{gauge transformation}} \mathcal{M}^\mu \; , \qquad \longrightarrow k_\mu \mathcal{M}^\mu = 0$$

Say $k^\mu = (k, 0, 0, k)$, then $0 = k_\mu \mathcal{M}^\mu = k(\mathcal{M}^0 - \mathcal{M}^3) \longrightarrow \mathcal{M}^0 = \mathcal{M}^3$

$$\sum_{\rm pol.} |\mathcal{M}|^2 = \underbrace{\sum_{\rm pol.} \epsilon_\mu(k)\epsilon^*_\nu(k)}_{-g_{\mu\nu}} \mathcal{M}^\mu \mathcal{M}^{*\nu} = -(\cancel{|\mathcal{M}^0|^2} - |\mathcal{M}^1|^2 - |\mathcal{M}^2|^2 - \cancel{|\mathcal{M}^3|^2})$$

only transverse photons matter!

- In QCD, covariance gauge fixing by $\Delta\mathcal{L}_{\rm QCD} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2$ is not enough.

  Unphysical gluon degrees of freedom are not automatically cancelled in cross sections and in internal loops.

- Extra fields: Faddeev-Popov ghosts

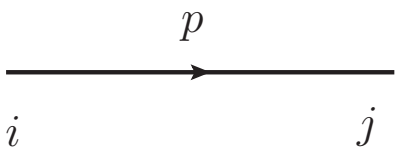
$$\Delta\mathcal{L}_{\rm QCD} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2 \; + \; \partial_\mu \eta^{a\dagger}(\underbrace{\partial^\mu \delta^{ab} + g f_{abc} A^{c,\mu}}_{\equiv D^\mu_{ab}})\eta^b$$

  They anti-commute as fermionic fields, but propagate as bosonic ones

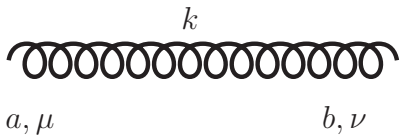
- In physical gauges, $n \cdot A^a = 0$, there are no ghosts, but the gluon propagator takes the form

$$D^{ab,\mu\nu}(k) = \delta^{ab}\,\frac{i}{k^2}\left(-g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n\cdot k} - (n^2 + \xi k^2)\frac{k^\mu k^\nu}{(n\cdot k)^2}\right)$$

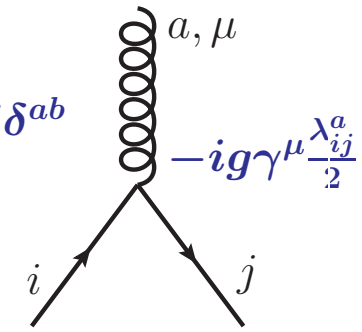
QCD Feynman rules - Covariant gauge, $\xi = 1$



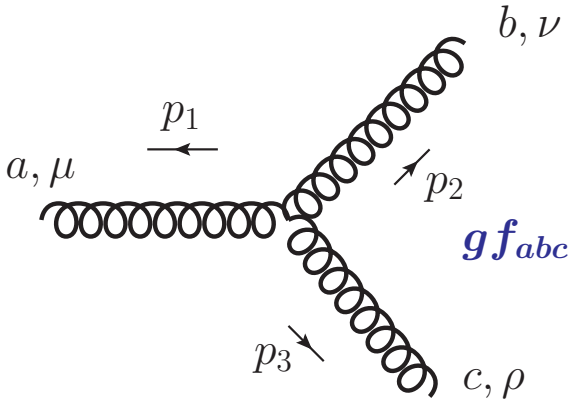
$$\frac{i}{\not{p}-m}\delta_{ij}$$



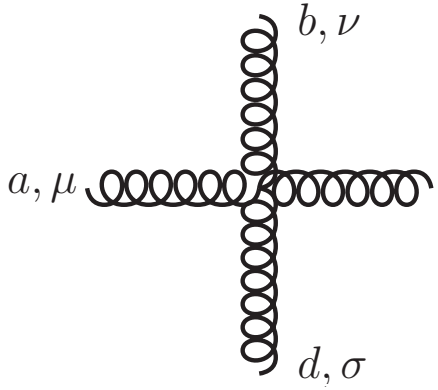
$$-\frac{i}{k^2}g^{\mu\nu}\delta^{ab}$$



$$-ig\gamma^{\mu}\frac{\lambda^a_{ij}}{2}$$



$$gf_{abc}[g_{\mu\nu}(p_1-p_2)_\rho + g_{\nu\rho}(p_2-p_3)_\mu + g_{\rho\mu}(p_3-p_1)_\nu]$$



$$ig^2\,[f_{abe}f_{cde}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) + f_{ace}f_{dbe}(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\nu}g_{\rho\sigma}) + f_{ade}f_{bce}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})]$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} ,$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} ,$$

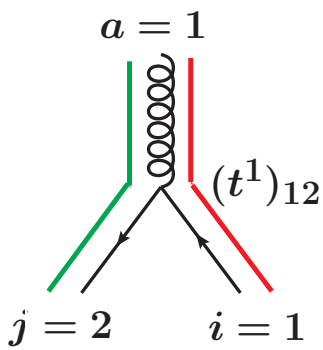
$$t^a = \frac{\lambda^a}{2}$$

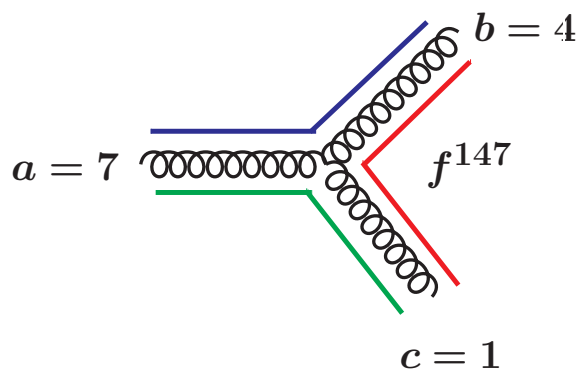
$$\mathrm{Tr}\left(\frac{\lambda^a}{2} \frac{\lambda^b}{2} \right) = \frac{\delta^{ab}}{2} , \quad \left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2} ,$$

$$f_{123} = 1 , \quad f_{458} = f_{678} = \sqrt{3}/2 , \quad f_{147} = f_{165} = f_{246} = f_{345} = f_{376} = f_{257} = 1/2$$

$$|R\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix} \ , \quad |G\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix} \ , \quad |B\rangle \doteq \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{aligned}
\lambda_1 &= |G\rangle\langle R| + |R\rangle\langle G| \ , \quad \lambda_2 = i|G\rangle\langle R| - i|R\rangle\langle G| \ , \quad \lambda_3 = |R\rangle\langle R| - |G\rangle\langle G| \ , \\
\lambda_4 &= |B\rangle\langle R| + |R\rangle\langle B| \ , \quad \lambda_5 = i|B\rangle\langle R| - i|R\rangle\langle B| \ , \\
\lambda_6 &= |B\rangle\langle G| + |G\rangle\langle B| \ , \quad \lambda_7 = i|B\rangle\langle G| - i|G\rangle\langle B| \ , \\
\lambda_8 &= \frac{1}{\sqrt{3}}|R\rangle\langle R| + \frac{1}{\sqrt{3}}|G\rangle\langle G| - \frac{2}{\sqrt{3}}|B\rangle\langle B|
\end{aligned}$$





$$\begin{pmatrix}0 & 1 & 0\end{pmatrix} \underbrace{\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{pmatrix}}_{\lambda_1} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

- Quark (say, red colored) splitting to quark-gluon

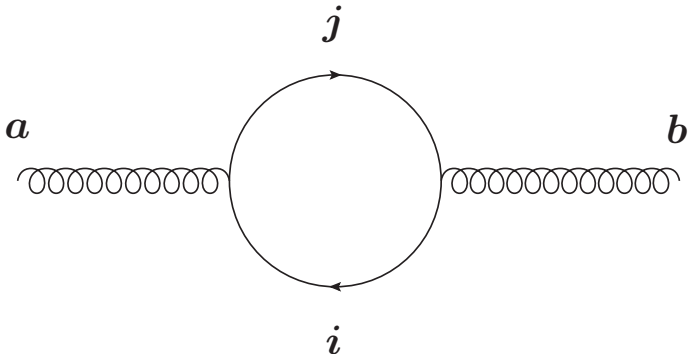
$$\begin{aligned}
\mathrm{Prob}(R \to \mathrm{qg}) &= \mathrm{Prob}(R \to R) + \mathrm{Prob}(R \to G) + \mathrm{Prob}(R \to B) \\
&= |\langle R|\frac{\lambda_3}{2}|R\rangle|^2 + |\langle R|\frac{\lambda_8}{2}|R\rangle|^2 \\
&+ |\langle G|\frac{\lambda_1}{2}|R\rangle|^2 + |\langle G|\frac{\lambda_2}{2}|R\rangle|^2 \\
&+ |\langle B|\frac{\lambda_4}{2}|R\rangle|^2 + |\langle B|\frac{\lambda_5}{2}|R\rangle|^2 = \ldots = \frac{4}{3} \equiv C_F \left(= \frac{N_c^2 - 1}{2N_c}\right)
\end{aligned}$$

- Gluon (say, no. 1) splitting to gluon-gluon

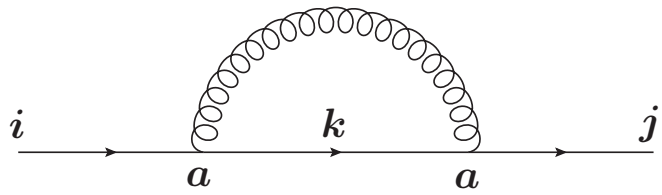
$$\begin{aligned}
\mathrm{Prob}(1 \to \mathrm{gg}) &= 2(\mathrm{Prob}(1 \to 2+3) + \mathrm{Prob}(1 \to 4+7) + \mathrm{Prob}(1 \to 5+6)) \\
&= 2(f_{123}^2 + f_{147}^2 + f_{156}^2) = 3 \equiv C_A (= N_c)
\end{aligned}$$

- Gluon (say, no. 1) splitting to quark-antiquark

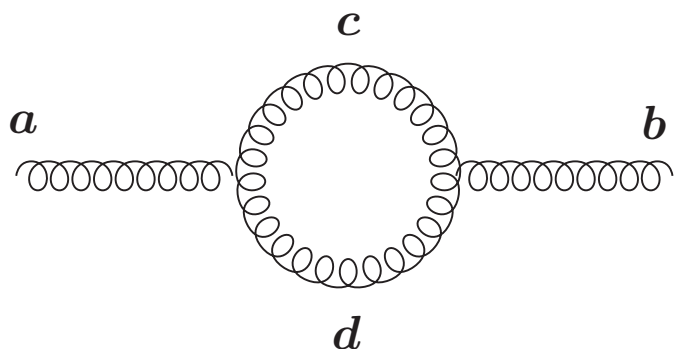
$$\mathrm{Prob}(1 \to \mathrm{q\bar{q}}) = \left|\frac{\lambda_{12}^1}{2}\right|^2 + \left|\frac{\lambda_{21}^1}{2}\right|^2 = \frac{1}{2} \equiv T_R$$



$$t^a_{ij} t^b_{ji} = \mathrm{Tr}(t^a t^b) = \tfrac{1}{2}\delta^{ab} \equiv T_R \delta^{ab}$$



$$t^a_{ik} t^a_{kj} = (t^a t^a)_{ij} = \tfrac{4}{3}\delta_{ij} \equiv C_F \delta_{ij}$$



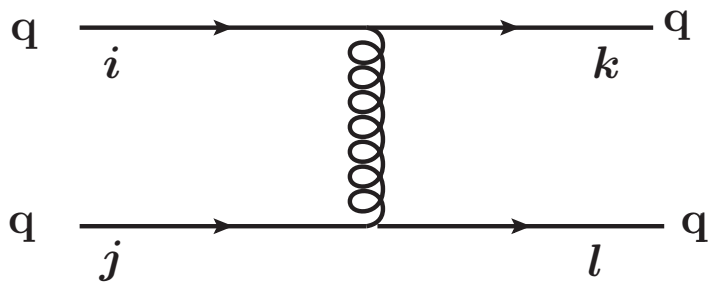
$$\underbrace{-f^{acd} f^{cbd}}_{\mathrm{Tr}(T^a T^b)} = f^{acd} f^{bcd} = 3\delta^{ab} \equiv C_A \delta^{ab}$$

$$t^a \equiv \frac{\lambda^a}{2}\,, \qquad T^a_{bc} = i f_{abc}$$

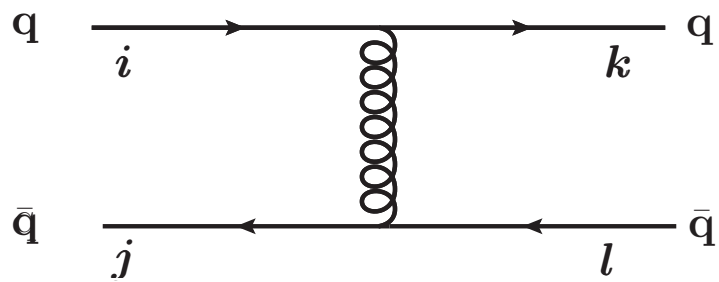
$$C_A = N_c\,, \quad C_F = \frac{N_c^2 - 1}{2N_c}\,, \quad C_A > C_F$$

Gluon charge is larger than quark charge!

- quark-quark and quark-antiquark scattering



$\frac{1}{4}\Sigma_a \lambda^a_{ik}\lambda^a_{jl}$



$\frac{1}{4}\Sigma_a \lambda^a_{ik}\lambda^a_{lj}$

- Examples:

qq, same color, $RR \to RR$: $\frac{1}{4}\left(\frac{\lambda^3_{11}}{2}\frac{\lambda^3_{11}}{2} + \frac{\lambda^8_{11}}{2}\frac{\lambda^8_{11}}{2}\right) = \frac{1}{4}\left(1+\frac{1}{3}\right) = \frac{1}{3}$

qq, two colors (no swap), $RB \to RB$: $\frac{1}{4}\frac{\lambda^8_{11}}{2}\frac{\lambda^8_{33}}{2} = \frac{1}{4}\frac{1}{\sqrt{3}}\frac{-2}{\sqrt{3}} = -\frac{1}{6}$

qq, two (with swap), $RG \to GR$: $\frac{1}{4}\left(\frac{\lambda^1_{12}}{2}\frac{\lambda^1_{21}}{2} + \frac{\lambda^2_{12}}{2}\frac{\lambda^2_{21}}{2}\right) = \frac{1}{4}(1+1) = \frac{1}{2}$

qq, three colors colors, $RB \to RG$: 0 (color conservation)

- Further examples:

  $q\bar{q}$, same as qq, just pay attention to position of color indices,

$$\underbrace{\sum_a \lambda^a_{ik}\lambda^a_{jl}}_{qq} \longrightarrow \underbrace{\sum_a \lambda^a_{ik}\lambda^a_{lj}}_{q\bar{q}}$$

  A useful shortcut, $\sum_a \frac{\lambda^a_{ik}}{2}\frac{\lambda^a_{jl}}{2} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ik}\delta_{jl}\right)$

- Problem: consider the $q\bar{q}$ scattering by one-gluon exchange for a pair in the

  1) singlet state, $\frac{1}{\sqrt{3}}\left(|R\bar{R}\rangle + |G\bar{G}\rangle + |B\bar{B}\rangle\right)$

  2) in an octet state, *e.g.* $|R\bar{G}\rangle$.

  Show, by comparison with the known attractive e^+e^- scattering by one-photon exchange, that case 1) is attractive, case 2) is repulsive.

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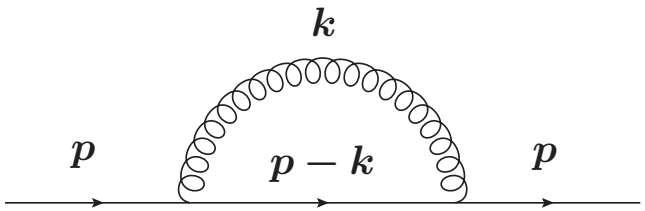
Renormalization

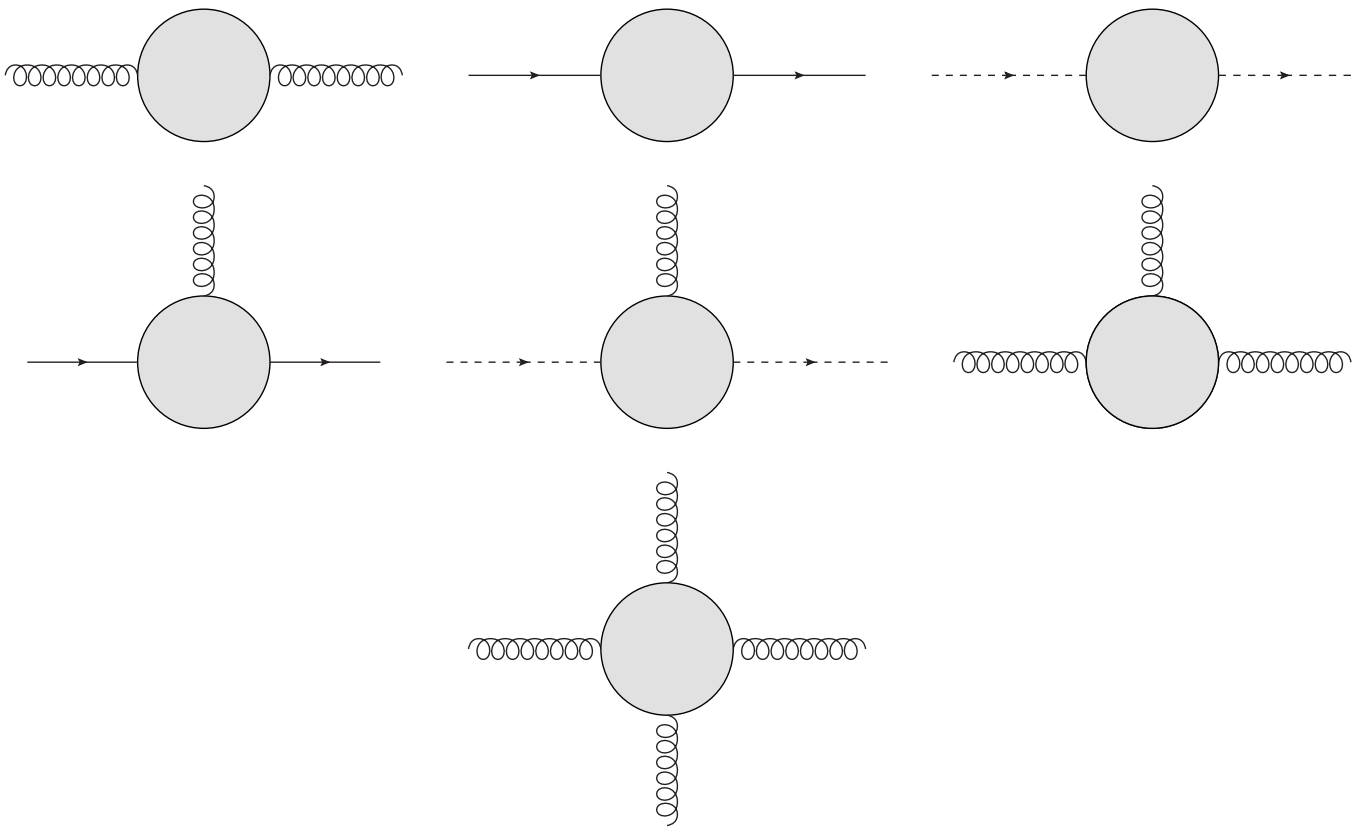


E.g., quark self-energy (up to a color factor):

$$-g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{p} - \not{k} + m)\gamma_\mu}{[(p-k)^2 - m^2]\, k^2}$$

divergent for large k, by power counting
(ultraviolet divergence)

Processes where UV divergences appear in QCD



- **Regularization**: gives meaning to div. integrals, by cutting-off large momenta

  Most commonly: **dimensional regularization** (space-time dimension D).

  E.g., $I(n) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-s)^n}$ does not exist for $n \leq 2$,

  but $I_D(n) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-s)^n}$ does exist for $n > D/2$,

  $I_D(n) = i\frac{(-1)^n}{\Gamma(n)} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{s^{n-D/2}}$, analytically continued to all complex values D

  The divergence for $D \to 4$ resurfaces as pole in $\epsilon \equiv \frac{4-D}{2}$, *e.g.*

$$I_D(n=2) = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{s^{2-D/2}} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{s^\epsilon} , \qquad \Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$

- Dimensional analysis:

$$[\mathcal{L}] = M^D , \quad [A^a_\mu] = M^{\frac{D-2}{2}} , \quad [\psi] = M^{\frac{D-1}{2}} \quad \longrightarrow \quad [g] = M^{\frac{4-D}{2}} = M^\epsilon$$

  To keep working with a dimensionless coupling, redefine g as $g\mu^\epsilon$.

  μ is an arbitrary mass scale.

- Re-normalize fields and parameters (original ones labeled by $^{(0)}$):

$$A^{(0)} = Z_A^{1/2} A\,, \quad \psi^{(0)} = Z_\psi^{1/2} \psi\,, \quad \eta^{(0)} = Z_\eta^{1/2} \eta\,,$$

$$m^{(0)} = Z_m m\,, \quad g^{(0)} = Z_g g\,, \quad \xi^{(0)} = Z_A \xi$$

Each term in $\mathcal{L}_{\rm QCD}$ can be recast as follows:

$$\bar\psi^{(0)} i\not\partial \psi^{(0)} = Z_\psi \bar\psi i\not\partial\psi = \bar\psi i\not\partial\psi + \underbrace{(Z_\psi - 1)\bar\psi i\not\partial\psi}_{\text{counterterm}}$$

$$g^{(0)}\bar\psi^{(0)}\not{A}^{(0)}\psi^{(0)} = \underbrace{Z_g Z_\psi Z_A^{1/2}}_{Z_{A\bar\psi\psi}}\, g\bar\psi\not{A}\psi = g\bar\psi\not{A}\psi + \underbrace{(Z_{A\bar\psi\psi} - 1) g\bar\psi\not{A}\psi}_{\text{counterterm}}$$

...

$$\mathcal{L}_{\rm QCD}[\psi^{(0)}, A^{(0)}, \eta^{(0)}, m^{(0)}, g^{(0)}, \xi^{(0)}] = \underbrace{\mathcal{L}_{\rm QCD}[\psi, A, \eta, m, g, \xi]}_{\text{same form as original}} + \underbrace{\mathcal{L}_{\rm counterterms}}_{\text{extra interactions}}$$

- Cancel ϵ-poles in divergent "processes", order by order in perturbation theory, by suitably choosing Z_ψ, Z_A, Z_η, $Z_{m\bar\psi\psi}$, $Z_{A\bar\psi\psi}$, $Z_{A\bar\eta\eta}$, Z_{A^3}, Z_{A^4}.

