Internal geometry of trajectories of charged particles in abelian and non-abelian external fields and the Hooft-Polyakov monopole in arbitrary gauge

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Purpose of work:

1) show that when a charged particle- including a relativistic particle - moves in some spherically or axially symmetric external fields or fields of magnetic monopoles, including non-Abelian monopoles, it is sufficient to know only the first integrals of motion to find the curvature k or some form-invariant combination of curvature and torsion χ in addition to external field, despite the fact that the trajectory is the second integral of motion.

2) Indicate examples of the corresponding external fields as classical solutions of gauge models of field theory.

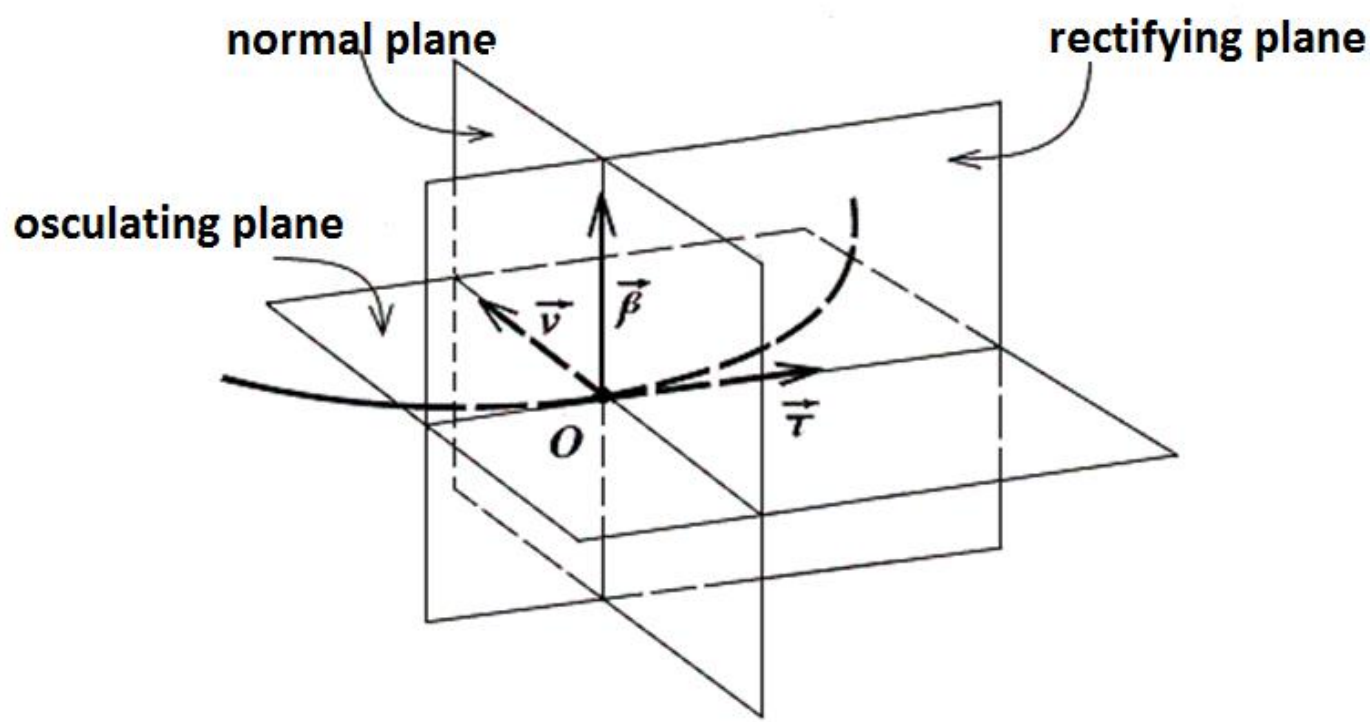
Frenet's equations relate curvature and torsion to the motion of the Frenet frame from the tangent, normal, and binormal

$$\mathbf{x}' = \frac{d\mathbf{x}}{d\ell} = \boldsymbol{\tau}$$

$$\boldsymbol{\tau}' = k\mathbf{v}$$

$$\boldsymbol{\beta}' = -\chi\mathbf{v}$$

$$\mathbf{v}' = \chi\boldsymbol{\beta} - k\boldsymbol{\tau}$$



Using general kinematic relations

$$\mathbf{x}(t) = r(t)\mathbf{n}(t), \qquad r = |\mathbf{x}|$$

$$\mathbf{V} = \dot{\mathbf{x}} = r\dot{\mathbf{n}} + \dot{r}\mathbf{n} = \upsilon\boldsymbol{\tau}, \qquad \upsilon = |\mathbf{V}|$$

