Quantum Trace Formulae for Integrals of the Hyperbolic RS model

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Classical Integrability

$$\dot{L} = [L, M]$$

$$I_k = \mathrm{Tr}\, L^k$$ ← Integrals of motion

$$L = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_N \end{pmatrix}$$

Various representations for integrals of motion

$$I_k = \mathrm{Tr}\, L^k = \sum_{i=1}^{N} x_i^k$$ ← power sums

$$\mathcal{S}_k = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}$$ ← elementary symmetric functions

$$\det(L - \zeta \mathbb{1}) = \sum_{k=0}^{N} (-\zeta)^{N-k} \mathcal{S}_k \,, \qquad \mathcal{S}_0 = 1$$

Outline

Result

- *Quantum trace formulae for integrals* I_k *of the hyperbolic Ruijsenaars-Schneider model*

Derivation

- *Classical Heisenberg double and Poisson reduction*
- *Quantization*
- *Relation of* I_k *to Macdonald operators*

Baxterisation of quantum R-matrix structures

with Rob Klabbers & Enrico Olivucci, 1902.06755 [hep-th]

Result

Hyperbolic Ruijsenaars-Schneider Model

$Q_i = e^{q_i}$, where q_i, $i = 1, \dots, N$ are coordinates

$$H = \frac{1}{2} \sum_{i=1}^{N} \left(b_i^{1/\omega} e^{p_i} + b_i^{\omega} e^{-p_i} \right) ,$$

$$b_i^{\omega} = \prod_{a \neq i}^{N} \frac{Q_a - \omega Q_i}{Q_a - Q_i} , \qquad \omega = e^{-\gamma}$$

coupling constant

Performing a canonical transformation

$$q_i \to q_i , \qquad p_i \to p_i + \frac{1}{2} \log \frac{b_i^{\omega}}{b_i^{1/\omega}} ,$$

together with $\gamma \to i\gamma$, the Hamiltonian turns into

$$H = \sum_{i=1}^{N} \prod_{j \neq i}^{N} \sqrt{1 + \frac{\sin^2 \frac{\gamma}{2}}{\sinh^2 \frac{1}{2} q_{ij}}} \cosh p_i \qquad q_{ij} = q_i - q_j$$

Ruijsenaars & Schneider, '86

Quantum theory

$$H = \sum_{j=1}^{N} b_j \top_j , \qquad \top_j = e^{-\hbar \frac{\partial}{\partial q_j}} , \qquad b_j \equiv b_j^{\omega}$$

On smooth functions $f(Q_1, \dots, Q_N)$ the operator $\top_j$ acts as

$$(\top_j f)(Q_1, \dots, Q_N) = f(Q_1, \dots, qQ_j , \dots Q_N) \qquad q = e^{-\hbar}$$

Commutative family is given by a set $\{\mathcal{S}_k\}$ of Macdonald operators

Quantum trace formulae for hyperbolic RS model

Quantum trace formulae

Quantum version of $I_k = \mathrm{Tr} L^k$

Quantum L-operator

$$\boxed{I_k^{\pm} = \mathrm{Tr}_{12}\left(C_{12}^{t_2} L_1 \bar{R}_{21}^{t_2} R_{\pm 12}^{t_2} L_1 \dots L_1 \bar{R}_{21}^{t_2} R_{\pm 12}^{t_2} L_1\right)}$$

$$L_1 = L \otimes \mathbb{1}$$

Quantum R-matrices

$$C_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}\,, \qquad C_{12}^{t_2} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ij}$$

Split Casimir

matrix unities

$$I_k^{\pm} I_m^{\pm} = I_m^{\pm} I_k^{\pm}\,, \qquad I_k^{\pm} I_m^{\mp} = I_m^{\mp} I_k^{\pm}$$