  $\overline{\rm MS}$ scheme: subtract not just $\frac{1}{\epsilon}$, but $\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E$.

- In particular, determine Z_g:

$$g^{(0)} = Z_g\, g\mu^\epsilon\ , \qquad Z_g = \frac{Z_{A\bar\psi\psi}}{Z_A^{1/2} Z_\psi} = \frac{Z_{A\bar\eta\eta}}{Z_A^{1/2} Z_\eta} = \frac{Z_{A^3}}{Z_A^{3/2}} = \sqrt{\frac{Z_{A^4}}{Z_A^2}}$$

[Taylor (1971); Slavnov (1972)]

$\overline{\text{MS}}$ scheme:

$$Z_g = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left[\frac{2}{3}T_R n_f - \frac{11}{6}C_A\right] + \mathcal{O}(\alpha_s^2) \equiv 1 + \frac{\beta_0}{\epsilon}\frac{\alpha_s}{2} + \mathcal{O}(\alpha_s^2)\ , \qquad \alpha_s \equiv \frac{g^2}{4\pi}$$

n_f is the number of "light" quarks, *i.e.* quarks with $m_q \ll \mu$

- The (dimensionless) renormalized coupling g evidently depends on μ. How?

$$\beta(g) \equiv \mu\frac{\partial g}{\partial \mu}\ , \quad 0 = \mu\frac{dg^{(0)}}{d\mu} = \mu\frac{d(Z_g\, g\mu^\epsilon)}{d\mu}$$

$$\longrightarrow \beta(g) = -\epsilon g - \frac{\mu}{Z_g}\frac{\partial Z_g}{\partial \mu} g = -\epsilon g - \beta_0 \frac{g^2}{4\pi}\ \frac{\beta(g)}{\epsilon} + \mathcal{O}(g^5) = -\beta_0\frac{g^3}{4\pi} + \mathcal{O}(g^5, \epsilon)$$

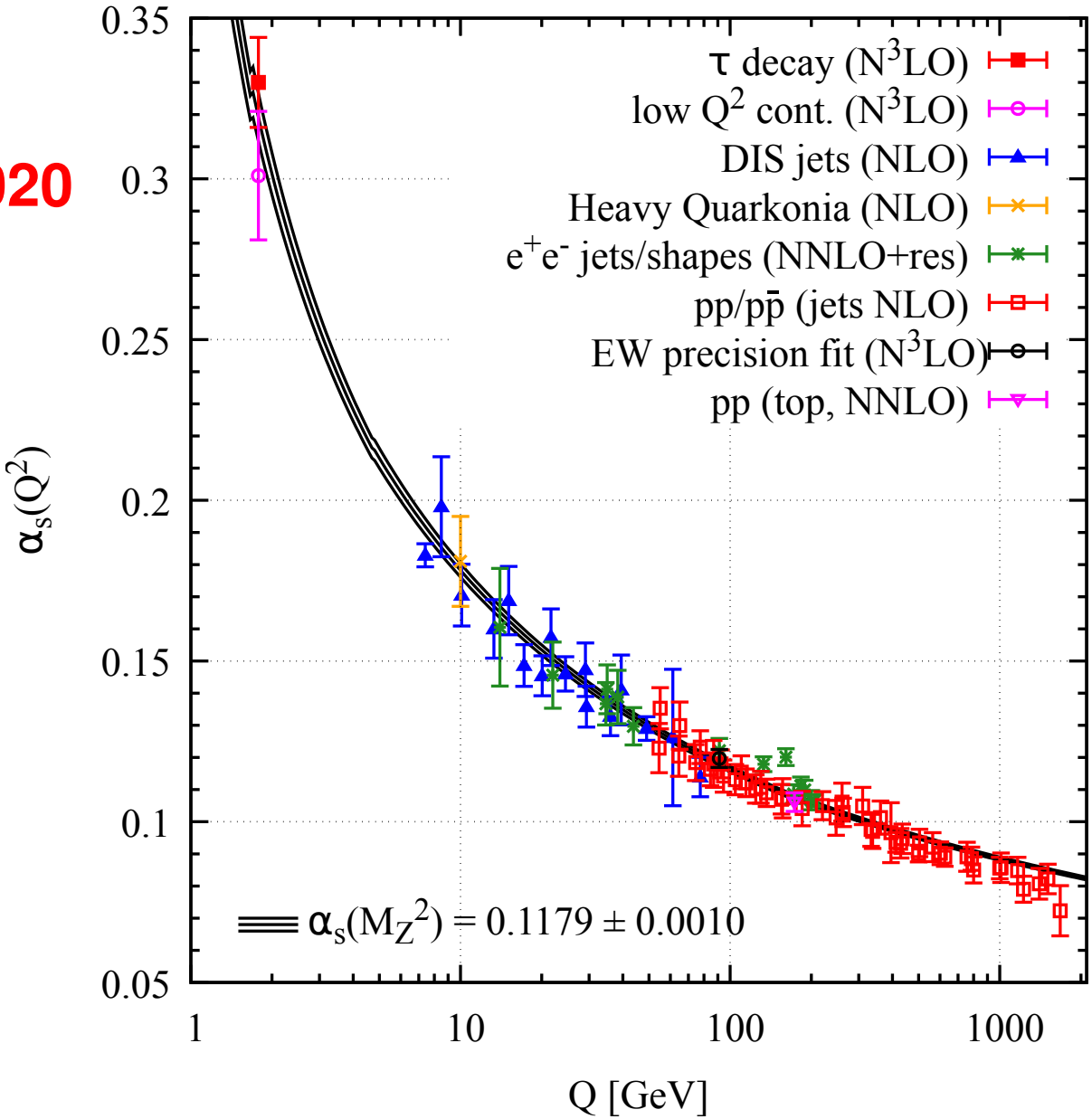
$$\beta(\alpha_s) \equiv \mu^2\frac{\partial \alpha_s}{\partial \mu^2} = -\alpha_s^2\beta_0 + \mathcal{O}(\alpha_s^3)$$

$$\frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + \beta_0 \ln\frac{\mu^2}{\mu_0^2} \quad \text{or} \quad \alpha_s(\mu^2) = \frac{1}{\beta_0 \ln\frac{\mu^2}{\Lambda^2_{\text{QCD}}}}$$

"Dimensional transmutation"

$$\beta(\alpha_s) = -\alpha_s^2(\beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + \beta_3\alpha_s^3 + \beta_4\alpha_s^4 + \ldots)$$

PDG2020



Renormalization group - an example

Let's consider again $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$, up to an irrelevant normalization:

$$R\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 1 + \sum_{n=1}^{\infty} r_n(Q^2/\mu^2)\,\alpha_s^n(\mu^2)\,, \quad Q^2 \text{ squared c.o.m. energy}$$

$$0 = \mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right] R\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

$$= \mu^2 \frac{dr_1}{d\mu^2}\alpha_s + \left(\mu^2 \frac{dr_2}{d\mu^2} - r_1\beta_0\right)\alpha_s^2 + \left(\mu^2 \frac{dr_3}{d\mu^2} - 2r_2\beta_0 - r_1\beta_1\right)\alpha_s^3 + \ldots$$

$$r_1 = c_1\,, \quad r_2 = c_2 + c_1\beta_0 t\,, \quad r_3 = c_3 + (2c_2\beta_0 + c_1\beta_1)t + c_1\beta_0 t^2\,, \quad t \equiv \ln\frac{\mu^2}{Q^2}$$

$$\begin{aligned} R(t, \alpha_s) &= 1 + c_1\left(1 + \beta_0\alpha_s t + (\beta_0\alpha_s t)^2 + \ldots\right)\alpha_s + c_2\alpha_s^2 + \ldots \\ &= 1 + c_1 \frac{\alpha_s(\mu^2)}{1 + \beta_0\alpha_s(\mu^2)\ln\frac{Q^2}{\mu^2}} + \ldots \end{aligned}$$

The choice $\mu = Q$ removes (possibly large) logarithms from the perturbative series:

$$R(1, \alpha_s(Q^2)) = 1 + c_1\alpha_s(Q^2) + c_2\alpha_s^2(Q^2) + \ldots$$

$\mu = Q$ not always the best choice, due to truncation: the first unknown c_i can be large.

In many applications, results are quoted for μ ranging in the interval $[\frac{Q}{2}, 2Q]$.

Optimization procedures have been designed to fix the μ scale.

Take R truncated at the order α_s^2:

$$R^{(2)}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 1 + c_1\alpha_s(\mu^2) + \left[c_2 - c_1\beta_0 \ln\frac{Q^2}{\mu^2}\right]\alpha_s^2(\mu^2)$$

1) Principle of Minimum Sensitivity [Stevenson (1981)]

$$\mu^2\frac{d}{d\mu^2}\left.R^{(2)}(\mu^2)\right|_{\mu^2_{\rm PMS}} = 0 \quad \longrightarrow \quad \mu^2_{\rm PMS} = Q^2\exp\left(-\frac{c_2}{\beta_0 c_1} - \frac{\beta_1}{2\beta_0^2}\right)$$

2) Fast apparent convergence [Grunberg (1980)]

$$R^{(1)}(\mu^2_{\rm FAC}) = R^{(2)}(\mu^2_{\rm FAC}) \quad \longrightarrow \quad \mu^2_{\rm FAC} = Q^2\exp\left(-\frac{c_2}{\beta_0 c_1}\right)$$

3) BLM procedure [Brodsky, Lepage, Mackenzie (1983)]

choose μ so to remove the n_f dependence in the perturbative coefficients

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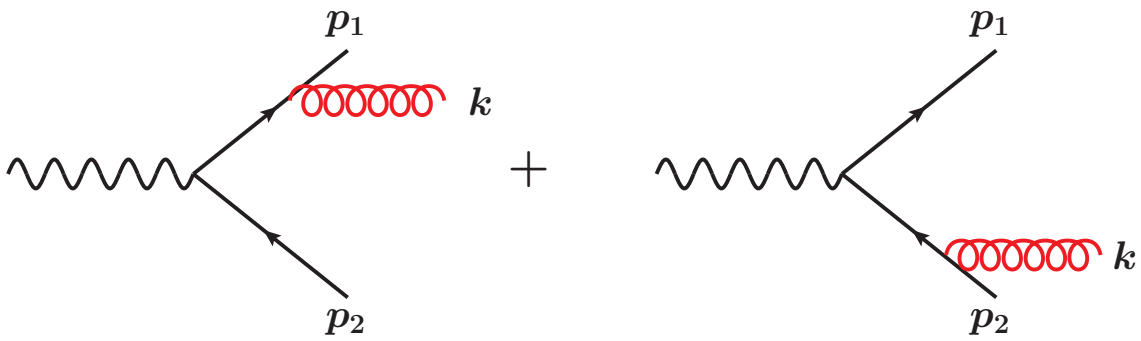
Infrared divergences

Consider (again) $e^+ + e^- \to \gamma^* \to$ hadrons, including QCD effects up to $\mathcal{O}(\alpha_s)$

Real corrections: gluon emission



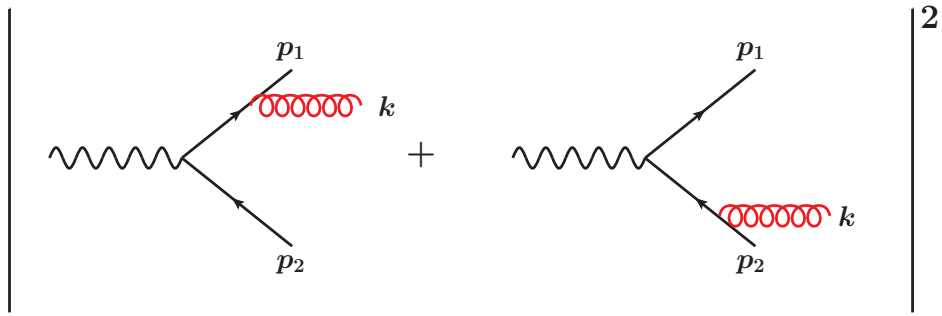
$$\mathcal{M}^{\mu}_{q\bar{q}g} = \bar{u}(p_1)\left[(-igt^a \not{\epsilon}(k))\, i\frac{\not{p}_1 + \not{k}}{(p_1+k)^2}(-ie_q\gamma^\mu) + (-ie_q\gamma^\mu)\, i\frac{-\not{p}_2 - \not{k}}{(p_2+k)^2}(-igt^a \not{\epsilon}(k))\right] v(p_2)$$

Soft limit, $k \to 0$, plus Dirac equation ($\bar{u}(p_1)\not{p}_1 = 0 = \not{p}_2 v(p_2)$, massless quarks)

$$\mathcal{M}^{\mu}_{q\bar{q}g} \simeq \underbrace{\bar{u}(p_1)(-ie_q\gamma^\mu)v(p_2)}_{\mathcal{M}^{\mu}_{q\bar{q}}}\ (gt^a)\ \left(\frac{p_1\cdot\epsilon}{p_1\cdot k} - \frac{p_2\cdot\epsilon}{p_2\cdot k}\right)$$

$$H^{\mu\nu}_{q\bar{q}g} \simeq \sum_{\text{color, pol}} \mathcal{M}^{\mu}_{q\bar{q}}(\mathcal{M}^{\nu}_{q\bar{q}})^* \left|(gt^a)\ \left(\frac{p_1\cdot\epsilon}{p_1\cdot k} - \frac{p_2\cdot\epsilon}{p_2\cdot k}\right)\right|^2$$

$$= -H^{\mu\nu}_{q\bar{q}} C_F g^2 \left(\frac{p_1}{p_1\cdot k} - \frac{p_2}{p_2\cdot k}\right)^2 = H^{\mu\nu}_{q\bar{q}} C_F g^2 \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}$$

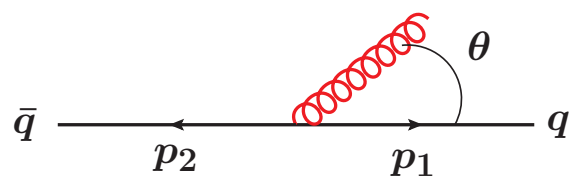


Include phase space:

$$H^{\mu\nu}_{q\bar{q}g}\, d\Phi_{q\bar{q}g} \simeq H^{\mu\nu}_{q\bar{q}}\, d\Phi_{q\bar{q}} \underbrace{\frac{d^3\vec{k}}{(2\pi)^3 2\omega}\, C_F g^2 \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}}_{\text{soft}-\text{gluon factor}\equiv dS}$$

In the quark-antiquark c.o.m. frame

$$dS = \omega d\omega\, d\cos\theta \frac{d\phi}{2\pi} \frac{2\alpha_s C_F}{\pi} \underbrace{\frac{2p_1\cdot p_2}{(2p_1\cdot k)(2p_2\cdot k)}}_{\frac{1}{\omega^2(1-\cos\theta^2)}}$$

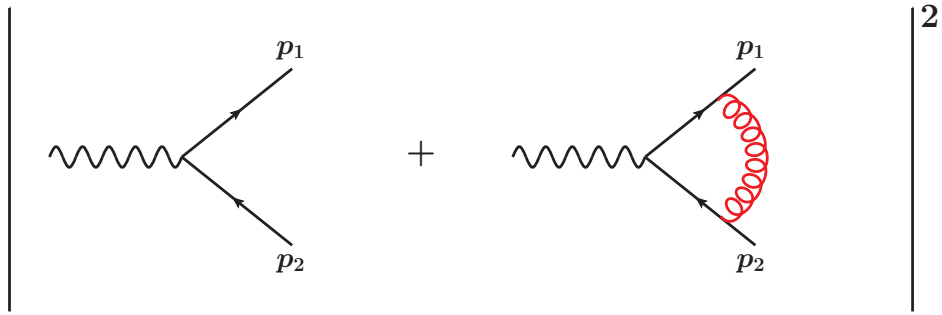


$$= \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

Divergence for $\omega \to 0$: **soft**

Divergence for $\theta = 0, \pi$: **collinear**

Virtual corrections: no gluon emission



Interference term, $\mathcal{O}(\alpha_s)$, in the region of soft gluon momenta:

$$H^{\mu\nu}_{q\bar{q},V} \simeq^{!} - H^{\mu\nu}_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

Total cross section

It must include both real and virtual gluon emission: $H^{\mu\nu}_{q\bar{q},\mathrm{tot}} = H^{\mu\nu}_{q\bar{q}g} + H^{\mu\nu}_{q\bar{q},V}$ infrared-finite!

Actual calculation: $\sigma_{\mathrm{tot}} = \sigma_B \left[1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right]$

Lessons

- Infrared divergences are related to long-distance/time physics: an internal line "goes on-shell".

  Propagator: $\frac{\not{p}_1}{2p_1\cdot k} \sim \frac{1}{\omega\theta^2}$

  Time scales: soft, $\sim 1/(\omega\theta^2)$, much larger than hard, $\sim 1/Q$

- The cancellation of infrared divergences occurs for "inclusive enough" final states - Kinoshita-Lee-Nauenberg theorem [Kinoshita (1962); Lee, Nauenberg (1964)]

  In our example, at the $\mathcal{O}(\alpha_s)$ two degenerate final states were considered, $q\bar{q}$ and $q\bar{q}g$, which are not experimentally distinguishable for a soft/collinear gluon.

- Infrared-safe observables must be insensitive to emissions of soft/collinear partons:

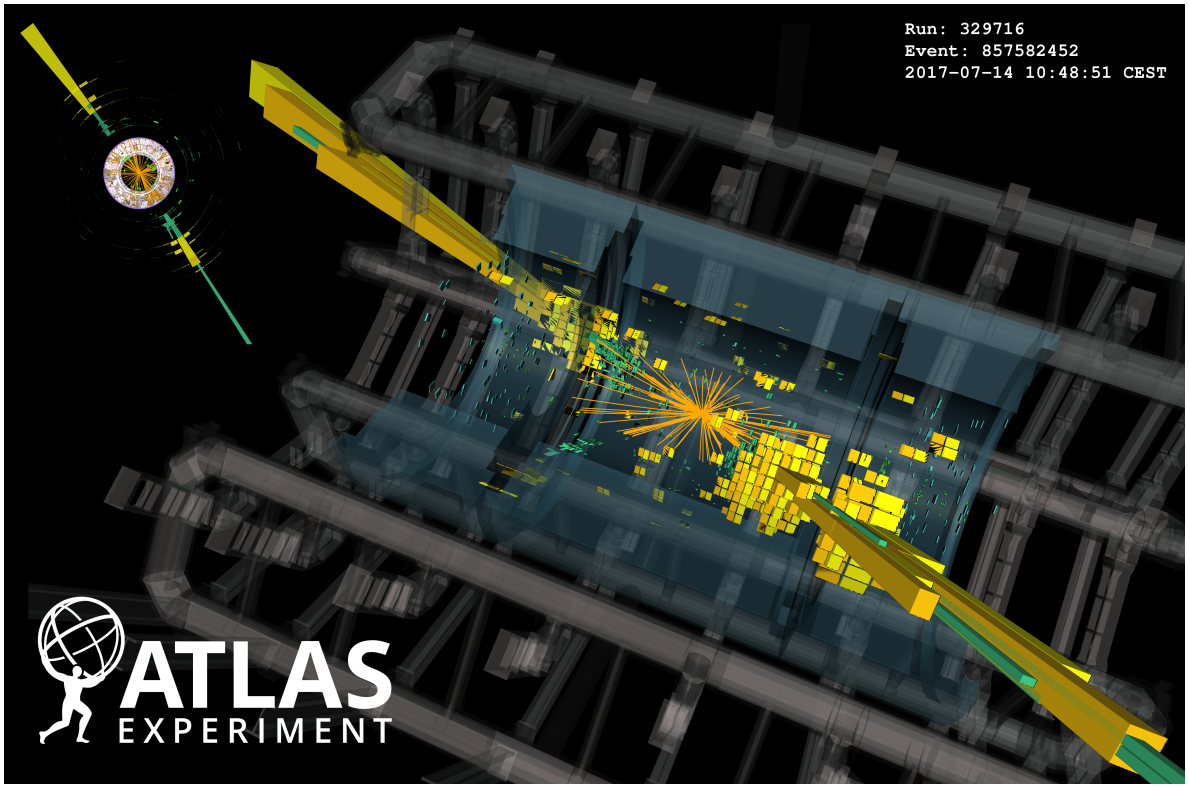
$$O_{n+1}(p_1,\ldots,\lambda p_n,(1-\lambda)p_n) = O_n(p_1,\ldots,p_n)$$
$$O_{n+1}(p_1,\ldots,p_n,p_{n+1}\to 0) = O_n(p_1,\ldots,p_n)$$

  Relevant for *event shape observables* and *jet algorithms*.

Jets

Hard QCD processes are perturbatively described in terms of scattered partons (quark and gluons) ...

... but detectors see just colorless hadrons, which, however, cluster into groups or jets, which can reveal the original hard scattering process.



Two-jet event, $\sqrt{s} = 13$ TeV [ATLAS Experiment @ 2021 CERN]

Jet reconstruction

- **Distance between particles *i* and *j***

$$(\Delta R_{ij})^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$$\underbrace{y = \frac{1}{2}\ln\frac{E + p_z}{E - p_z}}_{\text{rapidity}} \overset{m=0}{=} \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} = \underbrace{-\ln\tan\frac{\theta}{2} = \eta}_{\text{pseudo-rapidity}}$$

- **Recombination**

$$k_T: \qquad y_{ij} = \frac{\Delta R_{ij}}{R}\min(p_{T,i}, p_{T,j}) \qquad y_{iB} = p_{T,i}$$

$$\text{Cambridge/Aachen}: \qquad y_{ij} = \frac{\Delta R_{ij}}{R} \qquad y_{iB} = 1$$

$$\text{anti-}k_T: \qquad y_{ij} = \frac{\Delta R_{ij}}{R}\min(p_{T,i}^{-1}, p_{T,j}^{-1}) \qquad y_{iB} = p_{T,i}^{-1}$$

1) Find $\min_{ij}(y_{ij}, y_{iB}) \equiv y_{\min}$.
2a) If $y_{\min} = y_{ij} < y_{\text{cut}}$, then $i + j \to i$; go back to 1).
2b) If $y_{\min} = y_{iB} < y_{\text{cut}}$, remove subject i (beam radiation); go back to 1).
3) If $y_{\min} > y_{\text{cut}}$ all subjects are jets.