$$\mathbf{a} = \ddot{\mathbf{x}} = \dot{\mathbf{V}} = \upsilon\dot{\boldsymbol{\tau}} + \dot{\upsilon}\boldsymbol{\tau}$$

curvature and torsion can be written in terms of speed and acceleration

$$k = \frac{\left|[\mathbf{V}\times\mathbf{a}]\right|}{|\mathbf{V}|^3}, \qquad \chi = \frac{(\mathbf{V}\cdot[\mathbf{a}\times\dot{\mathbf{a}}])}{[\mathbf{V}\times\mathbf{a}]^2}$$

Motion under the action of the Lorentz force

$$\mathbf{a} = \frac{e}{m}\left(\frac{1}{c}\left[\mathbf{V}\times\mathbf{B}\right]+\mathbf{E}\right)$$

$$k = \frac{e}{m\left|\mathbf{V}\right|^{3}}\left\|\left[\mathbf{V}\times\left(\frac{1}{c}\left[\mathbf{V}\times\mathbf{B}\right]+\mathbf{E}\right)\right]\right\|$$

$$\chi = \frac{e}{m}\frac{\left(\left[\mathbf{V}\times\mathbf{a}\right]\cdot\left\{\frac{1}{c}\left[\mathbf{a}\times\mathbf{B}\right]+\frac{1}{c}\left[\mathbf{V}\times\dot{\mathbf{B}}\right]+\dot{\mathbf{E}}\right\}\right)}{\left[\mathbf{V}\times\mathbf{a}\right]^{2}}$$

Curvature in spherically symmetric potential fields

$$\mathbf{a} = \frac{e}{m}\mathbf{E} = -\frac{e}{m}U'(r)\frac{\mathbf{x}}{r}$$

$$k = \frac{eU'(r)b}{2\varepsilon r}\left(1 - \frac{U(r)}{\varepsilon}\right)^{-\frac{3}{2}}, \qquad \chi = 0$$

$$m\mathbf{V}^2 + 2U(r) = mV_0^2 = 2\varepsilon$$

$$|\mathbf{M}| = m|[\mathbf{x} \times \mathbf{V}]| = mbV_0$$

Movement in a magnetic field

$$\mathbf{a} = \frac{e}{mc}\left[\mathbf{V}\times\mathbf{B}\right], \quad \begin{cases} \mathbf{V}^2 = const, \\ \varepsilon\left(\mathbf{V}^2\right) = const, \end{cases}$$

$$k = \frac{e}{mc}\frac{\left|\left[\mathbf{V}\times\mathbf{B}\right]\right|}{\mathbf{V}^2},$$

$$\chi = -\frac{e}{mc}\frac{\left(\mathbf{V}\cdot\mathbf{B}\right)}{\mathbf{V}^2} + \left(\frac{e}{mc}\right)^2\frac{\left(\mathbf{V}\cdot\left[\mathbf{B}\times\dot{\mathbf{B}}\right]\right)}{\mathbf{a}^2}$$

let's introduce the unit vector of the direction of the external field

$$\mathbf{B} = |\mathbf{B}|\boldsymbol{\zeta} \longrightarrow \chi = -\frac{e}{mc}\frac{(\mathbf{V}\cdot\mathbf{B})}{\mathbf{V}^2} + \left(\frac{e}{mc}\right)^2 \mathbf{B}^2 \frac{\left(\mathbf{V}\cdot\left[\boldsymbol{\zeta}\times\dot{\boldsymbol{\zeta}}\right]\right)}{\mathbf{a}^2}$$

the second term disappears

$$\boldsymbol{\zeta} \mapsto \mathbf{n} \longrightarrow \left(\mathbf{V}\cdot\left[\mathbf{n}\times\dot{\mathbf{n}}\right]\right) = \left(\left(r\dot{\mathbf{n}} + \dot{r}\mathbf{n}\right)\cdot\left[\mathbf{n}\times\dot{\mathbf{n}}\right]\right) = 0$$