R and y_{cut} are parameters.

Jet reconstruction

- **Momenta combination**

  $\vec{p}_i + \vec{p}_j \to \vec{p}_i$

  Two options for the 0-th component:

  i) fix it so to make the new subject's invariant mass equal to zero
  (the ideal reconstruction should trace back to massless partons)

  ii) $p_i^0 + p_j^0 \to p_i^0$
  (allows to extend jet algorithms to massive particles, such as Z, W, ...)

The (anti-)k_t algorithm starts building a jet with (hard) soft constituents; Cambridge/Aachen is purely geometrical.

They are all infrared safe.

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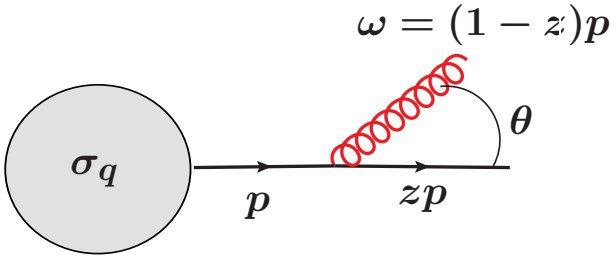
QCD-improved parton model

- Infrared safety for final-state quark splitting

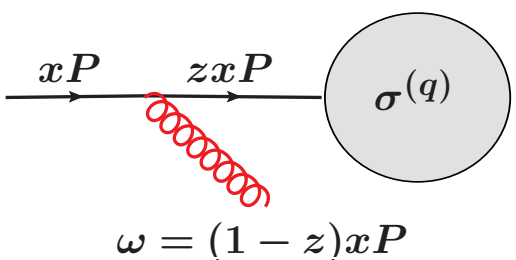


$$d\sigma_{q+g} \simeq \sigma_q \frac{\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta^2}{\theta^2}$$
$$\simeq \sigma_q \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}, \quad k_t \simeq \omega\theta$$

Virtual corrections to σ_q (only soft part):

$$d\sigma_{q,V} \simeq -\sigma_q \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

- Infrared safety for initial-state quark splitting



$$d\sigma_g^{(q)} \simeq \sigma^{(q)}(\overset{!}{zx}P) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

Virtual corrections to $\sigma^{(q)}$ (only soft part):

$$d\sigma_V^{(q)} \simeq -\sigma^{(q)}(\overset{!}{x}P) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

Integrate over the kinematics of the emitted gluon:

$$\sigma_g^{(q)} + \sigma_V^{(q)} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int \frac{dk_t^2}{k_t^2}}_{\text{infinite, collinear}} \underbrace{\int_0^1 \frac{dz}{1-z}\left[\sigma^{(q)}(zxP) - \sigma^{(q)}(xP)\right]}_{\text{finite, soft}}$$

- $$\frac{\alpha_s C_F}{\pi}\int_0^1 \frac{dz}{1-z}\left[\sigma^{(q)}(zxP) - \sigma^{(q)}(xP)\right] = \frac{\alpha_s}{2\pi}\int_0^1 dz \underbrace{P_{qq}(z)|_{\text{soft}}}_{C_F\left(\frac{2}{1-z}\right)_+}\sigma^{(q)}(zxP)$$

$$P_{qq} = C_F\left(\frac{1+z^2}{1-z}\right)_+ , \qquad \int_0^1 dz\, f(z)_+ g(z) \equiv \int_0^1 dz\, f(z)[g(z) - g(1)]$$

- $$\int \frac{dk_t^2}{k_t^2} \longrightarrow \int_{\mu_h^2}^{Q^2} \frac{dk_t^2}{k_t^2} = \int_{\mu_h^2}^{\mu_F^2} \frac{dk_t^2}{k_t^2} + \int_{\mu_F^2}^{Q^2} \frac{dk_t^2}{k_t^2} = \ln\frac{\mu_F^2}{\mu_h^2} + \ln\frac{Q^2}{\mu_F^2}$$

So that

$$\sigma_g^{(q)} + \sigma_V^{(q)} \simeq \frac{\alpha_s}{2\pi}\left(\ln\frac{\mu_F^2}{\mu_h^2} + \ln\frac{Q^2}{\mu_F^2}\right)\int_0^1 dz\, P_{qq}(z)|_{\text{soft}}\,\sigma^{(q)}(zxP)$$

Get the hadronic cross section, including leading order:

$$\sigma^{(h)} = \underbrace{\int_0^1 dx\ f_q(x)\left[\sigma^{(q)}(xP)\right.}_{\text{parton model}} + \sigma_g^{(q)} + \left.\sigma_V^{(q)}\right]$$

$$= \int_0^1 dx\ f_q(x)\left[\sigma^{(q)}(xP) + \frac{\alpha_s}{2\pi}\left(\ln\frac{\mu_F^2}{\mu_h^2} + \ln\frac{Q^2}{\mu_F^2}\right)\int_0^1 dz\ P_{qq}(z)|_{\text{soft}}\,\sigma^{(q)}(zxP)\right]$$

Observe that

$$\int_0^1 dx\ f_q(x)\left[\sigma^{(q)}(xP) + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\mu_h^2}\int_0^1 dz\ P_{qq}(z)|_{\text{soft}}\,\sigma^{(q)}(zxP)\right]$$

$$= \int_0^1 dx\ \left[f_q(x)\sigma^{(q)}(xP) + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\mu_h^2}\int_x^1 \frac{dz}{z}\ f_q\left(\frac{x}{z}\right)\ P_{qq}(z)|_{\text{soft}}\,\sigma^{(q)}(xP)\right] \overset{!}{}$$

$$= \int_0^1 dx\ \underbrace{\left[f_q(x) + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\mu_h^2}\int_x^1 \frac{dz}{z}\ f_q\left(\frac{x}{z}\right)\ P_{qq}(z)|_{\text{soft}}\right]}_{f_q(x,\mu_F)}\ \sigma^{(q)}(xP)$$

So, finally

$$\sigma^{(h)} = \int_0^1 dx\ f_q(x,\mu_F)\left[\sigma^{(q)}(xP) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_F^2}\int_0^1 dz\ P_{qq}(z)|_{\text{soft}}\,\sigma^{(q)}(zxP)\right] + \mathcal{O}(\alpha_s^2)$$

- Renormalized partonic distributions, such as $f_q(x, \mu_F)$ are non-perturbative objects, but their dependence on μ_F is perturbative:

$$\frac{\partial f_q(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)|_{\text{soft}} f_q\left(\frac{x}{z}\right)$$

- The actual calculation, taking also into account all splitting options, gives:

$$\frac{\partial f_q(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int_x^1 \frac{dz}{z} \left[P_{qq}(z) f_q\left(\frac{x}{z}\right) + P_{qg}(z) f_g\left(\frac{x}{z}\right)\right]$$

$$\frac{\partial f_g(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int_x^1 \frac{dz}{z} \left[P_{gg}(z) f_g\left(\frac{x}{z}\right) + \sum_{a=q,\bar{q}} P_{ga}(z) f_a\left(\frac{x}{z}\right)\right]$$

$$P_{qq} = C_F \left(\frac{1+z^2}{1-z}\right)_+ , \quad P_{gq} = C_F \left(\frac{1+(1-z)^2}{z}\right) , \quad P_{qg} = T_R[z^2+(1-z)^2] ,$$

$$P_{gg} = 2C_A \left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right) + \frac{11C_A - 4n_f T_R}{6}\delta(1-z)$$

[Dokshitzer (1977); Gribov, Lipatov (1972); Altarelli, Parisi (1977)]

P_{gg} and P_{gq} diverge for $z \to 0$: gluon dominance at small x.

How to solve DGLAP equations?

Simplified DGLAP equation:

$$\frac{\partial f(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int_x^1 \frac{dz}{z} P(z)\, f\left(\frac{x}{z}\right)$$

Move to moment space, through a Mellin transform:

$$f(n) = \int_0^1 dx\; x^{n-1} f(x)\;, \qquad f(x) = \int_{\delta - i\infty}^{\delta + i\infty} dn\; x^{-n} f(n)$$

with δ to the right of all singularities in the analytic continuation of $f(n)$.

In moment space the DGLAP equation becomes

$$\frac{\partial f(n,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi}\, \gamma(n)\, f(n) \;\longrightarrow\; f(n,\mu_F) = f(n,\mu_0) \left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{\alpha_s}{2\pi}\gamma(n)}$$

with $\gamma(n) = P(n)$ (anomalous dimension).

Leading logs $\alpha_s^n \log^n \mu_F^2$ resummed to all orders!

Example: $f_u - f_d \equiv f_{\rm NS}$

$$\frac{\partial f_{\rm NS}(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f_{\rm NS}\left(\frac{x}{z}\right)$$

$$f_{\rm NS}(n,\mu_F) = f_{\rm NS}(n,\mu_0) \left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{\alpha_s}{2\pi}\gamma_{qq}(n)}$$

Flavor number conservation (*e.g.* in the proton)

$$\int_0^1 dx\, (f_u(x) - f_d(x)) = 2\,, \quad \longrightarrow \quad f_{\rm NS}(n=1) = 2 \quad \longrightarrow \quad \gamma_{qq}(n=1) = 0$$

which holds true since

$$\gamma_{qq}(1) = \int_0^1 dx\, P_{qq}(x) = C_F \int_0^1 dx \left(\frac{1+x^2}{1-x}\right)_+ = 0$$

DGLAP at higher orders

$$\frac{\partial f_i(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s(\mu_F)}{2\pi} \sum_j P_{ij} \otimes f_j(\mu_F)$$

$$P_{ij} = P_{ij}^{(0)} + \frac{\alpha_s}{2\pi} P_{ij}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ij}^{(2)} + \ldots$$

- LO $P_{ij}^{(0)}$ [Dokshitzer (1977); Gribov, Lipatov (1972); Altarelli, Parisi (1977)]

  resum all leading logs, $\alpha_s^n \log^n \mu_F^2$

- NLO $P_{ij}^{(1)}$ [Curci, Furmanski, Petronzio (1980)]

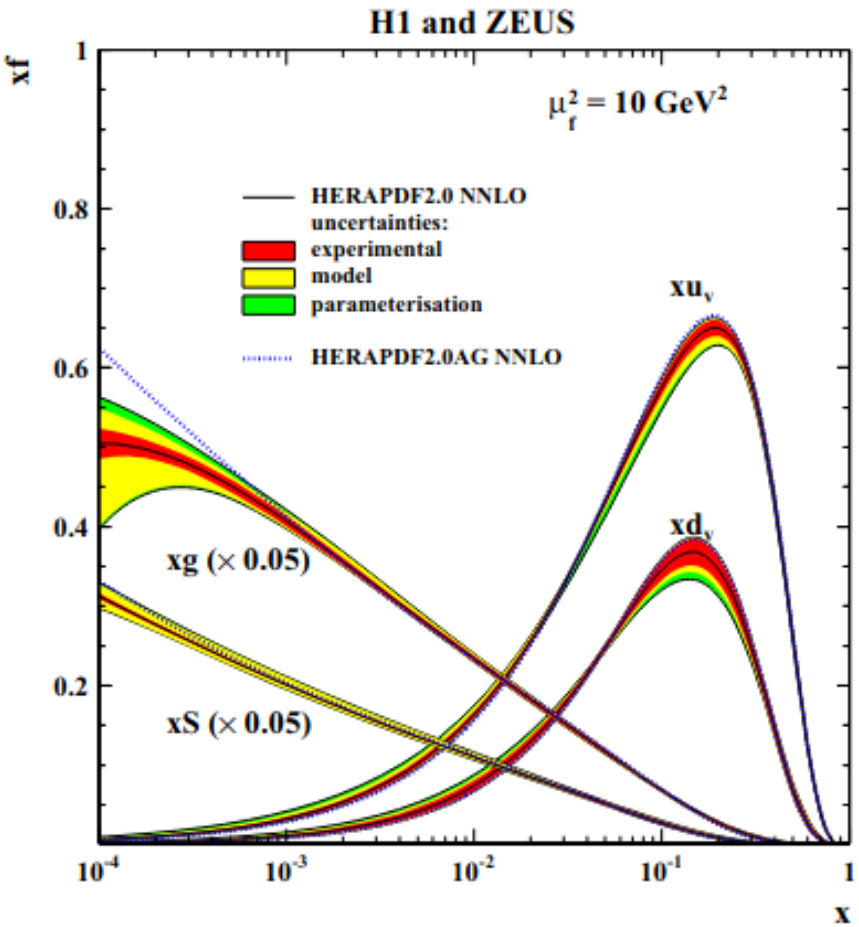
  resum all next-to-leading logs, $\alpha_s^{n+1} \log^n \mu_F^2$

- NNLO $P_{ij}^{(2)}$ [Moch, Vermaseren, Vogt (2004)]

  resum all next-to-next-to-leading logs, $\alpha_s^{n+2} \log^n \mu_F^2$

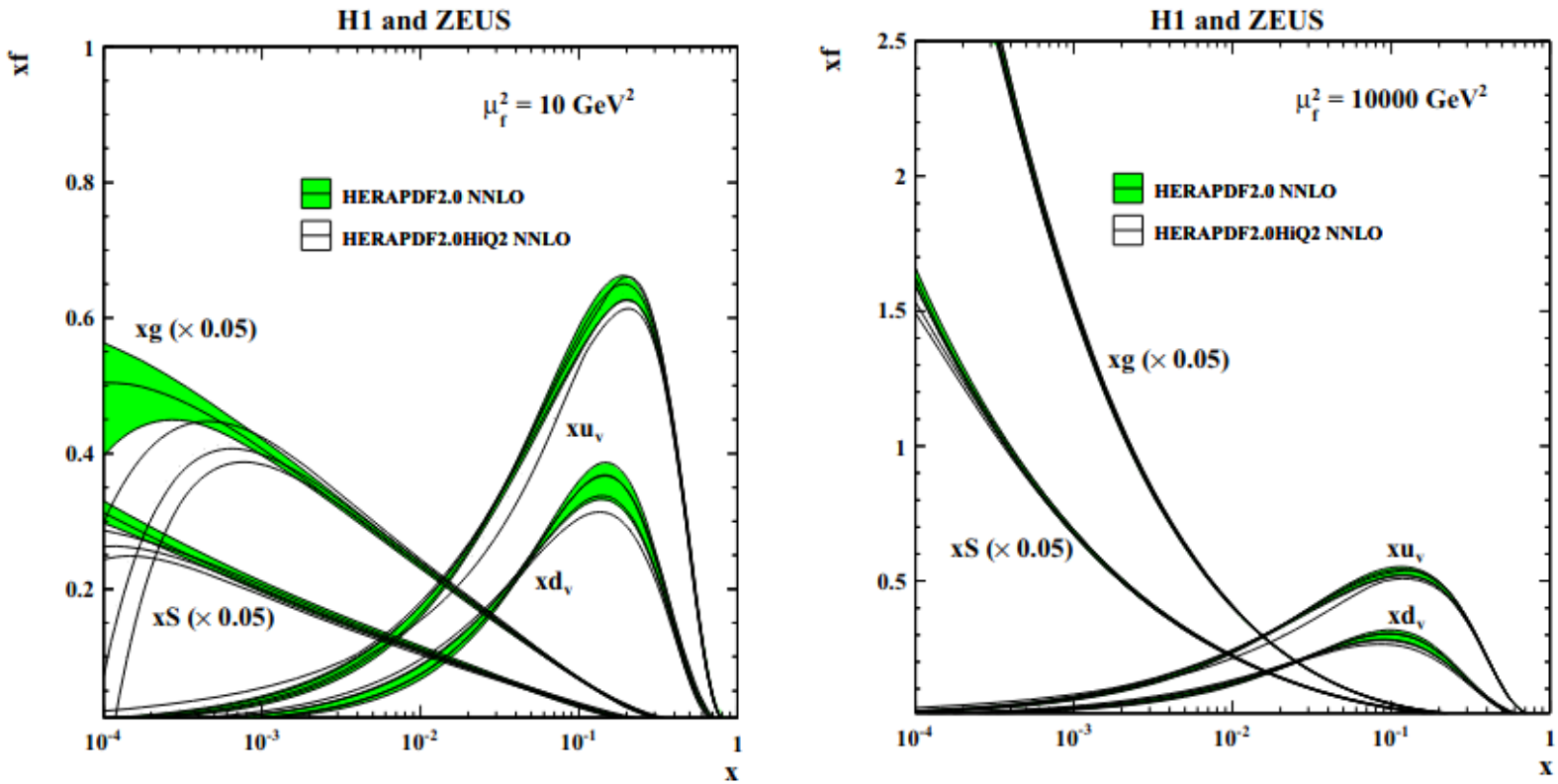
Proton PDFs

Partonic distribution functions (PDFs) are extracted from experiments, using an input parametrization at some (low) scale μ_0 and DGLAP-evolving it to higher scales.



$$xU = xf_u + xf_c$$
$$x\overline{U} = xf_{\bar{u}} + xf_{\bar{c}}$$
$$xD = xf_d + xf_s$$
$$x\overline{D} = xf_{\bar{d}} + xf_{\bar{s}}$$
$$xu_v = xU - x\overline{U}$$
$$xd_v = xD - x\overline{D}$$
$$xS = 2x(\overline{U} + \overline{D})$$

[H. Abramowicz *et al.*, Eur. Phys. J. C (2015) 75:580, [arXiv:1506.06042]]



[H. Abramowicz *et al.*, Eur. Phys. J. C (2015) 75:580, [arXiv:1506.06042]]

Factorization

Ingredients: inclusiveness, large s, hard scale $Q \sim \sqrt{s}$

- lepton-hadron (H) scattering (DIS):

$$\sigma(p, Q^2) = \sum_{a=q,\bar{q},g} \int_0^1 dz \quad \underbrace{f_{a/H}(z, \mu_F)}_{\text{process independent}} \quad \underbrace{\hat{\sigma}_a(zp, \alpha_s(Q^2), \mu_F)}_{\text{hard partonic cross section}}$$

- hadron (H_1)-hadron (H_2) scattering:

$$\sigma(p_1, p_2, Q^2) = \sum_{a,b} \int_0^1 dz_1 \int_0^1 dz_2 \; f_{a/H_1}(z_1, \mu_F) \; f_{b/H_2}(z_2, \mu_F)$$

$$\times \hat{\sigma}_{a,b}(z_1 p_1, z_2 p_2, \alpha_s(Q^2), \mu_F)$$

E.g., Drell-Yan ($H_1 + H_2 \to \mu^+ + \mu^- + X$, $e^+ + e^- + X$, $W + X$, $Z + X$)
$H_1 + H_2 \to$ jet $+ X$
$H_1 + H_2 \to$ heavy quark $+ X$

Corrections to factorization are power-suppressed, $\mathcal{O}\left(\frac{m_h}{Q}\right)^n$.

[Collins, Soper, Sterman (1981-1986)]

Outline

1. Why QCD?
2. Building the theory
   - Classical Lagrangian and its symmetries
   - Quantization and Feynman rules
   - Renormalization
3. Using the theory (perturbatively)
   - Infrared safety
   - QCD-improved parton model
4. QCD in the high-energy limit
   - Semihard processes and gluon Reggeization
   - BFKL approach
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Semihard processes

Collision processes with the following scale hierarchy:

$$s \gg Q^2 \gg \Lambda^2_{\mathrm{QCD}}$$

Q is the hard scale of the process (*e.g.* photon virtuality, heavy quark mass, jet/hadron transverse momentum, t, *etc.*)

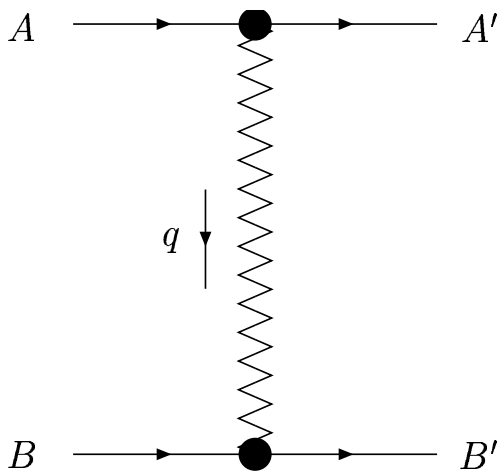
- large $Q \implies \alpha_s(Q) \ll 1 \implies$ perturbative QCD
- large $s \implies$ large energy logs $\implies \alpha_s(Q) \log s \sim 1 \implies$ resummation

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) approach provides with the general framework for this resummation.

Gluon Reggeization in perturbative QCD

Elastic scattering process $A + B \longrightarrow A' + B'$

- gluon quantum numbers in the t-channel: octet color representation, negative signature
- Regge limit: $s \simeq -u \to \infty$, t fixed (i.e. not growing with s)
- all-order resummation:
  leading logarithmic approximation (LLA): $\alpha_s^n (\ln s)^n$
  next-to-leading logarithmic approximation (NLA): $\alpha_s^{n+1} (\ln s)^n$



$$\left(\mathcal{A}_8^-\right)_{AB}^{A'B'} = \Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)}\right]\Gamma^c_{B'B}$$

$$j(t) = 1 + \omega(t)\,, \qquad j(0) = 1$$

$\omega(t)$ – Reggeized gluon trajectory

$$\Gamma^c_{A'A} = g\langle A'|T^c|A\rangle\Gamma_{A'A}$$

T^c fundamental (quarks) or adjoint (gluons)

Gluon Reggeization in perturbative QCD

Interlude: Sudakov decomposition

$$p = \beta p_1 + \alpha p_2 + p_\perp \,, \qquad p_\perp^2 = -\vec{p}^{\,2}$$

(p_1, p_2) light-cone basis of the initial particle momenta plane

$$p_A = p_1 + \frac{m_A^2}{s} p_2 \,, \qquad p_B = p_2 + \frac{m_B^2}{s} p_1 \,, \qquad 2\, p_1 \cdot p_2 = s \qquad \blacksquare$$

The gluon Reggeization has been first verified in fixed order calculations, then rigorously proved

- in the LLA [Ya.Ya. Balitsky, V.S. Fadin, L.N. Lipatov (1979)]

$$\Gamma^{(0)}_{A'A} = \delta_{\lambda_{A'}\lambda_A} \,, \quad \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{D-1}} \frac{N}{2} \int \frac{d^{D-2}k_\perp}{k_\perp^2 (q-k)_\perp^2} = -g^2 \frac{N\Gamma(1-\epsilon)}{(4\pi)^{D/2}} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (\vec{q}^{\,2})^\epsilon$$

$$D = 4 + 2\epsilon \,, \qquad t = q^2 \simeq q_\perp^2$$

- in the NLA [V.S. Fadin, R. Fiore, M.G. Kozlov, A.V. Reznichenko (2006)]

$$\Gamma^{(1)}_{A'A} = \delta_{\lambda_{A'}\lambda_A} \Gamma^{(+)}_{AA} + \delta_{\lambda_{A'},-\lambda_A} \Gamma^{(-)}_{AA} \,, \qquad \omega^{(2)}(t)$$

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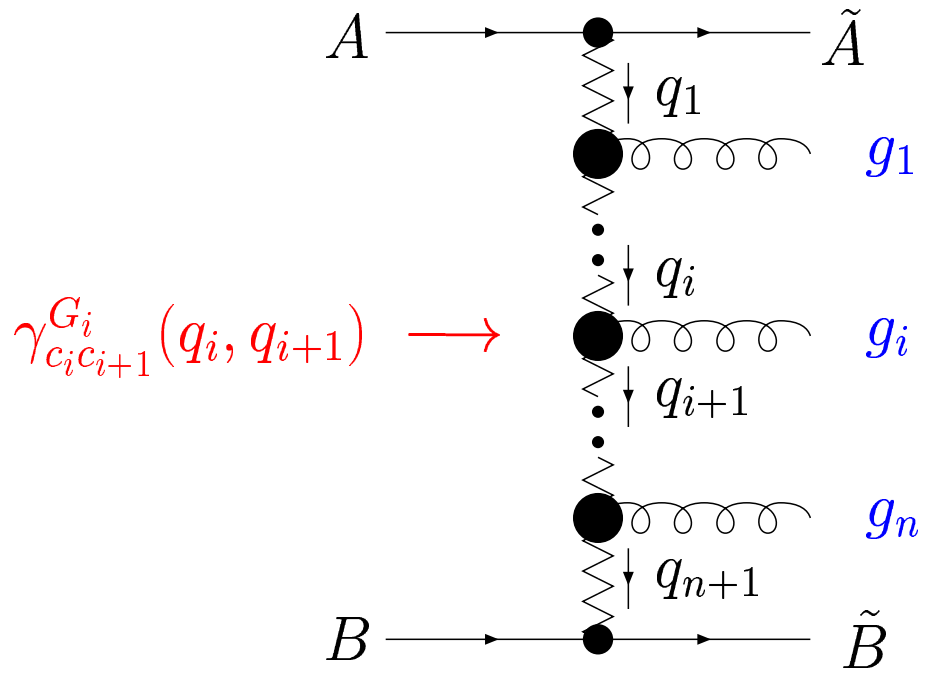
BFKL in leading accuracy

Inelastic scattering process $A + B \;\to\; \tilde{A} + \tilde{B} + n$ in the LLA



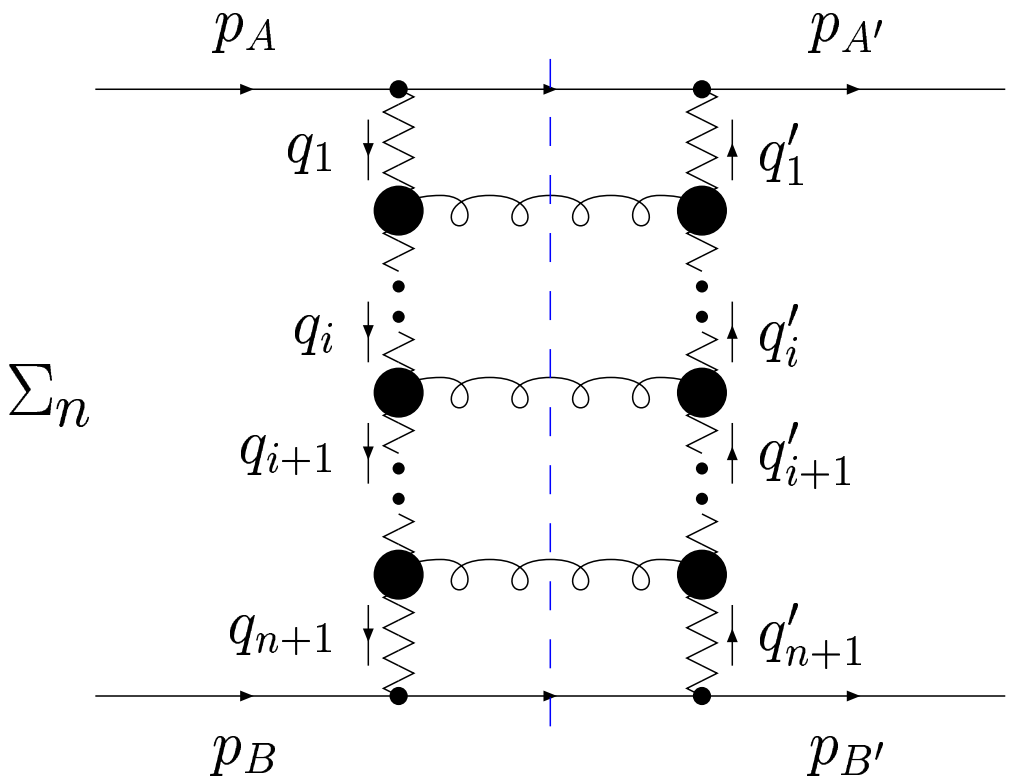
$$\mathrm{Re}\mathcal{A}^{\tilde{A}\tilde{B}+n}_{AB} = 2s\,\Gamma^{c_1}_{\tilde{A}A}\left(\prod_{i=1}^{n}\gamma^{P_i}_{c_i c_{i+1}}(q_i, q_{i+1})\left(\frac{s_i}{s_R}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})}\Gamma^{c_{n+1}}_{\tilde{B}B}$$

s_i invariant mass of the $\{g_{i-1}, g_i\}$ system, proportional to s

s_R energy scale, irrelevant in the LLA

BFKL in leading accuracy

Elastic amplitude $A + B \longrightarrow A' + B'$ in the LLA via s-channel unitarity

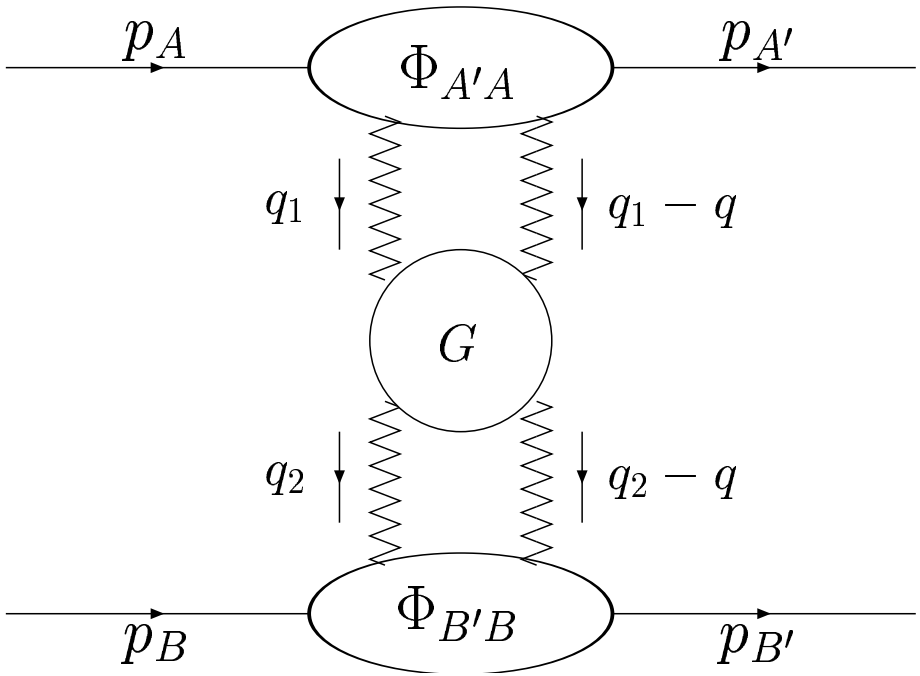


$$\mathcal{A}_{AB}^{A'B'} = \sum_{\mathcal{R}} (\mathcal{A}_{\mathcal{R}})_{AB}^{A'B'} , \qquad \mathcal{R} = 1 \text{ (singlet)}, 8^- \text{ (octet)}, \ldots$$

The 8^- color representation is important for the bootstrap, i.e. the consistency between the above amplitude and that with one Reggeized gluon exchange

BFKL in leading accuracy

Structure of the amplitude:



$$\mathrm{Im}_s(A_{\mathcal{R}})_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2} \int \frac{d^{D-2}q_2}{\vec{q}_2^{\,2}(\vec{q}_2-\vec{q})^2} \sum_{\nu} \Phi_{A'A}^{(\mathcal{R},\nu)}(\vec{q}_1;\vec{q})$$

$$\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0} \right)^{\omega} G_{\omega}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q}) \right] \Phi_{B'B}^{(\mathcal{R},\nu)}(-\vec{q}_2;-\vec{q})$$

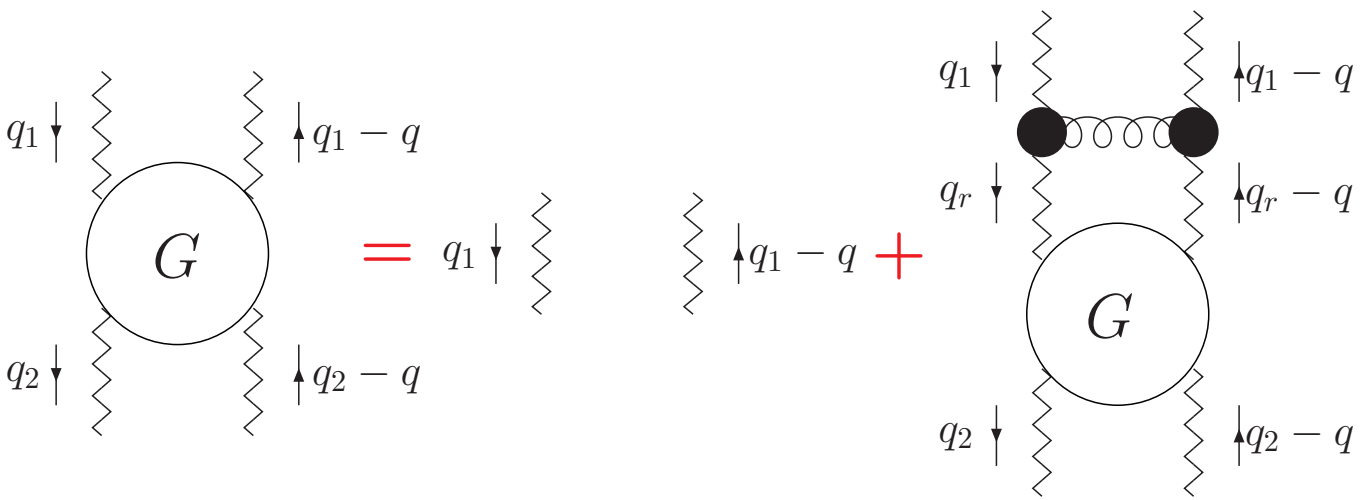
BFKL in leading accuracy

- $G_\omega^{(\mathcal{R})}$ – Mellin transform of the Green's functions for Reggeon-Reggeon scattering

$$\omega G_\omega^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2,\vec{q}\right) = \vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^{\,2}\delta^{(D-2)}\left(\vec{q}_1-\vec{q}_2\right) + \int \frac{d^{D-2}q_r}{\vec{q}_r^{\,2}(\vec{q}_r-\vec{q})^{\,2}} K^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_r;\vec{q}\right) G_\omega^{(\mathcal{R})}\left(\vec{q}_r,\vec{q}_2;\vec{q}\right)$$

BFKL equation: $t = 0$ and singlet color representation

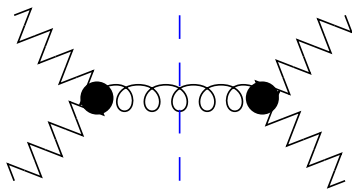
[Ya.Ya. Balitsky, V.S. Fadin, E.A. Kuraev, L.N. Lipatov (1975)]



BFKL in leading accuracy

$$\mathcal{K}^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right)=\left[\omega\left(-\vec{q}_1^{\,2}\right)+\omega\left(-(\vec{q}_1-\vec{q})^{\,2}\right)\right]\delta^{(D-2)}\left(\vec{q}_1-\vec{q}_2\right)+\mathcal{K}_r^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right)$$

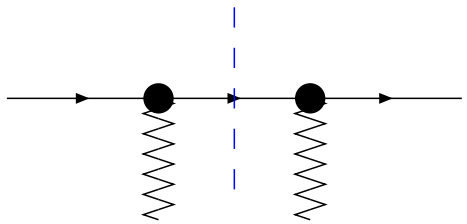
In the LLA: $\omega(t)=\omega^{(1)}(t), \quad \mathcal{K}_r=\mathcal{K}^{(B)}_{RRG}$



- $\Phi^{(\mathcal{R},\nu)}_{A'A}$ – impact factors in the t-channel color state $(\mathcal{R},\nu)$

$$\Phi_{A'A}=\sum_{\{f\}}\int\frac{d\kappa}{2\pi}d\rho_f\Gamma^{c}_{\{f\}A}(\Gamma^{c'}_{\{f\}A'})^{*}$$

constant in the LLA



BFKL in leading accuracy

Pomeron channel: $t = 0$ and singlet color representation in the t-channel

Redefinition: $G_\omega(\vec{q}_1, \vec{q}_2) \equiv \frac{G^{(0)}_\omega(\vec{q}_1, \vec{q}_2, 0)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}$, $\mathcal{K}(\vec{q}_1, \vec{q}_2) \equiv \frac{\mathcal{K}^{(0)}(\vec{q}_1, \vec{q}_2, 0)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}$

$$\omega G_\omega\left(\vec{q}_1, \vec{q}_2\right) = \delta^{(D-2)}\left(\vec{q}_1 - \vec{q}_2\right) + \int d^{D-2}q_r \mathcal{K}\left(\vec{q}_1, \vec{q}_r\right) G_\omega\left(\vec{q}_r, \vec{q}_2\right)$$

$$\mathcal{K}\left(\vec{q}_1, \vec{q}_2\right) = 2\omega(-\vec{q}_1^{\,2})\delta^{(D-2)}\left(\vec{q}_1 - \vec{q}_2\right) + \mathcal{K}_r\left(\vec{q}_1, \vec{q}_2\right)$$

Infrared divergences cancel in the singlet kernel

$\mathcal{K}\left(\vec{q}_1, \vec{q}_2\right)$ is scale-invariant $\longrightarrow$ its eigenfunctions are powers of $\vec{q}_2^{\,2}$:

$$\int d^{D-2}q_2 \mathcal{K}\left(\vec{q}_1, \vec{q}_2\right)(\vec{q}_2^{\,2})^{\gamma-1} = \frac{N\alpha_s}{\pi}\chi(\gamma)(\vec{q}_1^{\,2})^{\gamma-1}$$

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\,, \quad \psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)}$$

The set of functions $(\vec{q}_2^{\,2})^{\gamma-1}$, with $\gamma = 1/2 + i\nu$, $\nu \in (-\infty, +\infty)$ is complete.

BFKL in leading accuracy

Total cross section for the process $A + B \to$ all

$$\sigma_{AB}(s) = \frac{\mathcal{I}m_s\left(\mathcal{A}_{AB}^{AB}\right)}{s}$$

$$= \frac{1}{(2\pi)^{D-2}} \int \frac{d^{D-2}\vec{q}_A}{\vec{q}_A^{\,2}} \Phi_A(\vec{q}_A) \int \frac{d^{D-2}\vec{q}_B}{\vec{q}_B^{\,2}} \Phi_B(-\vec{q}_B) \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(\vec{q}_A, \vec{q}_B)$$

Using the complete set of kernel eigenfunctions, the BFKL equation and $D = 4$

$$\sigma_{AB}(s) = \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2(\omega - \frac{N\alpha_s}{\pi}\chi(1/2+i\nu))}$$

$$\times \int \frac{d^2\vec{q}_A}{2\pi} \int \frac{d^2\vec{q}_B}{2\pi} \left(\frac{s}{s_0}\right)^{\omega} \Phi_A(\vec{q}_A)(\vec{q}_A^{\,2})^{-i\nu-3/2} \Phi_B(-\vec{q}_B)(\vec{q}_B^{\,2})^{i\nu-3/2}$$

Infrared finiteness guaranteed for colorless colliding particles

[V.S. Fadin, A.D. Martin (1999)]

BFKL in leading accuracy

Contour integration over ω

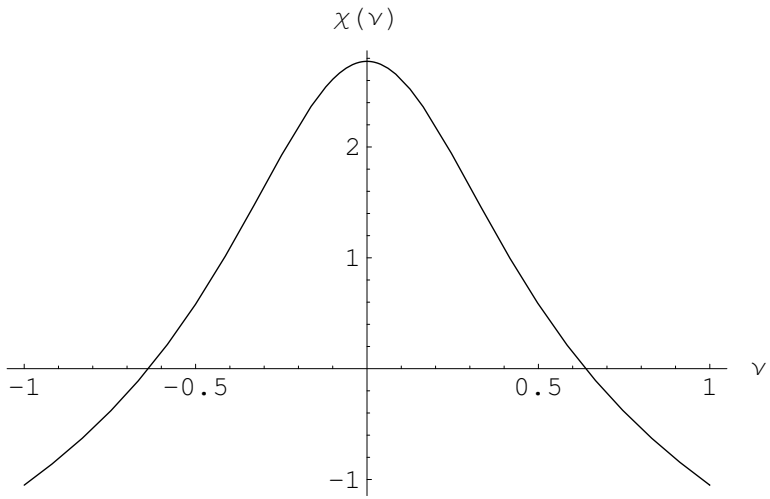
$$\sigma_{AB}(s) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2} \int \frac{d^2\vec{q}_A}{2\pi} \int \frac{d^2\vec{q}_B}{2\pi} \left(\frac{s}{s_0}\right)^{\bar{\alpha}_s \chi(\nu)} \Phi_A(\vec{q}_A)(\vec{q}_A^{\,2})^{-i\nu-3/2} \Phi_B(-\vec{q}_B)(\vec{q}_B^{\,2})^{i\nu-3/2}$$

$$\bar{\alpha}_s \equiv \frac{N\alpha_s}{\pi}\,, \qquad \chi(\nu) \equiv \chi(1/2 + i\nu)$$

Saddle point approximation:
$\chi(\nu) = 4\ln 2 - 14\zeta(3)\nu^2 + O(\nu^4)$

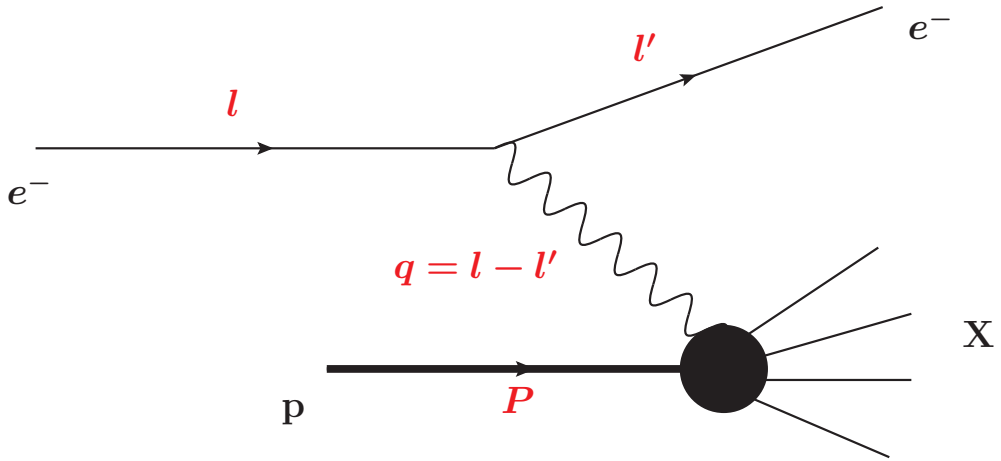
$$\sigma_{AB}(s) \sim \frac{s^{4\bar{\alpha}_s \ln 2}}{\sqrt{\ln s}}$$

$\omega_P = 4\bar{\alpha}_s \ln 2 \simeq 0.40$ for $\alpha_s = 0.15$



- unitarity is violated; BFKL cannot be applied at asymptotically high energies
- the scale of s and the argument of the running coupling constant are not fixed in the LLA $\longrightarrow$ NLA

BFKL and deep inelastic scattering



$$q^2 = (l - l')^2 \equiv -Q^2$$

$$s = (l + P)^2$$

$$W^2 = (P + q)^2$$

$$x_B = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{Q^2 + W^2}\,, \qquad W^2 \simeq \frac{Q^2(1 - x_B)}{x_B}$$

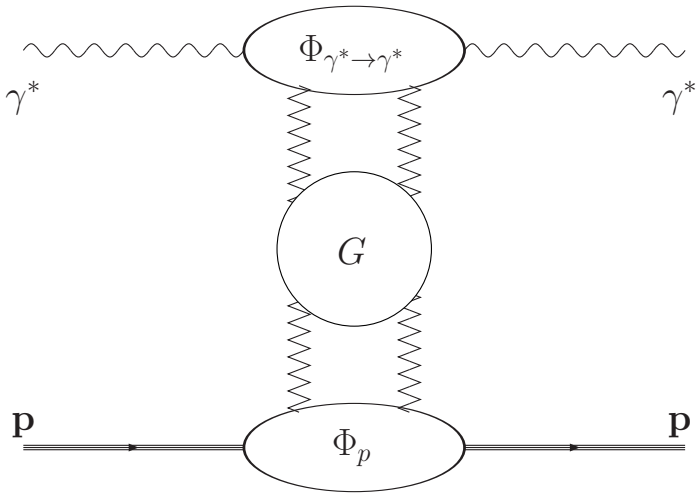
$$y = \frac{q \cdot P}{l \cdot P} \simeq \frac{Q^2}{x_B s}$$

$$\frac{d^2\sigma}{dx_B\, dQ^2} = \frac{2\pi\alpha_{\rm em}^2}{x_B Q^4}\left\{[1 + (1 - y)^2]F_2(x_B, Q^2) - y^2 F_L(x_B, Q^2)\right\}$$

$$F_2(x_B, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}\,[\sigma_T(x_B, Q^2) + \sigma_L(x_B, Q^2)]$$

$$F_L(x_B, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}\,\sigma_L(x_B, Q^2)$$

Low-x_B (or large W^2) regime: $W^2 \gg Q^2 \gg M_P^2 \quad \longrightarrow \quad x_B \ll 1$



$$F_2(x_B, Q^2) \sim \sigma_{\rm tot}(\gamma^* p) = \frac{\mathcal{I}m\mathcal{A}(\gamma^* p \to \gamma^* p)|_{t=0}}{W^2} \sim x_B^{-4\bar{\alpha}_s \ln 2}$$

$$\sigma_{T,L} = \Phi_{T,L} \otimes \underbrace{G \otimes \Phi_p}_{\rm UGD}$$

Straightforward emergence of the "unintegrated gluon distribution (UGD)":

- valid both in the LLA and in the NLA
- not unambiguous: normalization, s_0-scale setting, etc., follow from the definition of the (photon) impact factor
- non-perturbative ...
  - the proton impact factor Φ_p is intrinsically non-perturbative
  - unavoidable diffusion in the infrared of the transverse momentum integration
- ... but $(\ln x)$-resummation automatically encoded.