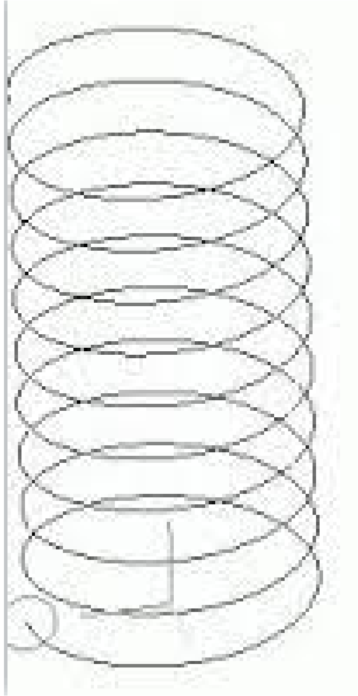
$$k^2 + \chi^2 = \left(\frac{e}{mc}\right)^2 \frac{\left[\mathbf{V}\times\mathbf{B}\right]^2 + \left(\mathbf{V}\cdot\mathbf{B}\right)^2}{\left(\mathbf{V}^2\right)^2} =$$

$$= \left(\frac{e}{mc}\right)^2 \frac{\mathbf{B}^2}{\mathbf{V}^2} = \left(\frac{e}{c}\right)^2 \frac{\mathbf{B}^2}{\mathbf{p}^2} = \frac{1}{\Lambda^2}$$

Motion in a stationary and uniform in the direction magnetic field

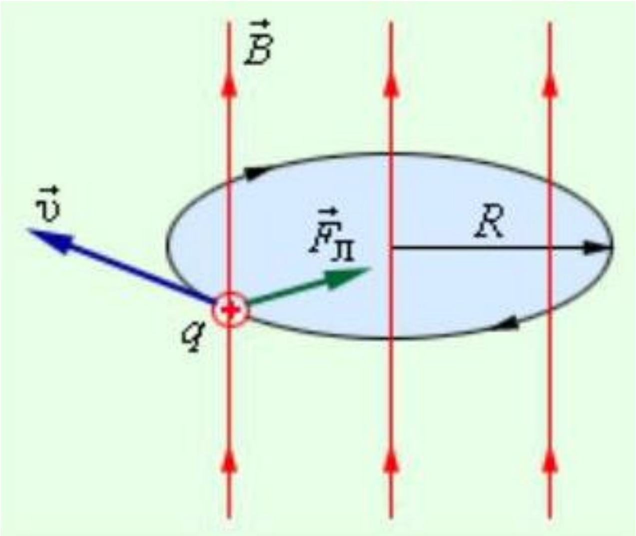
$$\dot{\zeta} = \frac{d\zeta(t,\mathbf{x})}{dt} = \frac{\partial\zeta(t,\mathbf{x})}{\partial t} + (\mathbf{V}\cdot\nabla_{\mathbf{x}})\zeta(t,\mathbf{x}) = 0$$

$$k^2 + \chi^2 = \left(\frac{e}{mc}\right)^2 \frac{\mathbf{B}^2}{\mathbf{V}^2} = \left(\frac{e}{c}\right)^2 \frac{\mathbf{B}^2}{\mathbf{p}^2} = \frac{1}{\Lambda^2}$$



$$\frac{1}{\Lambda} = \frac{|e|}{mc}\frac{|\mathbf{B}|}{|\mathbf{V}|} \equiv \frac{\omega}{|\mathbf{V}|}$$