Let's get a closer look (only in the LLA)...

$$\sigma_{T,L}(x,Q^2) = \frac{1}{(2\pi)^2}\int\frac{d^2\vec{k}}{\vec{k}^2}\int\frac{d^2\vec{k}'}{\vec{k}'^2}\Phi_{T,L}(Q^2,\vec{k})\,G(x,\vec{k},\vec{k}')\,\Phi_p(-\vec{k}')$$

$$= 2\pi\int\frac{d^2\vec{k}}{(\vec{k}^2)^2}\underbrace{\Phi_{T,L}(Q^2,\vec{k})}_{\text{known at LO}}\underbrace{\frac{\vec{k}^2}{(2\pi)^3}\int\frac{d^2\vec{k}'}{\vec{k}'^2}\,G(x,\vec{k},\vec{k}')\,\Phi_p(-\vec{k}')}_{\mathcal{F}(x,\vec{k})}$$

In the limit $Q^2\to\infty$ **(double leading-log approximation)**:

$$\frac{\partial F_2(x,Q^2)}{\partial\ln Q^2} = \sum_{q=1}^{n_f} e_q^2\frac{\bar{\alpha}_s}{9}\,G(x,Q^2)\,, \qquad G(x,Q^2)\equiv\int\frac{d^2\vec{k}}{\pi\vec{k}^2}\Theta(Q^2-\vec{k}^2)\mathcal{F}(x,\vec{k})$$

Using a specific model for the proton impact factor, $\Phi_p(\vec{k})\sim\left(\frac{\vec{k}^2}{\vec{k}^2+\mu^2}\right)^\delta$,

$$\frac{\mathcal{F}(x,\vec{k})}{\vec{k}^2}\propto\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\nu}{\sqrt{\mu^2\vec{k}^2}}\left(\frac{\vec{k}^2}{\mu^2}\right)^{i\nu}e^{\bar{\alpha}_s\chi(\nu)\ln 1/x}\,\frac{\Gamma(\delta-1/2-i\nu)\Gamma(1/2+i\nu)}{\Gamma(\delta)}$$

$$F_2(x,Q^2)\approx\mathcal{N}(\delta)\,\bar{\alpha}_s\sum e_q^2\left(\frac{Q^2}{\mu^2}\right)^{1/2}e^{\omega_P\ln 1/x}\,\exp\left(-\frac{\ln^2(Q^2/\mu^2)}{56\bar{\alpha}_s\zeta(3)\ln 1/x}\right)$$

by the saddle point method at $\nu=0$ (recall, $\omega_P=4\bar{\alpha}_s\ln 2$)

Mellin transforms

$$\mathcal{F}_N(\vec{k}) = \int_0^1 dx\, x^{N-1}\, \mathcal{F}(x,\vec{k})\ , \quad N \text{ stands for } \omega$$

$$\tilde{\mathcal{F}}_N(\gamma) = \int_1^\infty d\left(\frac{\vec{k}^2}{\mu^2}\right) \left(\frac{\vec{k}^2}{\mu^2}\right)^{-\gamma-1} \mathcal{F}_N(\vec{k})$$

Apply to

$$F_2(x,Q^2) \;=\; \frac{Q^2}{4\pi^2\alpha}\left[\sigma_T(x,Q^2)+\sigma_L(x,Q^2)\right]$$

$$\sigma_{T,L}(x,Q^2) = 2\pi \int \frac{d^2\vec{k}}{(\vec{k}^2)^2}\Phi_{T,L}(Q^2,\vec{k})\,\mathcal{F}(x,\vec{k})$$

$$\longrightarrow F_{2,N}(Q^2) = \int_{1/2-i\infty}^{1/2+i\infty} \frac{d\gamma}{2\pi i}\frac{h_{2,N}(\gamma)}{\gamma^2} \underbrace{\frac{1}{(2\pi)^3}\frac{\Phi_p(\gamma,\mu)}{N}}_{\tilde{\mathcal{F}}_N^0(\gamma)} \frac{N}{N-\bar{\alpha}_s\chi(\gamma)} \left(\frac{Q^2}{\mu^2}\right)^\gamma$$

with $\gamma = 1/2 + i\nu$ and $\chi(\gamma)$ stands for the previous $\chi(\nu)$

Contour integral in the γ-plane (left half plane):
(a) simple (!) pole at $\gamma = 0$ (contributes with Q^2-independent term)
(b) pole at $\bar{\gamma}$ such that $N = \bar{\alpha}_s\chi(\bar{\gamma})$
(c) other poles (from $h_{2,N}$) contribute with terms suppressed in Q^2

Up to a Q^2-independent additive term,

$$\frac{\partial F_{2,N}(Q^2)}{\partial \ln Q^2} = h_{2,N}(\bar{\gamma}) R_N \tilde{\mathcal{F}}_N^0(\bar{\gamma}) \left(\frac{Q^2}{\mu^2}\right)^{\bar{\gamma}} , \qquad R_N = \frac{1}{-\bar{\alpha}_s \bar{\gamma} \chi'(\bar{\gamma})/N}$$

i) Expanding $\chi(\bar{\gamma})$ around $\bar{\gamma} = 1/2$ (i.e. $\nu = 0$) and solving $N = \bar{\alpha}_s \chi(\bar{\gamma})$, one gets

$$\bar{\gamma} \simeq \frac{1}{2} - \sqrt{\frac{N - \omega_P}{14 \bar{\alpha}_s \zeta(3)}} , \qquad R_N \simeq -\frac{\omega_P}{\sqrt{14 \bar{\alpha}_s \zeta(3)(N - \omega_P)}}$$

and the Mellin anti-transform gives back

$$F_2(x, Q^2) \approx \mathcal{N}(\delta)\, \bar{\alpha}_s \sum_q e_q^2 \left(\frac{Q^2}{\mu^2}\right)^{1/2} e^{\omega_P \ln 1/x} \exp\left(-\frac{\ln^2(Q^2/\mu^2)}{56 \bar{\alpha}_s \zeta(3) \ln 1/x}\right)$$

ii) Scaling violations even at asymptotically large Q^2:

$$N = \bar{\alpha}_s \chi(\bar{\gamma}) , \quad \chi(\gamma) = \frac{1}{\gamma} + 2 \sum_{r=1}^{\infty} \zeta(2r+1)\, \gamma^{2r} , \quad |\gamma| < 1$$

$$\longrightarrow \bar{\gamma} = \frac{\bar{\alpha}_s}{N} + 2\zeta(3) \left(\frac{\bar{\alpha}_s}{N}\right)^4 + 2\zeta(5) \left(\frac{\bar{\alpha}_s}{N}\right)^6 + \mathcal{O}\left(\frac{\bar{\alpha}_s}{N}\right)^7$$

BFKL anomalous dimension

Contact with the DGLAP formalism

$$G(x,Q^2) \equiv xg(x,Q^2)\,, \quad \Sigma(x,Q^2) \equiv \sum_{i=1}^{n_f}[xq_i(x,Q^2) + x\bar{q}_i(x,Q^2)]$$

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix} \Sigma_N(Q^2) \\ G_N(Q^2) \end{pmatrix} = \begin{pmatrix} \gamma_{qq}^N & 2n_f\gamma_{qg}^N \\ \gamma_{gq}^N & \gamma_{gg}^N \end{pmatrix}\begin{pmatrix} \Sigma_N(Q^2) \\ G_N(Q^2) \end{pmatrix}$$

$$F_{2,N}(Q^2) = \sum_i e_i^2 C_{i,N}(Q^2/\mu_F^2, \alpha_s(\mu_F^2))Q_{i,N}(\mu_F^2) + C_{g,N}(Q^2/\mu_F^2, \alpha_s(\mu_F^2))G_N(\mu_F^2)$$

(i) Leading order, (ii) $\mu_F = Q$, (iii) neglect quark densities at low x, (iv) neglect derivatives wrt $\ln Q^2$ of coefficient functions (subleading):

$$\frac{\partial F_{2,N}(Q^2)}{\partial \ln Q^2} = (\langle e_q^2\rangle 2n_f\gamma_{qg}^N + C_g^N(1,\alpha_s)\gamma_{gg}^N)G_N(Q^2)$$

$$G_N(Q^2) = G_N(\mu^2)\left(\frac{Q^2}{\mu^2}\right)^{\gamma_{gg}^N}$$

Compare with

$$\frac{\partial F_{2,N}(Q^2)}{\partial \ln Q^2} = h_{2,N}(\bar{\gamma})R_N\tilde{\mathcal{F}}_N^0(\bar{\gamma})\left(\frac{Q^2}{\mu^2}\right)^{\bar{\gamma}}$$

$$\longrightarrow \gamma_{gg}^N = \bar{\gamma} = \frac{\bar{\alpha}_s}{N}$$

Note also that $\gamma_{qg}^N|_{N\to 0} = \frac{\bar{\alpha}_s}{18}$ and $h_{2,N}(\bar{\gamma}) = \bar{\alpha}_s\langle e_q^2\rangle\frac{n_f}{9}$.

BFKL predicts the most singular part of γ_{gg}^N to all orders in α_s.

BFKL in next-to-leading accuracy

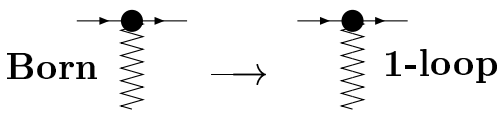
Production amplitudes keep the simple factorized form

$$\mathrm{Re}\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s\,\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{P_i}(q_i,q_{i+1})\left(\frac{s_i}{s_R}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}}$$

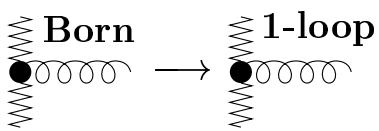
but, with respect to the LLA case, one replacement is allowed among the following:

multi-Regge kinematics

- $\omega^{(1)} \longrightarrow \omega^{(2)}$
- $\Gamma_{P'P}^{c}\,(\text{Born}) \longrightarrow \Gamma_{P'P}^{c}\,(\text{1-loop})$



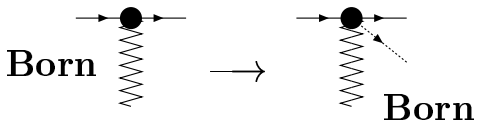
- $\gamma_{c_i c_{i+1}}^{G_i}(\text{Born}) \longrightarrow \gamma_{c_i c_{i+1}}^{G_i}(\text{1-loop})$



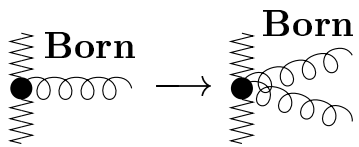
BFKL in next-to-leading accuracy

quasi-multi-Regge kinematics

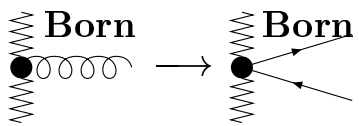
- $\Gamma^{c}_{P'P}(\text{Born}) \longrightarrow \Gamma^{c}_{\{f\}P}(\text{Born})$



- $\gamma^{G_i}_{c_i c_{i+1}}(\text{Born}) \longrightarrow \gamma^{Q\overline{Q}}_{c_i c_{i+1}}(\text{Born})$



- $\gamma^{G_i}_{c_i c_{i+1}}(\text{Born}) \longrightarrow \gamma^{GG}_{c_i c_{i+1}}(\text{Born})$



This is the program of calculation of radiative corrections to the LLA BFKL

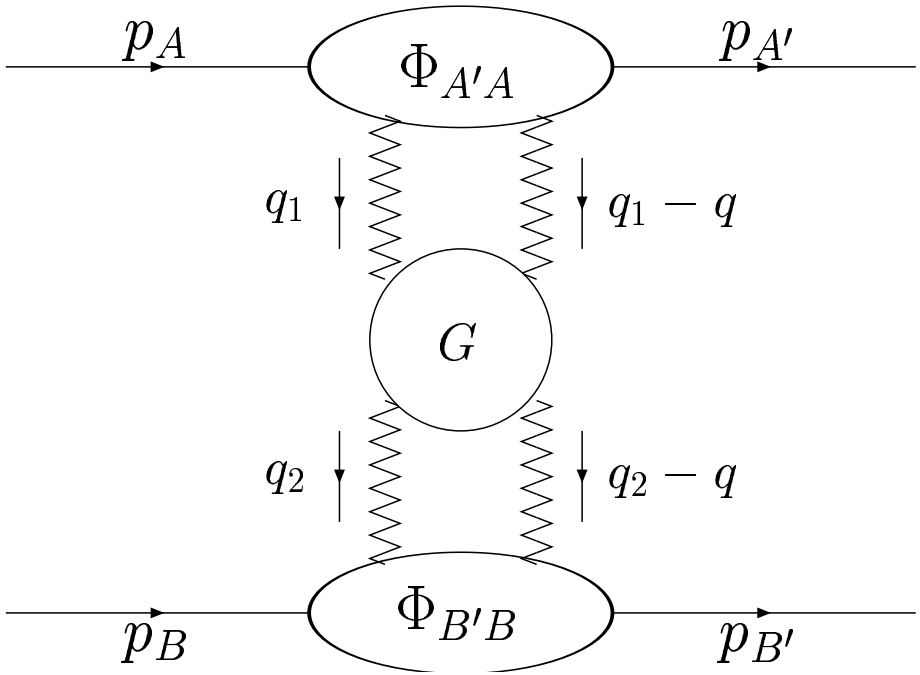
[V.S. Fadin, L.N. Lipatov (1989)]

BFKL in next-to-leading accuracy

- $\omega^{(2)}(t)$ [V.S. Fadin (1995)] [V.S. Fadin, R. Fiore, M.I. Kotsky (1995)] [V.S. Fadin, R. Fiore, A. Quartarolo (1996)] [V.S. Fadin, R. Fiore, M.I. Kotsky (1996)] [V.S. Fadin, M.I. Kotsky (1996)]
- $\gamma^{G_i}_{c_i c_{i+1}}$ (1-loop) [V.S. Fadin, L.N. Lipatov (1993)] [V.S. Fadin, R. Fiore, A. Quartarolo (1994)] [V.S. Fadin, R. Fiore, M.I. Kotsky (1996)] [V.S. Fadin, R. Fiore, A. P. (2001)]
- $\Gamma^{c}_{P' P}$ (1-loop) [V.S. Fadin, R. Fiore (1992)] [V.S. Fadin, L.N. Lipatov (1993)] [V.S. Fadin, R. Fiore, A. Quartarolo (1994)] [V.S. Fadin, R. Fiore, M.I. Kotsky (1995)]
- $\gamma^{Q\overline{Q}}_{c_i c_{i+1}}$ (Born) [V.S. Fadin, R. Fiore, A. Flachi, M.I. Kotsky (1997)] [S. Catani, M. Ciafaloni, F. Hautmann (1990)] [G. Camici, M. Ciafaloni (1996)]
- $\gamma^{GG}_{c_i c_{i+1}}$(Born) [V.S. Fadin, L.N. Lipatov (1996)] [V.S. Fadin, M.I. Kotsky, L.N. Lipatov (1997)]

BFKL in next-to-leading accuracy

Structure of the amplitude:



$$\mathrm{Im}_s(A_{\mathcal{R}})_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2} \int \frac{d^{D-2}q_2}{\vec{q}_2^{\,2}(\vec{q}_2-\vec{q})^2} \sum_{\nu} \Phi_{A'A}^{(\mathcal{R},\nu)}(\vec{q}_1;\vec{q};s_0)$$

$$\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0}\right)^{\omega} G_{\omega}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q})\right] \Phi_{B'B}^{(\mathcal{R},\nu)}(-\vec{q}_2;-\vec{q};s_0)$$

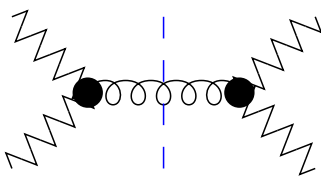
BFKL in next-to-leading accuracy

- $G_\omega^{(\mathcal{R})}$ – Mellin transform of the Green's functions for Reggeon-Reggeon scattering

$$\omega G_\omega^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2,\vec{q}\right) = \vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^{\,2}\delta^{(D-2)}\left(\vec{q}_1-\vec{q}_2\right) + \int \frac{d^{D-2}q_r}{\vec{q}_r^{\,2}(\vec{q}_r-\vec{q})^{\,2}} \mathcal{K}^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_r;\vec{q}\right) G_\omega^{(\mathcal{R})}\left(\vec{q}_r,\vec{q}_2;\vec{q}\right)$$

$$\mathcal{K}^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right) = \left[\omega\left(-\vec{q}_1^{\,2}\right)+\omega\left(-(\vec{q}_1-\vec{q})^{\,2}\right)\right]\delta^{(D-2)}\left(\vec{q}_1-\vec{q}_2\right) + \mathcal{K}_r^{(\mathcal{R})}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right)$$

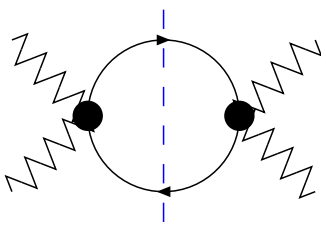
In the NLA: $\omega(t)=\omega^{(1)}(t)+\omega^{(2)}(t)$, $\mathcal{K}_r = \mathcal{K}^{(B)}_{RRG}+\mathcal{K}^{(1)}_{RRG}+\mathcal{K}^{(B)}_{RRQ\overline{Q}}+\mathcal{K}^{(B)}_{RRGG}$



$\mathcal{K}^{(1)}_{RRG}$

$t=0$: [V.S. Fadin, L.N. Lipatov (1993)]
[V.S. Fadin, R. Fiore, A. Quartarolo (1994)]
[V.S. Fadin, R. Fiore, M.I. Kotsky (1996)]

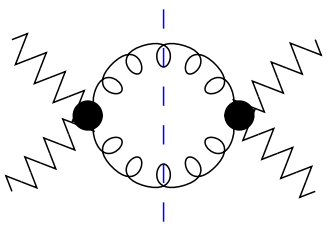
$t\neq 0$: [V.S. Fadin, R. Fiore, A. P. (2001)]



$\mathcal{K}^{(B)}_{RRQ\overline{Q}}$

$t=0$: [V.S. Fadin, R. Fiore, A. Flachi, M.I. Kotsky (1997)]

$t\neq 0$: [V.S. Fadin, R. Fiore, A. P. (1999)]



$\mathcal{K}^{(B)}_{RRGG}$

$t=0$: [V.S. Fadin, L.N. Lipatov, M.I. Kotsky (1997)]

$t\neq 0$: [V.S. Fadin, D.A. Gorbachev (2000)]
[V.S. Fadin, R. Fiore (2005)]

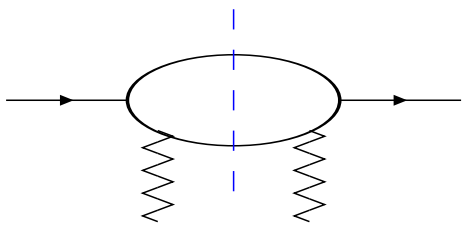
– **counterterm**

BFKL in next-to-leading accuracy

- $\Phi_{A'A}^{(\mathcal{R},\nu)}$ – impact factors in the t-channel color state $(\mathcal{R},\nu)$

$$\Phi_{A'A} = \sum_{\{f\}} \int \frac{d\kappa}{2\pi} d\rho_f \Gamma^c_{\{f\}A} (\Gamma^{c'}_{\{f\}A'})^*$$

$$\times \left(\frac{s_0}{\vec{q}_1^{\,2}}\right)^{\frac{\omega(-\vec{q}_1^{\,2})}{2}} \left(\frac{s_0}{(\vec{q}_1-\vec{q})^2}\right)^{\frac{\omega(-(\vec{q}_1-\vec{q})^2)}{2}}$$



– **counterterm**

non-trivial momentum and scale-dependence

BFKL in next-to-leading accuracy

Pomeron channel: $t = 0$ and singlet color representation in the t-channel

$$\left(\mathcal{K}(\vec{q}_1,\vec{q}_2) \equiv \frac{\mathcal{K}^{(0)}(\vec{q}_1,\vec{q}_2,0)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\ ,\ \gamma = \frac{1}{2} + i\nu\right)$$

$$\int d^{D-2}q_2 \mathcal{K}\left(\vec{q}_1,\vec{q}_2\right)\left(\vec{q}_2^{\,2}\right)^{\gamma-1} = \frac{N\alpha_s(\vec{q}_1^{\,2})}{\pi}\left(\chi(\gamma) + \frac{N\alpha_s(\vec{q}_1^{\,2})}{\pi}\chi^{(1)}(\gamma)\right)\left(\vec{q}_1^{\,2}\right)^{\gamma-1}$$

$$\chi^{(1)}(\nu) = -\frac{\beta_0}{8\,N_c}\left(\chi^2(\nu) - \frac{10}{3}\chi(\nu) - i\chi'(\nu)\right) + \bar{\chi}(\nu)\ , \qquad \beta_0 = \frac{11\,N_c}{3} - \frac{2n_f}{3}$$

$$\begin{aligned}\bar{\chi}(\nu) \;=\; & -\frac{1}{4}\left[\frac{\pi^2-4}{3}\chi(\nu) - 6\zeta(3) - \chi''(\nu) - \frac{\pi^3}{\cosh(\pi\nu)}\right.\\ & + \left.\frac{\pi^2\sinh(\pi\nu)}{2\nu\cosh^2(\pi\nu)}\left(3 + \left(1+\frac{n_f}{N_c^3}\right)\frac{11+12\nu^2}{16(1+\nu^2)}\right) + 4\phi(\nu)\right]\end{aligned}$$

$$\phi(\nu) \;=\; 2\int_0^1 dx\,\frac{\cos(\nu\ln(x))}{(1+x)\sqrt{x}}\left[\frac{\pi^2}{6} - \mathrm{Li}_2(x)\right]$$

- broken scale invariance
- large corrections: $\left.-\frac{\chi^{(1)}(\gamma)}{\chi(\gamma)}\right|_{\gamma=1/2} \simeq 6.46 + 0.05\,\frac{n_f}{N} + 0.96\,\frac{n_f}{N^3}$

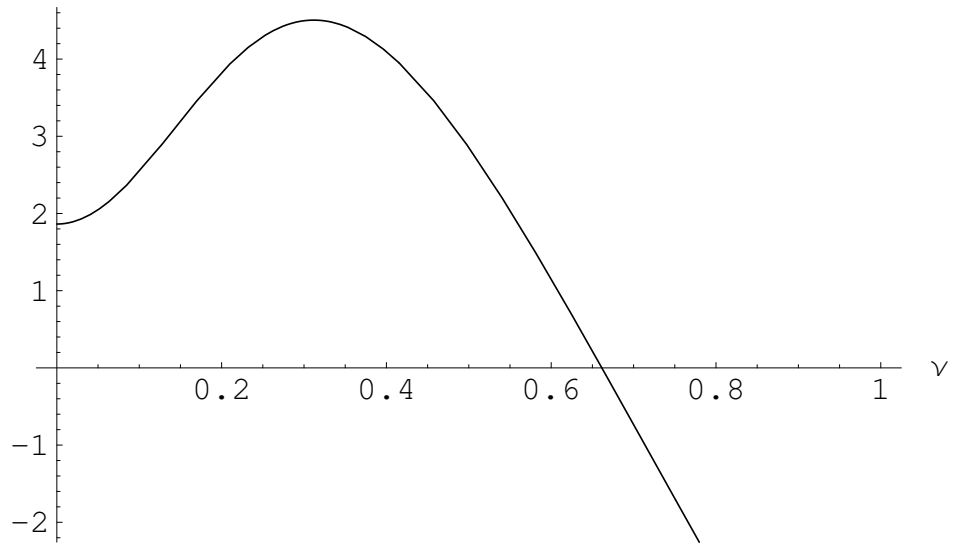
[V.S. Fadin, L.N. Lipatov (1998)] [G. Camici, M. Ciafaloni (1998)]

BFKL in next-to-leading accuracy

$\chi(\nu) + \bar{\alpha}_s(\vec{q}_1^{\,2})\chi^{(1)}(\nu)$ *vs* ν

$\bar{\alpha}_s(\vec{q}_1^{\,2}) \equiv \frac{\alpha_s(\vec{q}_1^{\,2})N}{\pi} = 0.15$

(omitted terms with the first derivative)



Double maxima $\longrightarrow$ oscillations in momentum space after ν-integration

Ways out:

- rapidity veto [C.R. Schmidt (1999)] [J.R. Forshaw, D.A. Ross, A. Sabio Vera (1999)]
- collinear improvement [G. Salam (1998)] [M. Ciafaloni, D. Colferai (1999)]
- renormalization with a physical scheme [S.J. Brodsky, V.S. Fadin, V.T. Kim, L.N. Lipatov, G.B. Pivovarov (1999)]
- ...

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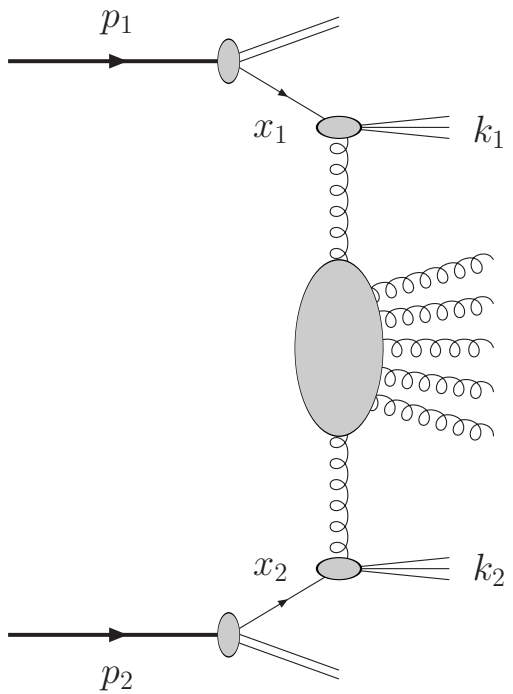
Mueller-Navelet jets

$$\text{proton}(p_1) + \text{proton}(p_2) \to \text{jet}_1(k_1) + \text{jet}_2(k_2) + X$$

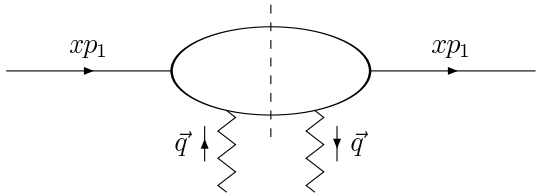


- large jet transverse momenta (hard scales): $\vec{k}_1^{\,2} \sim \vec{k}_2^{\,2} \gg \Lambda^2_{\text{QCD}}$
- large rapidity gap between jets, $\Delta y \equiv Y = y_{J_1} - y_{J_2}$,
  which requires large c.m. energy of the proton collisions, $s = 2p_1 \cdot p_2 \gg \vec{k}_{1,2}^{\,2}$

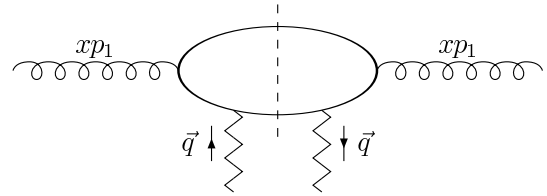
[A.H. Mueller, H. Navelet (1987)]

- Step 0: take the impact factors for colliding partons

[V.S. Fadin, R. Fiore, M.I. Kotsky, A.P. (2000)]
[M. Ciafaloni and G. Rodrigo (2000)]

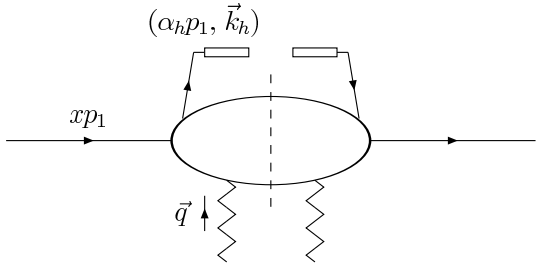


quark impact factor

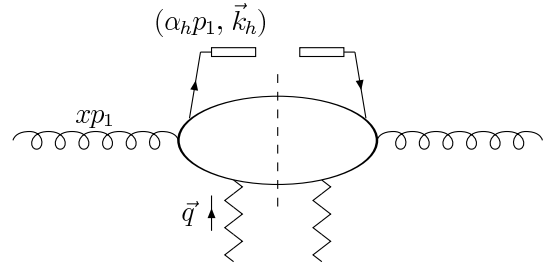


gluon impact factor

- Step 1: "open" one of the integrations over the phase space of the intermediate state to "allow" one parton to generate the jet



quark jet vertex

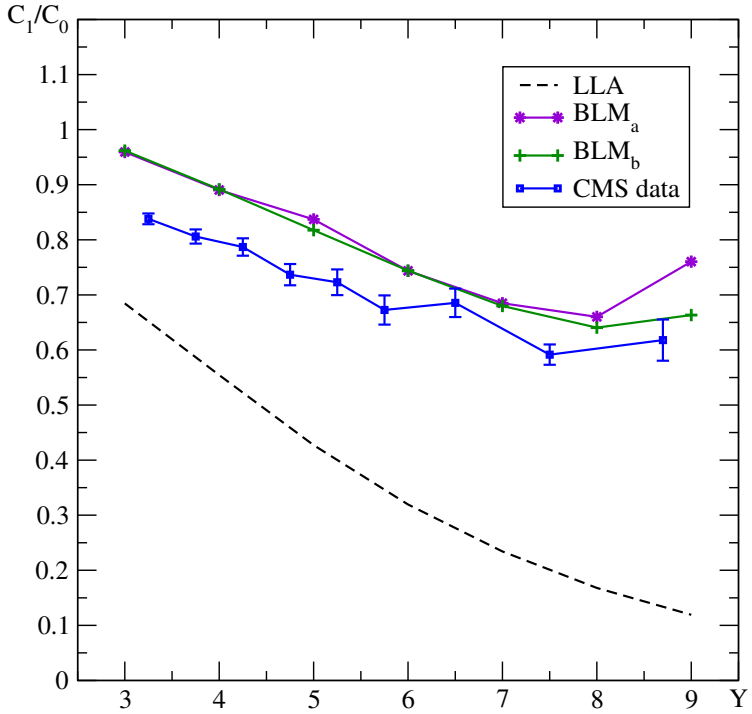


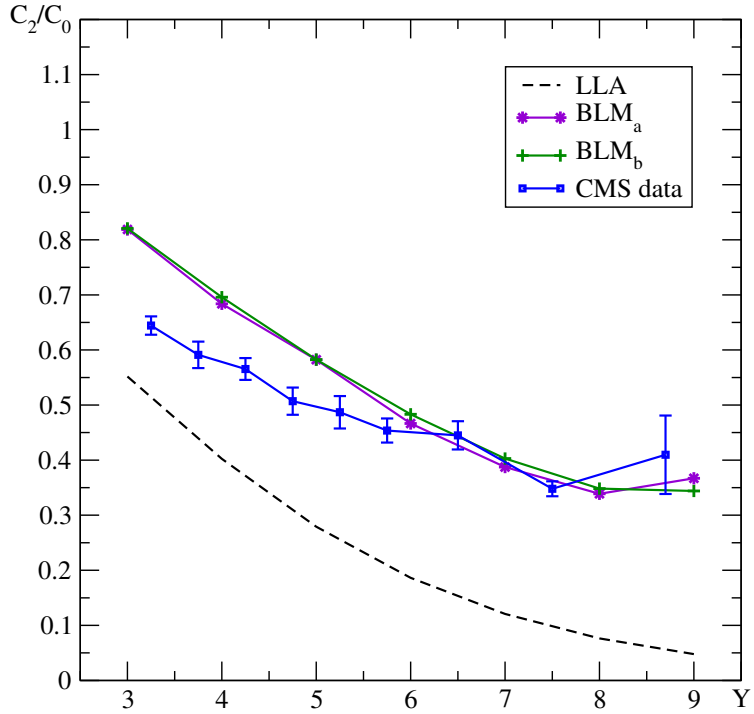
gluon jet vertex

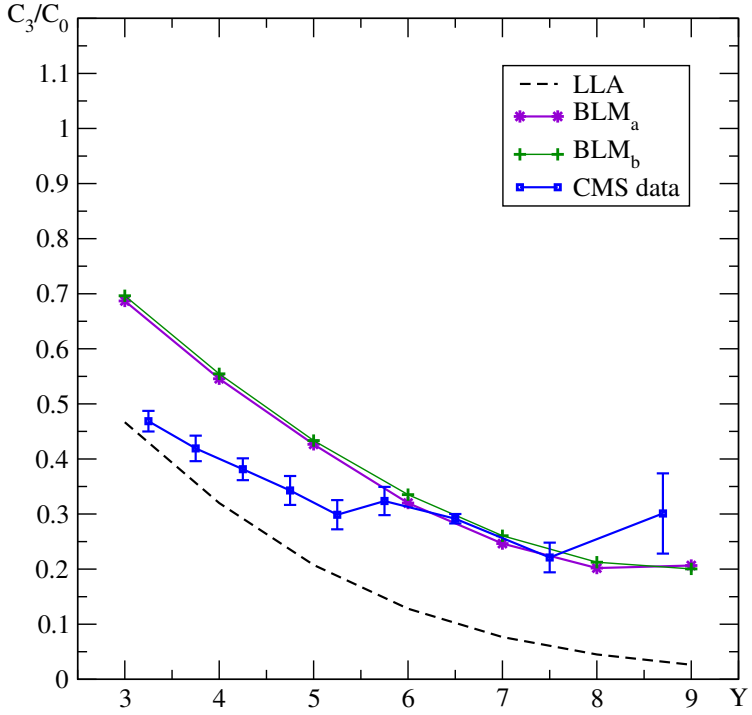
- Step 2: take the convolution with leading-twist PDFs

$$\sum_{a=q,\bar{q}} f_a \otimes (\text{quark jet vertex}) \quad + \quad f_g \otimes (\text{gluon jet vertex})$$

[J. Bartels, D. Colferai, G.P. Vacca (2003)]
[F. Caporale, D.Yu. Ivanov, B. Murdaca, A.P., A. Perri (2011)]
[D.Yu. Ivanov, A.P. (2012)] (small-cone approximation)





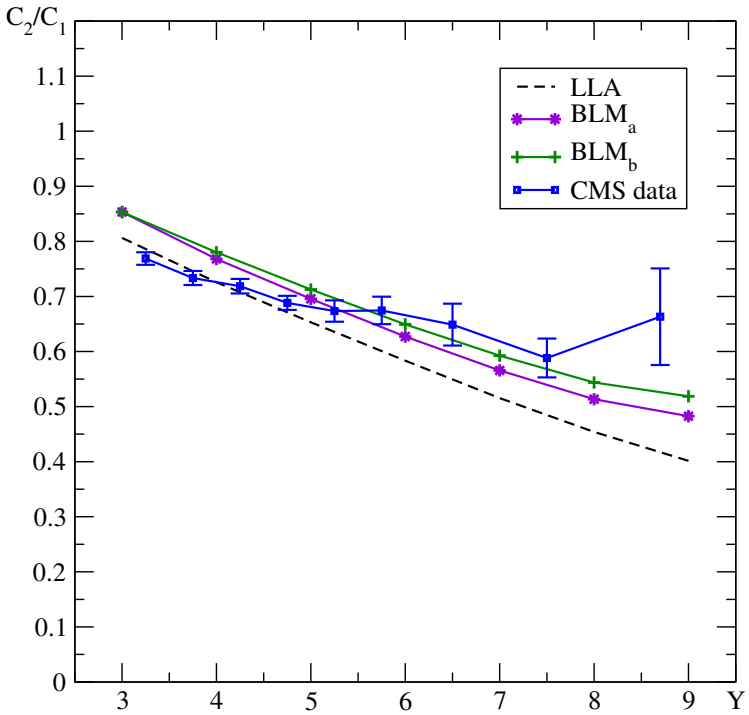


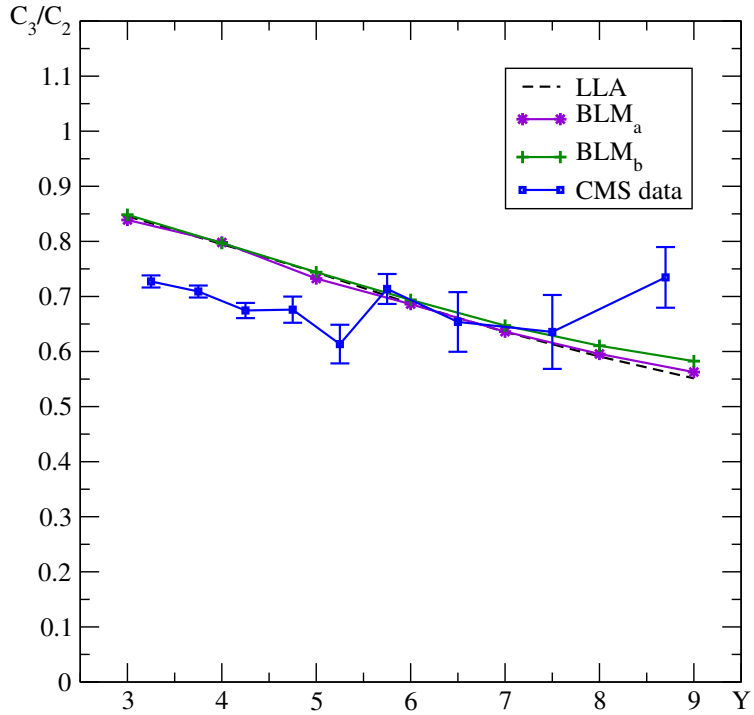
$C_n/C_0 = \langle \cos[n(\phi_{J_1} - \phi_{J_2} - \pi)] \rangle$
vs $Y = y_{J_1} - y_{J_2}$

small-cone approximation
BLM scale setting

□ CMS (7 TeV)

[F. Caporale, D.Yu. Ivanov, B. Murdaca, A.P. (2014)]





$$C_n/C_m = \frac{\langle \cos(n\phi) \rangle}{\langle \cos(m\phi) \rangle} \; vs \; Y = y_{J_1} - y_{J_2}$$

[A. Sabio Vera (2006)]
[A. Sabio Vera, F. Schwennsen (2007)]

small-cone approximation
BLM scale setting

□ CMS (7 TeV)

Similar results obtained with the exact jet vertices
[B. Ducloué, L. Szymanowski, S. Wallon (2013,2014)]

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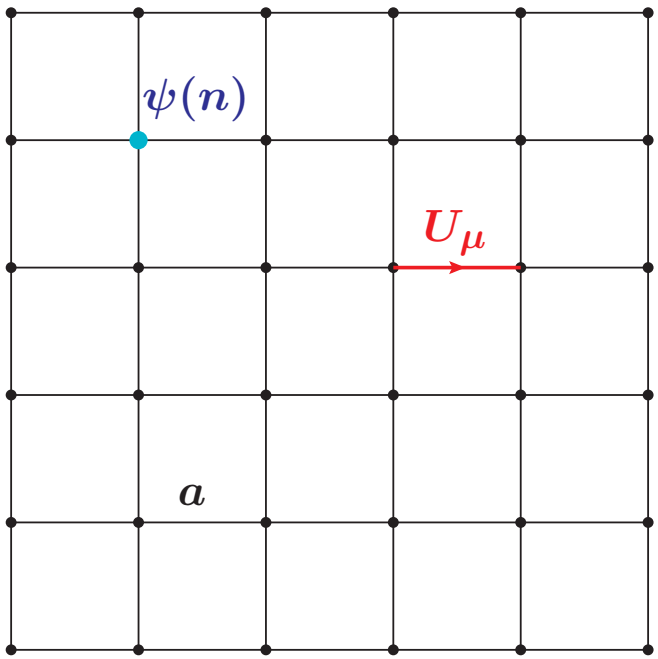
Path integral formalism

$$Z = \int D\bar{\psi} D\psi \; DA_\mu \, e^{iS_{\rm QCD}} = \int D\bar{\psi} D\psi DA_\mu \; \exp\left[i \int dt \int d^3\vec{x} \mathcal{L}_{\rm QCD} \right]$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\bar{\psi} D\psi DA_\mu \; \mathcal{O}[\bar{\psi}, \psi, A_\mu] \; e^{iS_{\rm QCD}}$$



1. Euclidean space-time:

$$t \to -i\tau \; , \quad iS_{\rm QCD} \to -S^{(E)}_{\rm QCD}$$

2. space-time lattice (spacing a)
   ψ fields on sites, gauge fields on links

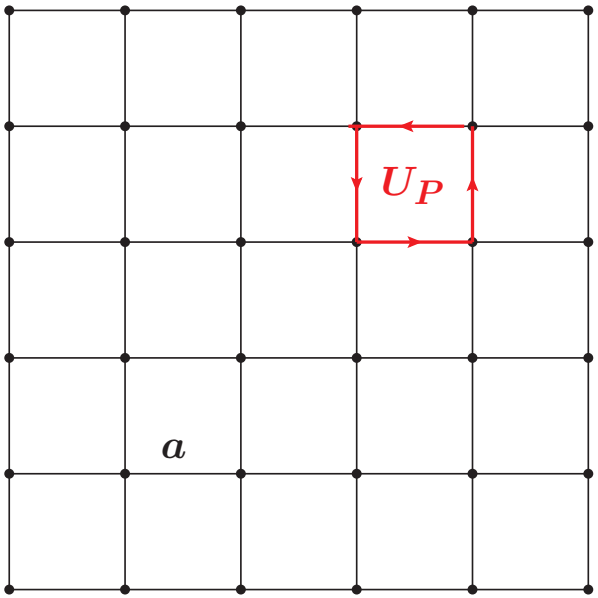
$$U_\mu(n) = \mathrm{P} \exp\left(ig \int_x^{x+\hat{\mu}a} dx^\mu A_\mu(x) \right)$$

3. $Z \to \int D\bar{\psi} D\psi \; DU \; e^{-S^{(E)}_{\rm QCD}}$
   suitable for Monte-Carlo methods

Space-time lattice acts as a regulator, cutting off momenta above $\sim 1/a$.

Also non-perturbative physics gets accessible, just “measure” the appropriate $\mathcal{O}$.

Lattice action



$$S_{\mathrm{QCD}} = S_G[U] + S_F[U, \psi, \overline{\psi}]$$

$$U_{P,\mu\nu} = 1 + iga^2 F_{\mu\nu} - \frac{1}{2} g^2 a^4 F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^6)$$

$$S_G[U] = \frac{2N_c}{g^2} \sum_P \left[1 - \frac{1}{N_c} \mathrm{Re\ Tr}\ U_P \right]$$

[Wilson (1974)]

$$g = g(a) \overset{a \to 0}{\longrightarrow} 0$$

$$\begin{aligned} S_F &= \frac{1}{2} \sum_{n,\mu} \sum_{f=1}^{n_f} \left[\overline{\psi}^f(n) \gamma_\mu U_\mu(n) \psi^f(n+\hat{\mu}) - \overline{\psi}^f(n+\hat{\mu}) \gamma_\mu U_\mu^\dagger(n) \psi^f(n) \right] \\ &+ a \sum_n \sum_{f=1}^{n_f} m_f \overline{\psi}^f(n) \psi^f(n) \end{aligned}$$

$\longrightarrow$ Doublers ($n_f = 1 \implies 2^4$ fermions)

Ways out: staggered, Wilson, overlap, domain-wall or twisted-mass fermions

Lattice setup

How to get to the continuum limit?

- Tune bare parameters ($g(a)$, $m_f(a)$) in the direction where a decreases, keeping the physics "constant", *e.g.*

$$\left(\frac{m_\pi}{m_p}\right)_{\text{lat}} = \left(\frac{m_\pi}{m_p}\right)_{\text{exp}}$$

- Scale fixing: take a reference observable, *e.g.* m_p, "measure" it on the lattice for a choice of bare parameters and compare with the experimental value:

$$(m_p)_{\text{lat}} = a(m_p)_{\text{exp}} \quad \longrightarrow \quad a = a(g, m_f) = \frac{(m_p)_{\text{lat}}}{(m_p)_{\text{exp}}}$$

- Increase accordingly the number of lattice sites, to keep the physical volume fixed at a size large enough to accommodate the relevant physics.