$$\Lambda \mapsto R$$



Dirac's Abelian monopole

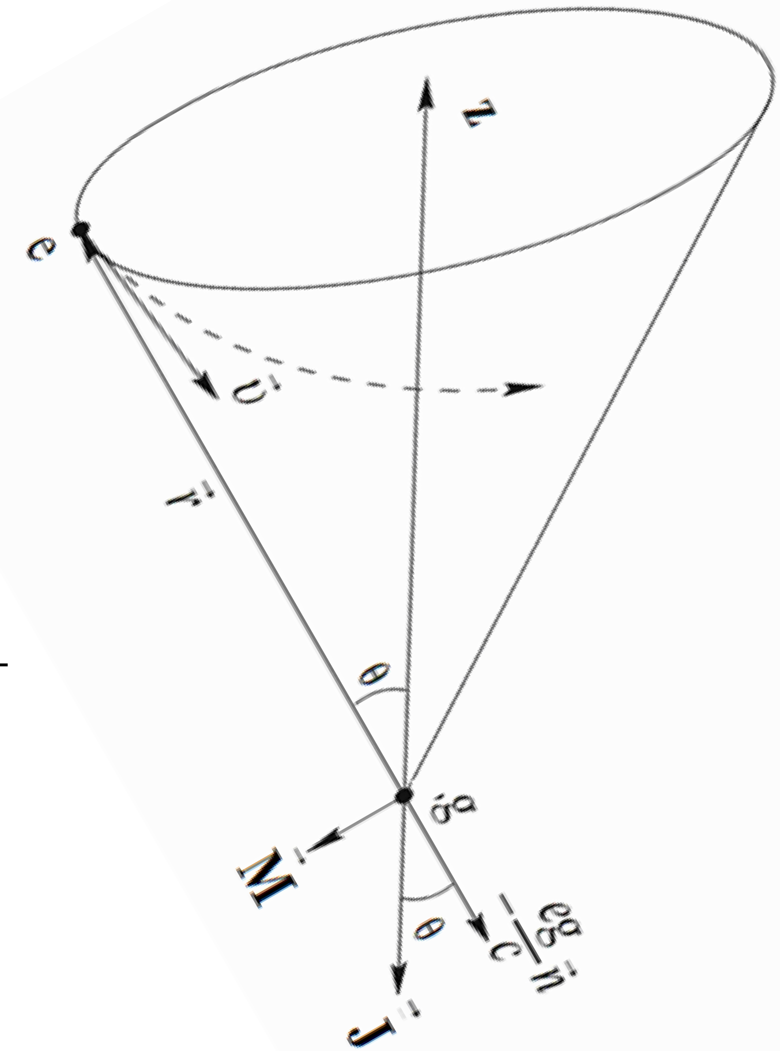
$$\mathbf{B} = \frac{g\mathbf{x}}{r^3} = \frac{g\mathbf{n}}{r^2}$$

$$Q = \frac{eg}{c} = |\mathbf{M}| ctg\,\theta = mb|\mathbf{V}| ctg\,\theta$$

$$k(r) = \frac{|Q|\,|[\mathbf{V}\times\mathbf{x}]|}{mr^3\mathbf{V}^2} = \frac{b^2 ctg\,\theta}{r^3}$$

$$\chi(r) = -\frac{Q(\mathbf{V}\cdot\mathbf{x})}{mr^3\mathbf{V}^2} = -\frac{b\,ctg\,\theta}{r^2}\sqrt{1-\left(\frac{b}{r}\right)^2}$$

$$k^2+\chi^2 = \left(\frac{Q}{mr^2|\mathbf{V}|}\right)^2 = \left(\frac{e}{c}\right)^2\frac{\mathbf{B}^2}{\mathbf{p}^2}$$



Relativistic case

$$(ds)^2 = (cd\tau)^2 = dx^\mu dx_\mu = c^2(dt)^2 - (d\mathbf{x})^2 = \frac{c^2(dt)^2}{\gamma^2}$$

$$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{ds} = (u^0, \mathbf{u}) = \gamma\left(1, \frac{\mathbf{V}}{c}\right) \quad , \qquad d\mathbf{x} = \mathbf{V}dt$$

$$w^\mu = \dot{u}^\mu = \ddot{x}^\mu = (w^0, \mathbf{w}) \, , \qquad p^\mu = mc\dot{x}^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$$

$$E = p^0 c = \gamma mc^2 = \sqrt{(mc^2)^2 + \mathbf{p}^2 c^2}$$

$$k = \frac{\left|[\mathbf{u} \times \mathbf{w}]\right|}{|\mathbf{u}|^3} \quad , \qquad \chi = \frac{(\mathbf{u} \cdot [\mathbf{w} \times \dot{\mathbf{w}}])}{[\mathbf{u} \times \mathbf{w}]^2}$$