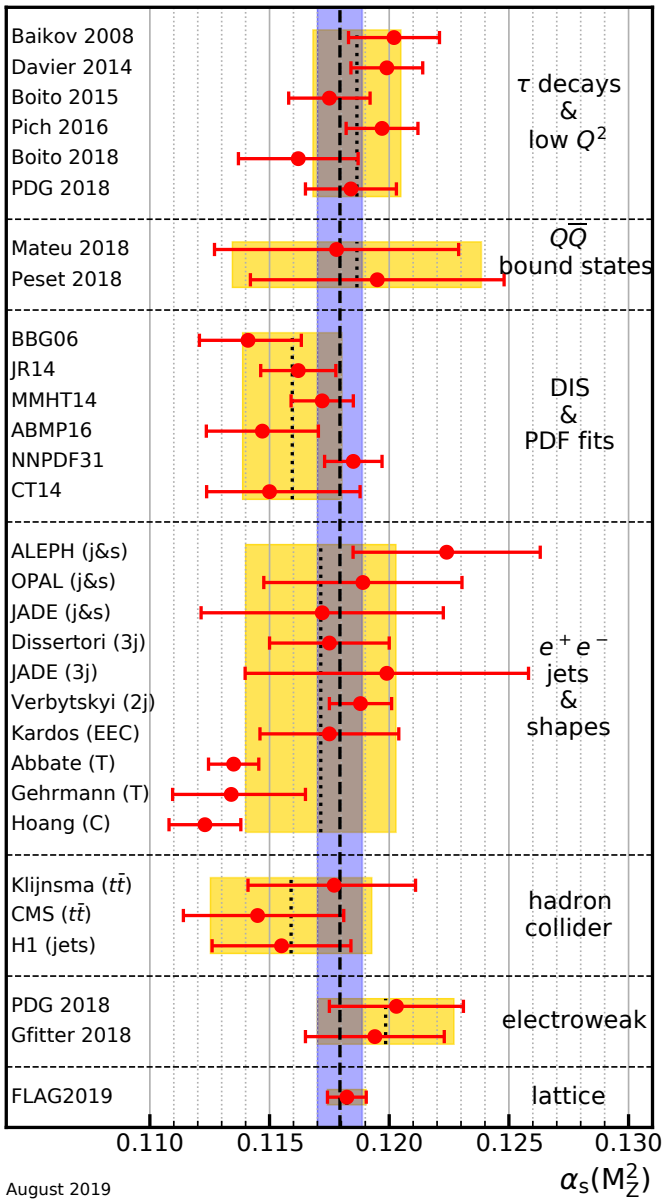
Current best settings:

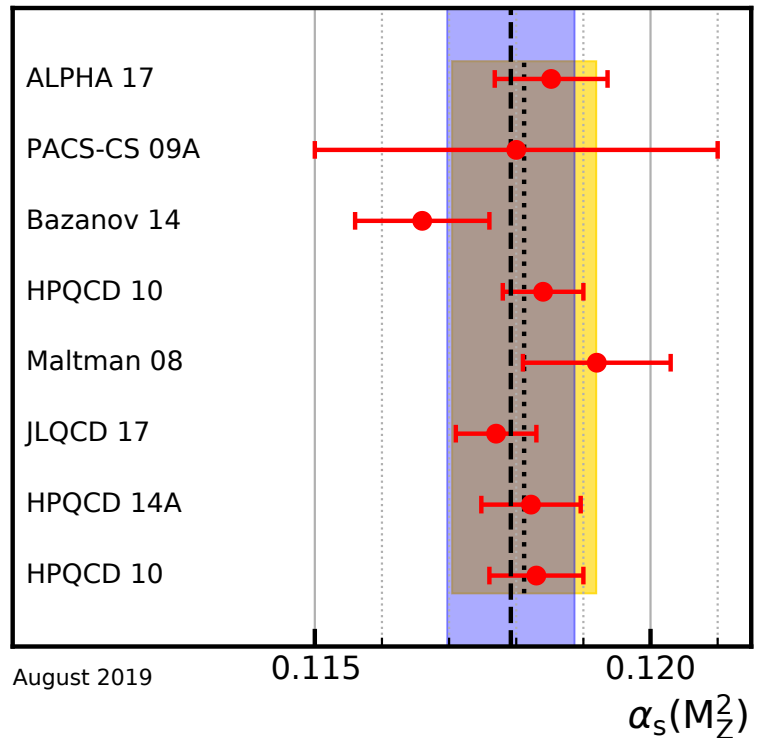
- $V_4 = (aN)^4$, $N \lesssim \mathcal{O}(100)$, $aN = (\text{a few})\ \text{fm}$, $a \gtrsim 0.04$ fm

- degenerate up and down quarks, with physical masses, plus physical strange plus (possibly) charm (2+1+1)

Outline

1. Why QCD?
2. Building the theory
   - Classical Lagrangian and its symmetries
   - Quantization and Feynman rules
   - Renormalization
3. Using the theory (perturbatively)
   - Infrared safety
   - QCD-improved parton model
4. QCD in the high-energy limit
   - Semihard processes and gluon Reggeization
   - BFKL approach
   - LHC phenomenology - an example
5. Using the theory (non-perturbatively)
   - General setup
   - Some selected results
6. Conclusions



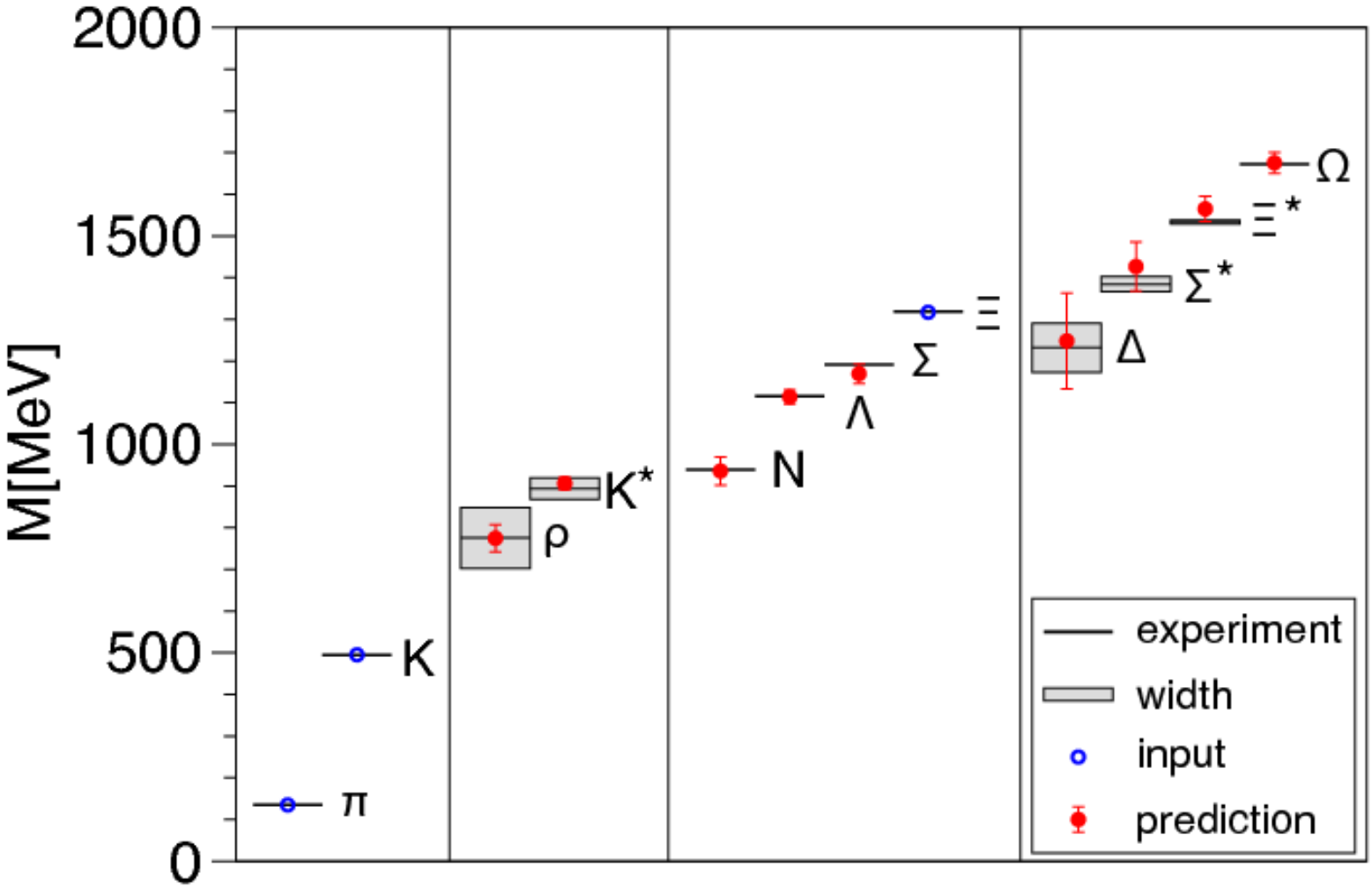


Through $Q\bar{Q}$ potential, current-current correlators, QCD vertices, Schrödinger functional, ...

$$\alpha_s(M_Z^2) = 0.1182(8) \quad \text{[FLAG2019]}$$

$$\alpha_s(M_Z^2) = 0.1176(11) \quad \text{world w/o latt.}$$

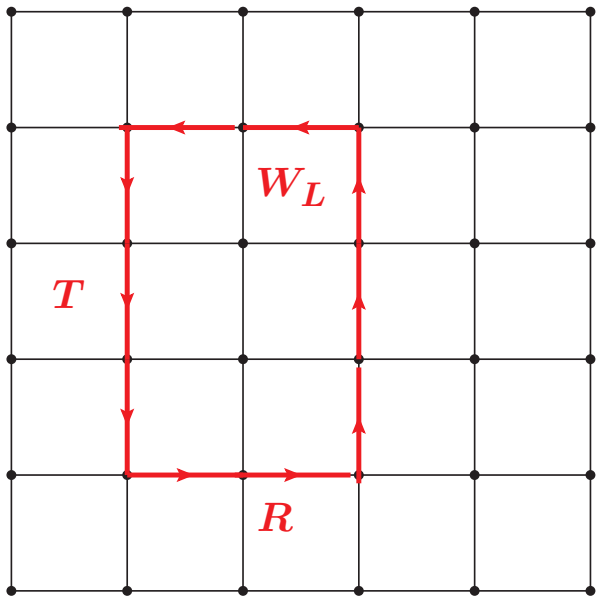
Hadron spectrum



(2+1)-QCD with almost physical quark masses

[Dürr *et al.*, Science **322** (2008), 1224 [arXiv:0906.3599 [hep-lat]]]

Static quark potential - pure gauge case

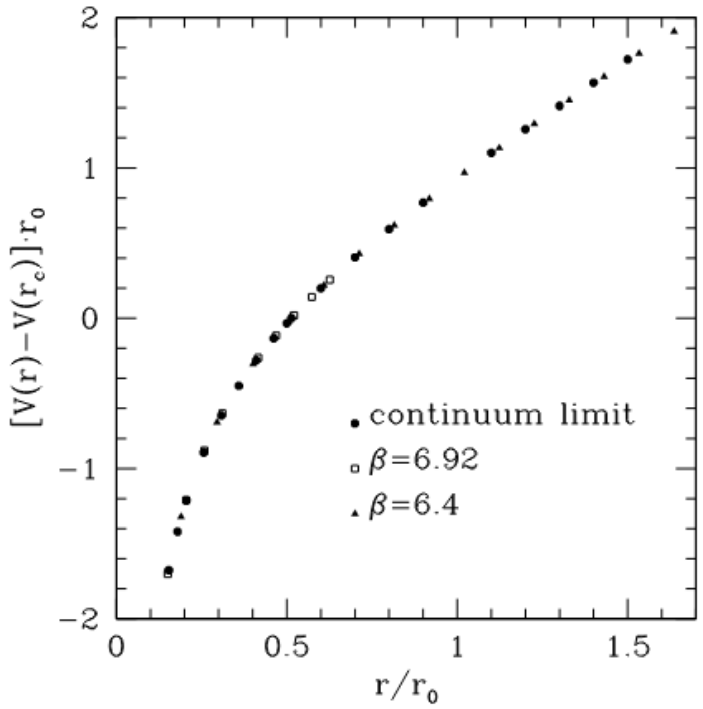


$$\langle W_L \rangle \overset{T\to\infty}{\propto} e^{-TV(R)}$$

$V(R)$ - potential between a static quark-antiquark pair

$$V(R) = A + \frac{B}{R} + \sigma R$$

$$\sigma \approx 900 \text{ MeV/fm}$$

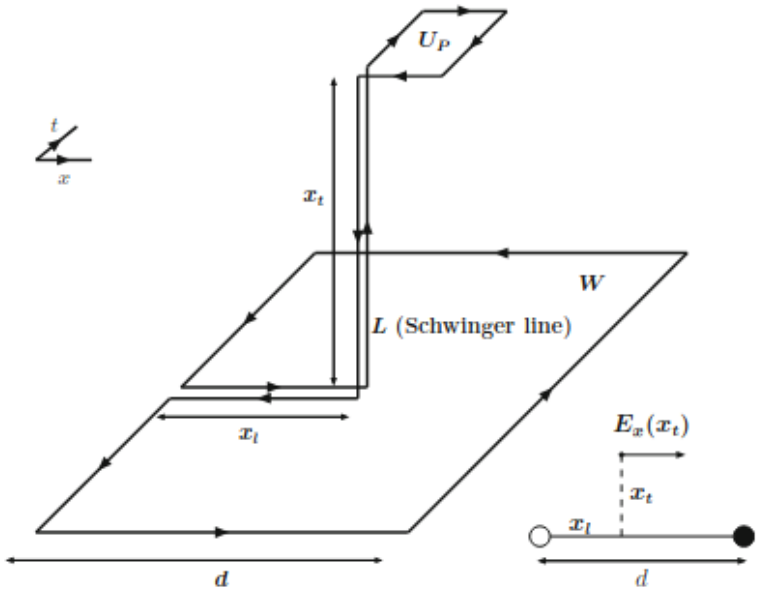


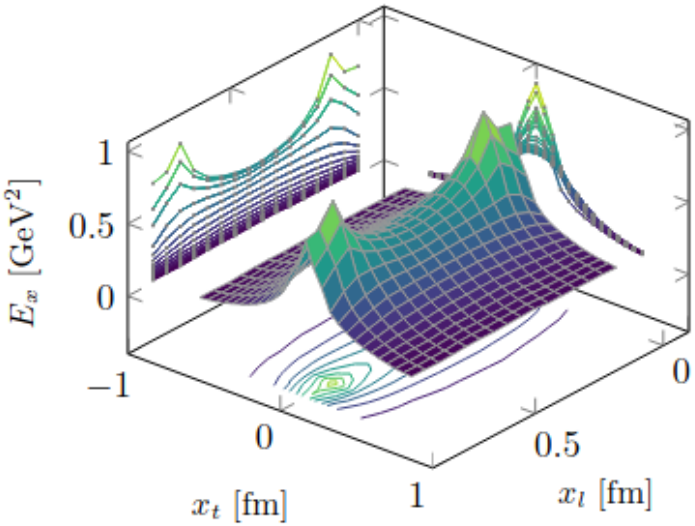
[S. Necco, hep-lat/0306005]

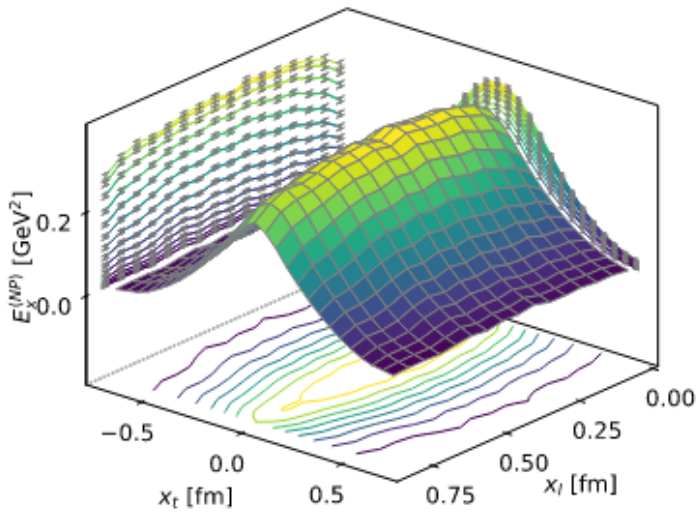
$$r_0^2 F(r_0) = 1.65$$

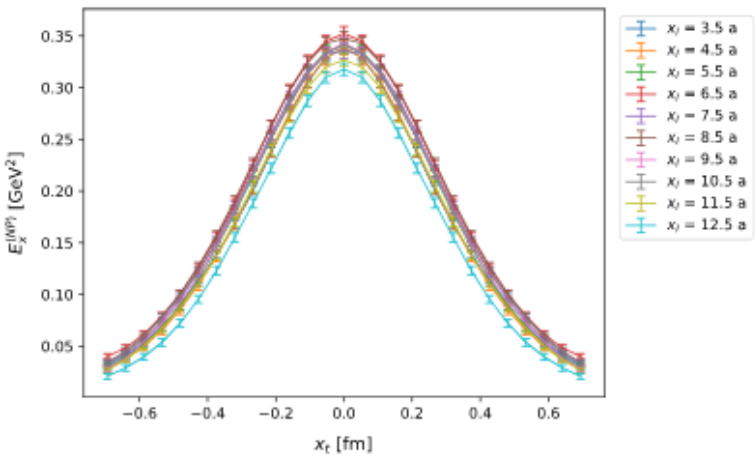
$$r_0 \approx 0.5 \text{ fm}$$

Flux-tube anatomy - pure gauge case





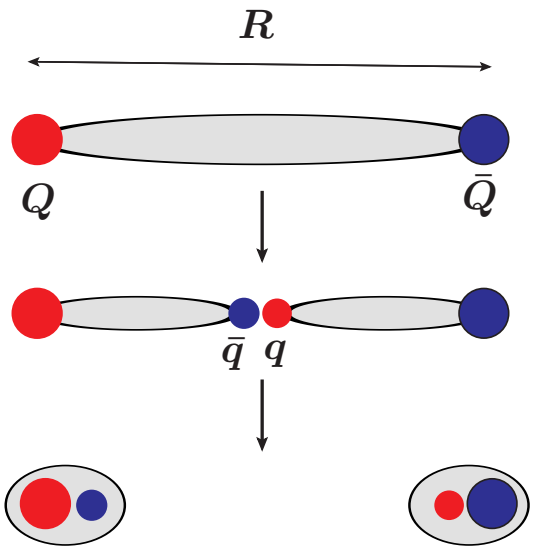


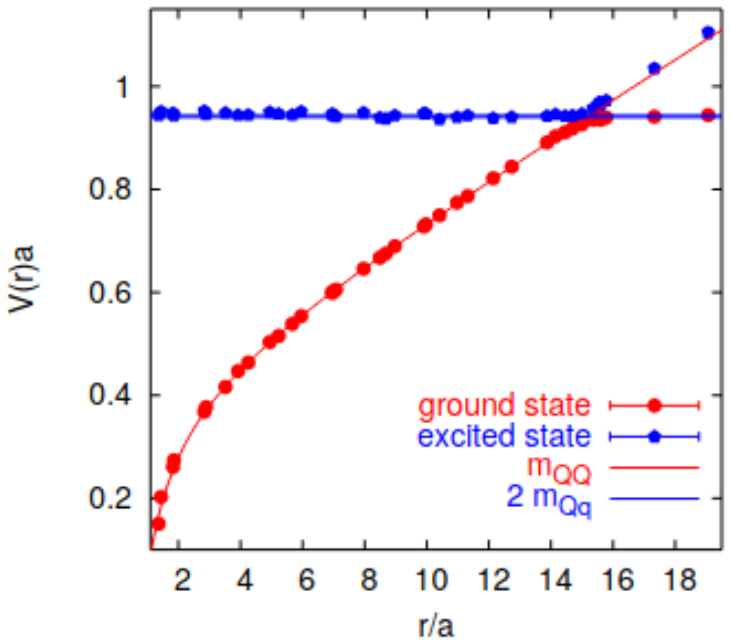


$$E_x^{(NP)} = E_x - E_x^{(\text{Coulomb})}$$

[M. Baker *et al.*, Eur. Phys. J. C **80** (2020) 514 [arXiv:1912.04739]]

String breaking in QCD





QCD with $N_f = 2$ Wilson fermions

$r_{\text{break}} \simeq 1.25$ fm

[G.S. Bali *et a.*, Nucl. Phys. B Proc. Suppl. **140** (2005), 609 [arXiv:hep-lat/0409137]]

Partonic distributions

Formal definition:

$$q(x) = \int \frac{dz}{2\pi} e^{-ixP^+z} \langle P|\bar{\psi}(z)\frac{\gamma^+}{2} \underbrace{\mathcal{A}[z,0]}_{\text{Wilson line}} \psi(0)|P\rangle \,, \qquad P^+ = \frac{P^0 + P^3}{\sqrt{2}}$$

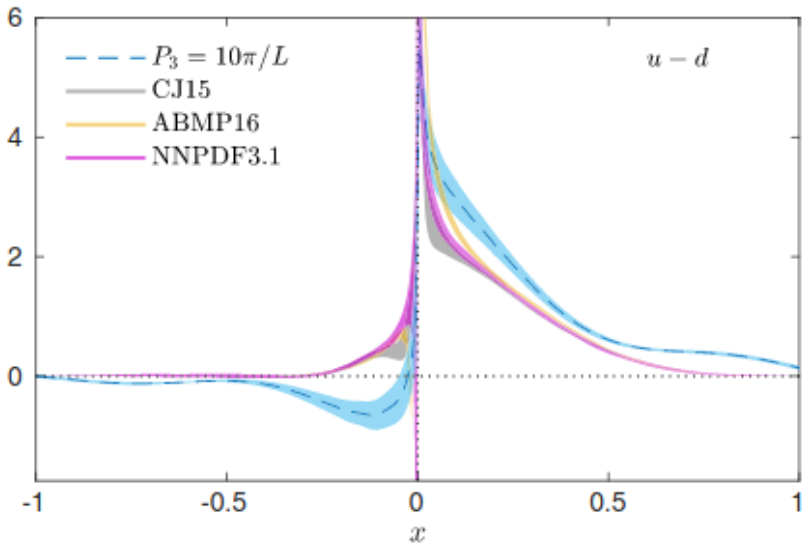
z a light-cone vector, hard to implement on a lattice with finite spacing