Stationary non-abelian fields

For fields independent of x^0 i.e. $\left(\partial^0 \to 0\right)$, in gauge $A_a^0(\mathbf{x}) = 0$:

$$\vec{I} = \eta_a \vec{e}_a, \qquad \breve{\mathbf{B}} = \mathbf{B}_a \eta_a$$

Wong's equations:

$$\begin{cases} mc^2 \mathbf{w} = e\left[\mathbf{u} \times \breve{\mathbf{B}}\right] \\ \dot{\eta}_a = e\varepsilon^{abc}\left(\mathbf{A}_b \cdot \mathbf{u}\right)\eta_c \end{cases}$$

$$k = \frac{e}{mc}\frac{\left[\mathbf{u} \times \breve{\mathbf{B}}\right]}{\mathbf{u}^2}, \quad \chi = -\frac{e}{mc}\frac{\left(\mathbf{u} \cdot \breve{\mathbf{B}}\right)}{\mathbf{u}^2} + \left(\frac{e}{mc}\right)^2 \frac{\left(\mathbf{u} \cdot \left[\breve{\mathbf{B}} \times \dot{\breve{\mathbf{B}}}\right]\right)}{\mathbf{w}^2}$$

$$\breve{\mathbf{B}} = \left|\breve{\mathbf{B}}\right| \boldsymbol{\xi}, \quad \chi = -\frac{e}{mc}\frac{\left(\mathbf{u} \cdot \breve{\mathbf{B}}\right)}{\mathbf{u}^2} + \left(\frac{e}{mc}\right)^2 \frac{\breve{\mathbf{B}}^2\left(\mathbf{u} \cdot \left[\boldsymbol{\xi} \times \dot{\boldsymbol{\xi}}\right]\right)}{\mathbf{w}^2}$$

Cases of existence of a form-invariant

1) $\boldsymbol{\xi} \Rightarrow \mathbf{n}, \quad \left(\mathbf{u}\cdot\left[\mathbf{n}\times\dot{\mathbf{n}}\right]\right) = \frac{\gamma}{c}\left(\mathbf{V}\cdot\left[\mathbf{n}\times\dot{\mathbf{n}}\right]\right) = 0$

2) $\dot{\boldsymbol{\xi}}(t,\mathbf{x}) \equiv \frac{d\boldsymbol{\xi}}{ds} = u^{\mu}\partial_{\mu}\boldsymbol{\xi} \Rightarrow \left(\mathbf{u}\cdot\nabla_{\mathbf{x}}\right)\boldsymbol{\xi} = 0$

$$k^2 + \chi^2 = \left(\frac{e}{mc^2}\right)^2 \frac{\breve{\mathbf{B}}^2}{\mathbf{u}^2} = \left(\frac{e}{c}\right)^2 \frac{\breve{\mathbf{B}}^2}{\mathbf{p}^2} = \frac{e^2\breve{\mathbf{B}}^2}{E^2 - m^2c^4}$$

Abelization for the nonrelativistic limit and for the field of a non-abelian Wu-Yang monopole

$$d\tau \Rightarrow dt, \quad u^{\mu} \Rightarrow (1,\mathbf{0}), \qquad A_a^0 = 0$$

$$\frac{d\eta^a}{dt} \Rightarrow e\varepsilon^{abc} A_a^0 \eta^c = 0, \quad \eta^a = \delta^{a3}\eta^3 = \delta^{a3}\sqrt{\vec{I}^2}$$

Abelization : $e\breve{\mathbf{B}} \Rightarrow q\mathbf{B}_3, \quad q = e\eta_3$

Non-abelian Wu-Yang monopole: $B_a^j(\mathbf{x}) = \dfrac{n^j n^a}{er^2}$

And its abelization : $e\breve{\mathbf{B}} = q\overline{\mathbf{B}}, \quad \overline{\mathbf{B}}(\mathbf{x}) = \dfrac{\mathbf{n}}{er^2}$

$$q = en^a \eta_a$$

Young-Mills Lagrangian George-Gleshaw Lagrangian

$$L_{YM} = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a, \qquad L = L_{YM} + \frac{1}{2}\left(\hat{D}^\mu \hat{h}\right)_a \left(\hat{D}_\mu \hat{h}\right)_a - \frac{\lambda}{4}\left(h_a h_a - \mathrm{f}^2\right)^2$$

$$\hat{A}^\mu = A_a^\mu T^a, \qquad \hat{h} = h_a T^a, \qquad \hat{D}^\mu = \mathbf{1}\partial^\mu - ie\hat{A}^\mu,$$

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\varepsilon^{abc} A_b^\mu A_c^\nu, \qquad G_a^{jk} = -\varepsilon^{jkl} B_a^l,$$

$$\left(\hat{D}^\mu \hat{h}\right)_a = \partial^\mu h_a + e\varepsilon^{abc} A_b^\mu h_c$$

Wu-Yang's monopole Hooft-Polyakov monopole

$$B_a^l(\mathbf{x}) = \frac{n^l n^a}{er^2}, \qquad B_a^l = B_a^l[Y, \alpha, \beta] \mapsto B_a^l[K, \omega] \leftrightarrow B_a^l[N, \omega]$$

General stationary "ss" calibration field

$$A_a^j(\mathbf{x}) = -n^j n^a \frac{\alpha(r)}{er} + \varepsilon^{jka} n^k \frac{\gamma(r)}{er} - \left(\delta^{ja} - n^j n^a\right)\frac{\beta(r)}{er}$$

$$(\mathbf{n}\cdot\mathbf{A}_a) = -n^a \frac{\alpha(r)}{er} \neq 0 \qquad\qquad (\nabla\cdot\mathbf{A}_a) = \frac{n^a}{er^2}\left(2\beta - \partial_\rho(\rho\alpha)\right)$$

$$B_a^j(\mathbf{x}) = \frac{n^j n^a}{er^2}\left[1-(1+\gamma)^2 - \beta^2\right] - \frac{\left(\delta^{ja} - n^j n^a\right)}{er^2}\left[r\gamma' + \alpha\beta\right] -$$

$$-\frac{\varepsilon^{jka} n^k}{er^2}\left[r\beta' - \alpha(1+\gamma)\right]$$

The non-Abelian Wu-Yang monopole corresponds to the values $\gamma = -1, \alpha = \beta = 0$, i.e. purely longitudinal contribution in the interior and configuration spaces

Equations of motion for general fields in the Georgie-Gleshaw model

$$\langle 0 | h_3 | 0 \rangle = \mathrm{f}\,, \qquad M^2 = e^2\,\mathrm{f}^2\,, \qquad Y = \gamma + 1$$

$$h_a(\mathbf{x}) = n^a\,\mathrm{f}\left[1 - S(r)\right] \equiv n^a\,\frac{H(r)}{er}$$

$$r^2\frac{d}{dr}\left(\frac{rY' + \alpha\beta}{r}\right) = Y\left(Y^2 + \beta^2 + H^2 - 1\right) - \alpha\left(r\beta' - \alpha Y\right),$$

$$r^2\frac{d}{dr}\left(\frac{r\beta' - \alpha Y}{r}\right) = \beta\left(Y^2 + \beta^2 + H^2 - 1\right) + \alpha\left(rY' + \alpha\beta\right),$$

$$\beta\left(rY' + \alpha\beta\right) = Y\left(r\beta' - \alpha Y\right),$$

$$r^2\frac{d^2H}{dr^2} = H\left\{\frac{\lambda}{e^2}\left[H^2 - (Mr)^2\right] + 2\left(Y^2 + \beta^2\right)\right\}$$

Substituting: $Y(r) = K(r)\cos\omega(\mathrm{r}),\qquad \beta(r) = K(r)\sin\omega(r),$

for which $Y^2 + \beta^2 = K^2 > 0,\qquad r\partial_r\,\omega(r) = \alpha(r),$

leads to the well-known Hooft-Polyakov equations:

$$r^2\frac{d^2K}{dr^2} = K\left[K^2 + H^2 - 1\right],\qquad r^2\frac{d^2H}{dr^2} = H\left\{\frac{\lambda}{e^2}\left[H^2 - (Mr)^2\right] + 2K^2\right\}$$

with boundary conditions at $r \to 0$:

$$K(r) \to 1 + O(r),\qquad H(r) \to O(r)$$

and $r \to \infty$: $K(r) \sim e^{-Mr},\quad S(r) \sim e^{-2Mr}$, if $\lambda > 0$,

$S(r) \sim 1/(Mr)$ if $\lambda = 0$.