- Traditional approach: evaluate moments of partonic distributions
  signal-to-noise drops fast, only $n = 3$ reachable, theoretical issues
- Recent development: express PDFs as Fourier transforms of equal-time correlators

Quasi-PDFs: the Fourier variable is z [X. Ji (2013)]

Pseudo-PDFs: the Fourier variable is $P \cdot z$ ("Ioffe time"), dual to x_B
[A. Radyushkin (2017)]

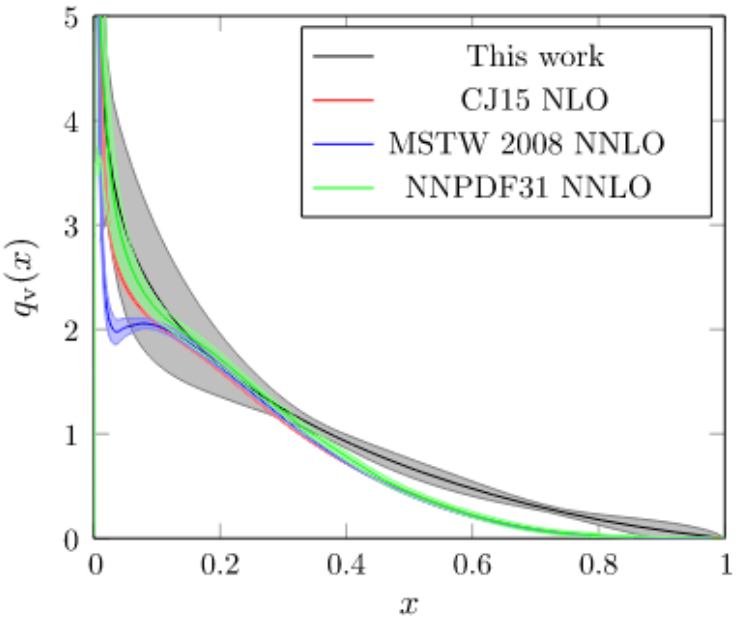
Partonic distributions



Quasi-PDFs

$N_f = 2$ twisted Wilson fermions
physical quark masses

[C. Alexandrou *et al.*, Phys. Rev. D **99** (2019) 114504 [arXiv:1902.00587]]

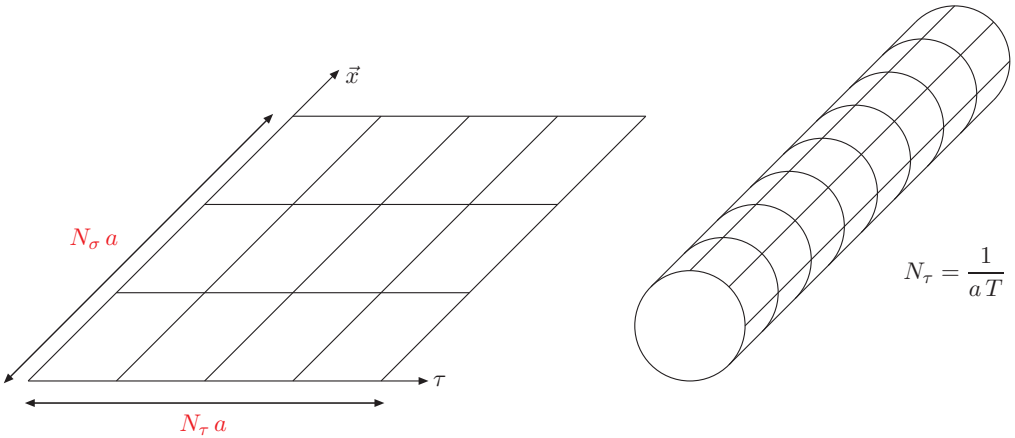


Pseudo-PDFs

QCD with 2+1 Wilson fermions
extrapolated to physical quark mass

[B. Joo *et al.*, Phys. Rev. Lett. **125** (2020) 232003 [arXiv:2004.01687]]

QCD at non-zero temperature



(Anti-)periodic boundary conditions:

$$U_\mu(\vec{n}, 0) = U_\mu(\vec{n}, N_\tau)$$

$$\psi(\vec{n}, 0) = -\psi(\vec{n}, N_\tau)$$

$$Z^{(E)} = \int D\bar{\psi} D\psi\, DU\, \exp\left[-\int_0^T d\tau \int d^3\vec{x}\, \mathcal{L}^{(E)}\right], \qquad \beta \equiv \frac{1}{aN_\tau}$$

- $m_q \to \infty$: breaking of the center symmetry, Z(N$_c$); $T_d \approx 260$ MeV

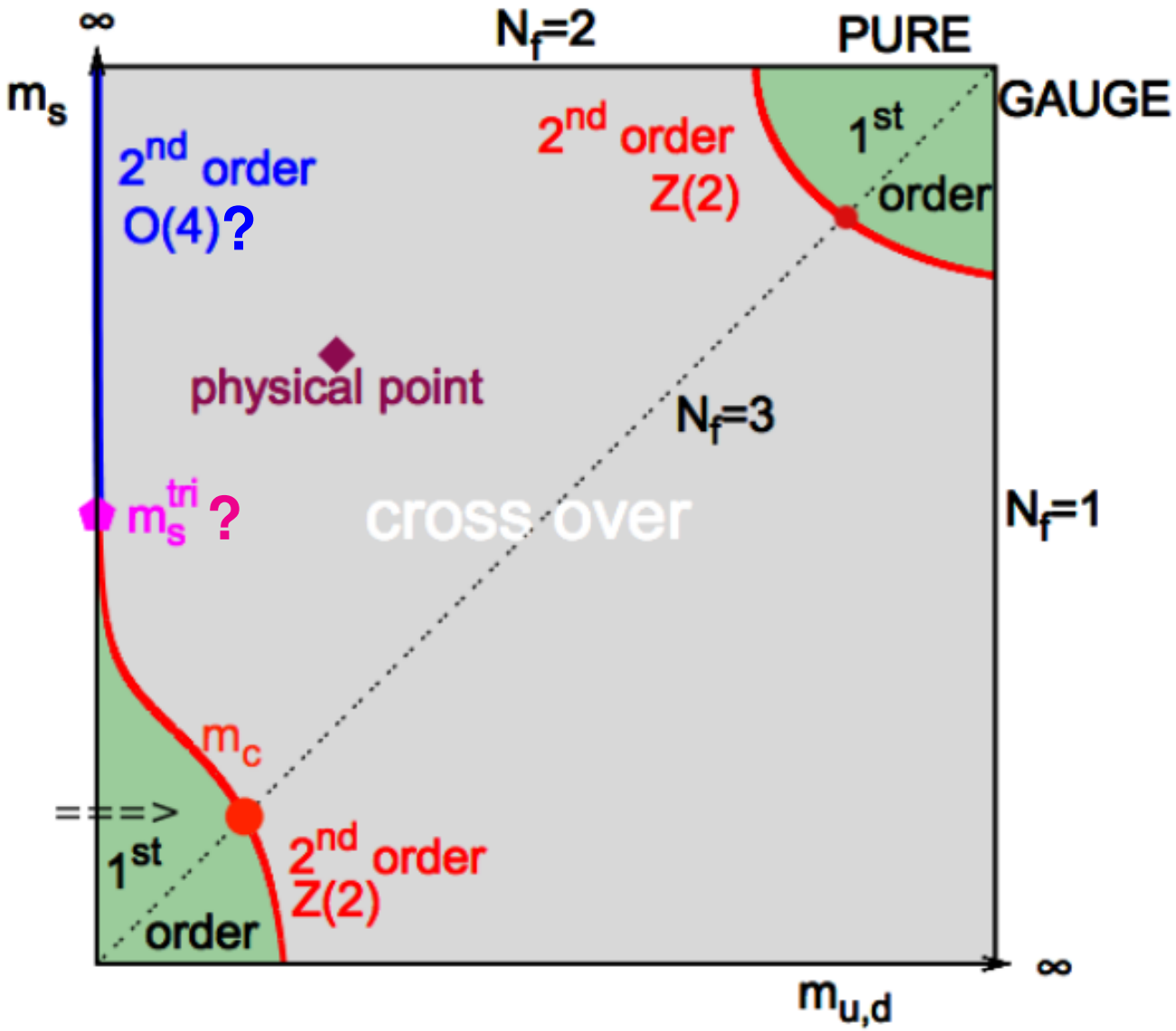
  $P(\vec{n}) = \mathrm{Tr} \prod_{n_4=1}^{N_\tau} U_4(\vec{n}, n_4)$, $\quad e^{-\beta F_q} = \langle P(\vec{n}) \rangle = \langle L \rangle$

  $T < T_d$: $\langle L \rangle = 0$ confinement; $\quad T > T_d$: $\langle L \rangle \neq 0$ deconfinement

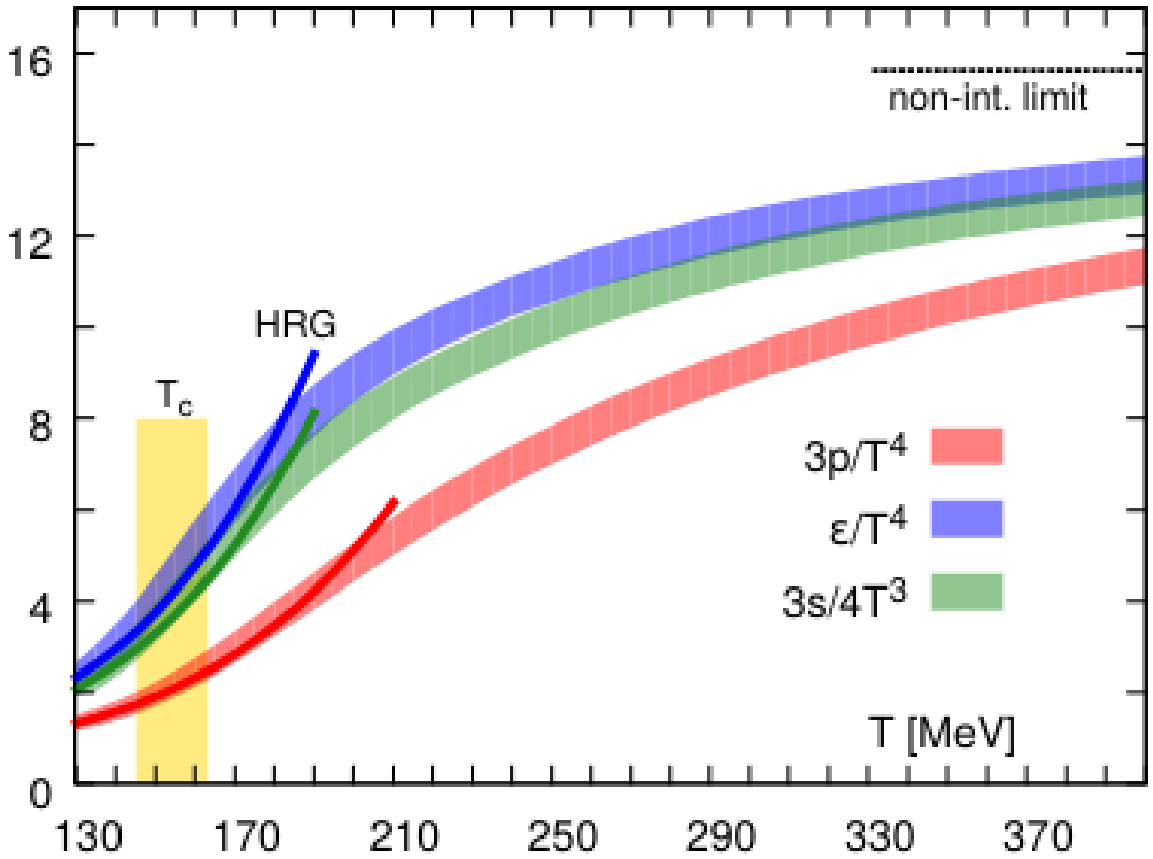
- $m_q \to 0$: breaking of the chiral symmetry;

  $T < T_{ch}$: $\langle \bar{\psi}\psi \rangle \neq 0$ broken symm.; $\quad T > T_{ch}$: $\langle \bar{\psi}\psi \rangle = 0$ restored symm.

QCD at non-zero temperature: Columbia plot



QCD equation of state



(2+1)-QCD with physical-mass staggered fermions

[HotQCD Collaboration, Phys. Rev. D **90** (2014) 094503, [arXiv:1407.6387]]

High-temperature QCD matter (quark-gluon plasma) as a low-viscosity fluid.

QCD at non-zero baryon density

Take, for simplicity, $n_f = 1$:

$$Z^{(E)}(T,\mu) = \mathrm{Tr}\exp\left[-\frac{H-\mu\hat{N}}{T}\right] , \qquad \hat{N} = \int d^3\vec{x} J^0 , \qquad J^\mu = \bar{\psi}\gamma^\mu\psi$$

$\hat{N}$ quark number operator; its eigenvalues give the excess of quarks over anti-quarks.

$$\mathcal{L}_{\mathrm{QCD}}(\mu) \to \mathcal{L}_{\mathrm{QCD}} + \mu J^0$$

$$Z^{(E)}(T,\mu) = \int DU\, D\overline{\psi} D\psi \;\; e^{-S_F[U,\psi,\overline{\psi},\mu]-S_G[U]}$$

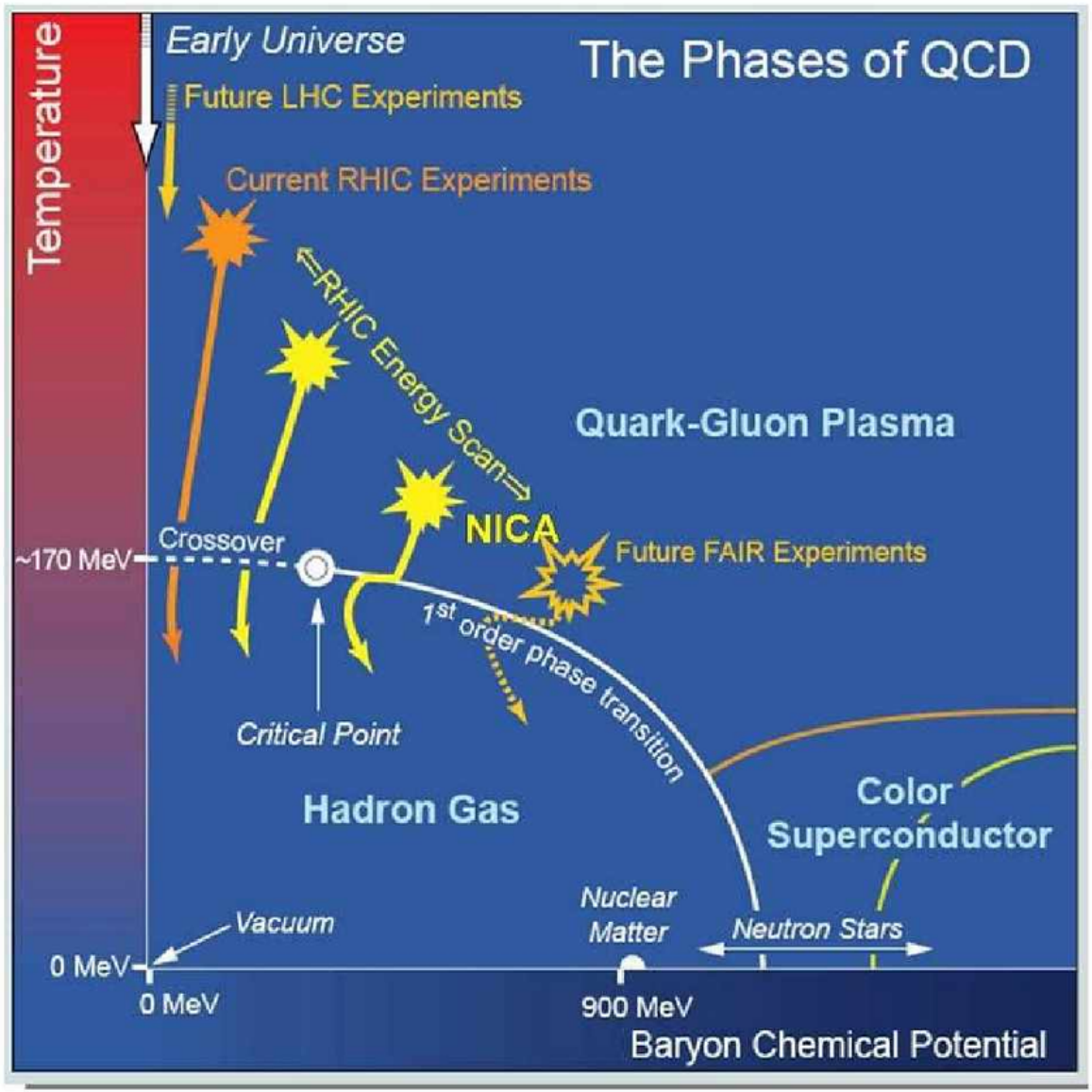
$$S_F = \sum_{n,m} \overline{\psi}(n) M_{nm}(\mu)\psi(m) , \qquad \int D\overline{\psi} D\psi \; e^{-S_F[U,\psi,\overline{\psi},\mu]} = \det M(\mu)$$

$$Z^{(E)}(T,\mu) = \int DU\, e^{-S_{\mathrm{eff}}[U]} , \qquad S_{\mathrm{eff}}[U] = S_G[U] - \ln \underbrace{\det M[U,\mu]}_{\text{complex for } \mu\neq 0}$$

Sign problem! Monte-Carlo methods unfeasible.

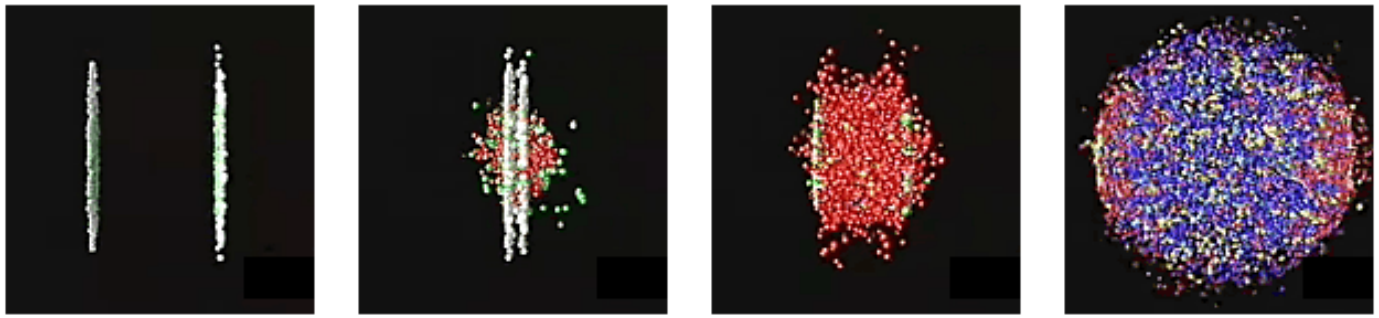
Ways out: reweighting, Taylor expansion around $\mu = 0$, imaginary chemical potential, field complexification, dual transformations, ...

QCD phase diagram



[BNL: www.bnl.gov (2010)]

QCD phase diagram: heavy-ion collisions

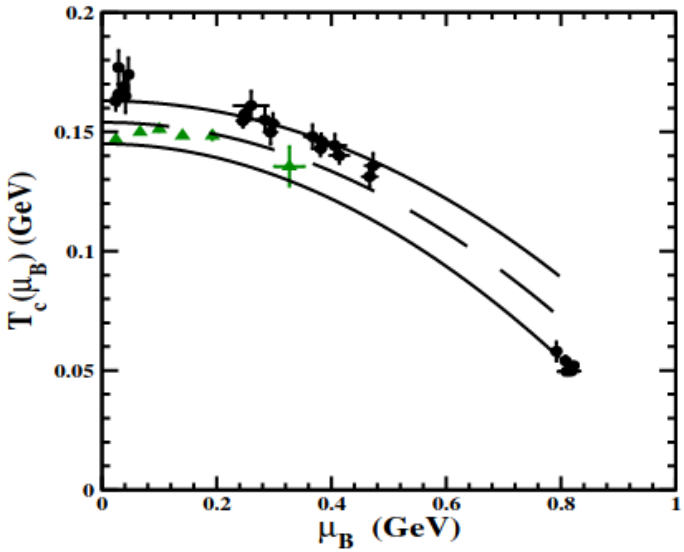


1. Ions about to collide* 2. Ion collision 3. Quarks, gluons freed 4. Plasma created

[BNL: www.bnl.gov]

5. Re-hadronization 6. Chemical freeze-out

Thermo-statistical models can describe particle abundances in terms of T and μ



[P. Cea, L. Cosmai, A.P.,
Phys. Rev. D **93** (2016) 014507 [arXiv:1508.07599]]

Lattice data (critical line):
(2+1)-QCD with almost physical staggered fermions,
$\mu_u = \mu_d = \mu_s$

Exp. data (freeze-out, BNL):
AGS (black), STAR (green)

Conclusions

QCD is a fascinating, multi-faceted theory, answering many questions in hadron physics, but posing also hard challenges.

It is the ideal place for creative minds.

Acknowledgments

- Previous QCD lectures at this school, in particular those from I. Ivanov (2010) and D. Ivanov (2019) have been useful guidelines for mine.
- Many figures and plots were taken either from the Particle Data Group web site [`https://pdg.lbl.gov`] or from Wikipedia.
- Most Feynman diagram were drawn with the help of the interface JaxoDraw [Binosi *et al.*, Comput. Phys. Commun. 180 (2009) 1709 [arXiv:0811.4113 [hep-ph]]].

Bibliography

- T. Muta,
  Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories,
  World scientific

- G. Dissertori, I. Knowles, M. Schmelling,
  Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories,
  Oxford Science Publications

- J. Collins,
  Foundations of Perturbative QCD,
  Cambridge monographs

- C. Gattringer, C.B. Lang,
  Quantum Chromodynamics on the Lattice,
  Springer-Verlag

- A. Kronfeld, C. Quigg,
  Resource Letter: Quantum Chromodynamics,
  Am. J. Phys. 78 (2010) 1081 [arXiv:1002.5032 [hep-ph]]

  More than 500 references (with comments) to books, reviews and original papers about QCD.