Chromo-magnetic field in an arbitrary gauge:

$$er^2 B_a^j(\mathbf{x}) = n^j n^a \left[1 - K^2\right] - \left(\delta^{ja} - n^j n^a\right) rK' \cos\omega -$$

$$- \varepsilon^{jka} n^k rK' \sin\omega$$

Taking $K(r)$ as a constant $K(r) = C$ at $C = 0$ we get to particular solutions of the system with boundary conditions at $r \to \infty$ in the form of the Higgs field

$$H(r) = Mr$$

and the Wu-Yang monopole with the same abelization

$$e\breve{\mathbf{B}} = q\overline{\mathbf{B}}, \qquad q = en^a \eta_a$$

Substitution $\beta = \eta Y$ at $Y_{\pm} = N e^{\pm i\omega}$ and $\eta = \mp i$

gives the equations :

$$r^2 N'' - \left(H^2 - 1\right) N = 0, \quad r^2 H'' = \frac{\lambda}{e^2}\left[H^2 - (Mr)^2\right] H$$

for which we have exact solutions :

$$H(r) = \pm Mr = \pm\zeta, \qquad N(r) = N_0 \left(\frac{2\zeta}{\pi}\right)^{1/2} K_{i\rho}(\zeta)$$

$$N(r) \to N_0 e^{-\zeta} \text{ at } r \to \infty, \qquad \rho = \sqrt{3}/2$$

$$N(r) \to \sqrt{\zeta} \quad \text{at} \quad r \to 0$$

The final solution for the fields is written in terms of the transverse tensor :

$$\mathrm{T}_{\pm}^{ja} = e^{\pm i\omega}\left[\left(\delta^{ja} - n^j n^a\right) \mp i\varepsilon^{jka} n^k\right]$$

in the form :

$$A_{a\pm}^{j}(\mathbf{x}) = -\frac{\varepsilon^{jka} n^k}{er} - n^j n^a \frac{\alpha(r)}{er} \pm i \frac{N}{er} \mathrm{T}_{\pm}^{ja}$$

$$B_{a\pm}^{j}(\mathbf{x}) = \frac{n^j n^a}{er^2} - \frac{rN'}{er^2} \mathrm{T}_{\pm}^{ja}$$

Longitudinal component of the field exactly coincides with the field of the non-Abelian Wu-Yang monopole.

The found solution also leads to the classical Dirac monopole

$$\overline{\mathbf{B}}(\mathbf{x}) = \frac{\mathbf{n}}{er^2}$$

Conclusion

- The existence of a form-invariant combination of curvature and torsion, which is the same for Abelian and non-Abelian external magnetic fields with different types of symmetry, is established.
- A new analytical monopole solution of the equations of the George-Glashow model is obtained in an arbitrary gauge with an explicitly "split off" contribution of the Wu-Yang monopole, and containing the Dirac monopole in already known form
- The results are published in the work : E. A. Voronova, S. É. Korenblit, Russian Physics Journal, Vol. 64, No. 1 (2021) pp 35—42.

Thanks for attention

Дополнения

Substituting any of the found "chromo-magnetic" fields together with the Higgs field into the Hooft tensor:

$$F^{jk} = \frac{h_a G_a^{jk}}{\left(h_b h_b\right)^{1/2}} - \frac{\varepsilon^{abc} h^a}{e\left(h_b h_b\right)^{3/2}} \left(\hat{D}^j \hat{h}\right)_b \left(\hat{D}^k \hat{h}\right)_c = -\varepsilon^{jkl} \overline{B}^l \ ,$$

we obtain to the field of the classical Dirac monopole

$$\overline{\mathbf{B}}(\mathbf{x}) = \frac{\mathbf{n}}{er^2}$$

$$\dot{q} = \frac{dq}{ds} = \varphi_a u_\mu \left(\hat{D}^\mu \hat{I}\right)_a + \eta_a u_\mu \left(\hat{D}^\mu \hat{\mathrm{q}}\right)_a \mapsto 0$$

$$u_\mu \left(\hat{D}^\mu \hat{\mathrm{q}}\right)_a \mapsto Y(r)\dot{n}^a + \beta(r)\varepsilon^{abc}\dot{n}^b n^c$$

$$B_a^j(\mathbf{x}) = \frac{n^j n^a}{er^2}\left[1 - C^2\right]$$

$$\overline{\mathbf{B}}(\mathbf{x}) = \frac{\mathbf{n}}{er^2}\left[1 - C^2\right]$$