In the classical limit $R, \overline{R} \to \mathbb{1}$

$$I_k^{\pm} \to \sum_{i,j=1}^{N} \mathrm{Tr}(E_{ij} L^k)\, \mathrm{Tr}(E_{ij}) = \mathrm{Tr}\left(\underbrace{\sum_{i=1}^{N} E_{ii}}_{\mathbb{1}} L^k\right) = \mathrm{Tr} L^k$$

Derivation

Heisenberg Double

Consider $\mathrm{Fun}(G \times G)$

$A, B \in G = \mathrm{GL}(N, \mathbb{C})$

Kazhdan, Kostant & Sternberg '78
Olshanetsky & Perelomov '81
Gorsky & Nekrasov, '95
Frolov & G.A. '98

$$
\begin{aligned}
\{A_1, A_2\} &= -r_-\, A_1 A_2 - A_1 A_2\, r_+ + A_1\, r_- A_2 + A_2\, r_+ A_1\,, \\
\{A_1, B_2\} &= -r_-\, A_1 B_2 - A_1 B_2\, r_- + A_1\, r_- B_2 + B_2\, r_+ A_1\,, \\
\{B_1, A_2\} &= -r_+\, B_1 A_2 - B_1 A_2\, r_+ + B_1\, r_- A_2 + A_2\, r_+ B_1\,, \\
\{B_1, B_2\} &= -r_-\, B_1 B_2 - B_1 B_2\, r_+ + B_1\, r_- B_2 + B_2\, r_+ B_1\,.
\end{aligned}
$$

$$
\begin{aligned}
r_+ &= +\frac{1}{2}\sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \sum_{i<j}^{N} E_{ij} \otimes E_{ji}\,, \\
r_- &= -\frac{1}{2}\sum_{i=1}^{N} E_{ii} \otimes E_{ii} - \sum_{i>j}^{N} E_{ij} \otimes E_{ji}\,,
\end{aligned}
$$

Semenov-Tian-Shansky '85

(A, B) monodromies of a flat connection on a punctured torus

Fock & Rosly '99

If G is a Poisson-Lie group with the Sklyanin bracket

$$\{h_1, h_2\} = -[r_\pm, h_1 h_2]$$

then its adjoint action on the Heisenberg double

$$A \to hAh^{-1}\,, \qquad B \to hBh^{-1}$$

is Poisson map. The corresponding (non-abelian) moment map is

$$m = BA^{-1}B^{-1}A$$

Fix the moment map to the following value

coupling constant

$$\mathcal{m} = \exp(\gamma n)\,,$$

where n is the Lie algebra element $\quad n = \mathrm{e}\otimes\mathrm{e}^t - \mathbb{1}\,,\quad \mathrm{e}^t = \underbrace{(1,\ldots,1)}_{N}$

One needs to solve the following matrix equation (the moment map equation)

$$\boxed{BA^{-1}B^{-1}A = \omega\mathbb{1} + \beta\,\mathrm{e}\otimes\mathrm{e}^t}$$

$$\omega = e^{-\gamma}\,,\qquad \beta = -e^{-\gamma}\frac{1-e^{N\gamma}}{N}$$

$\det \mathcal{m} = 1$
N^2-1 equations

Special parametrisation

$$A = TQT^{-1}\,,$$
$$B = UP^{-1}T^{-1}$$

Frolov & G.A. '98

Matrix P decouples

Q, P are diagonal matrices

$T, U \in G$ are two Frobenius matrices, *i.e.* they satisfy the Frobenius condition

$$T\,\mathrm{e} = \mathrm{e}\,,\qquad U\,\mathrm{e} = \mathrm{e}$$

$T, U \in F \subset G$

Frobenius group

Introducing $W = T^{-1}U \in F$, $\qquad Q^{-1}W^{-1}Q - \omega W^{-1} = \beta\,\mathrm{e}\otimes\mathrm{e}^t\,T\,,\qquad W\,\mathrm{e} = \mathrm{e}$

W is invariant under
$T \to hT\,, U \to hU\,, h \in F$

Stability group $\quad F = \{h \in G:\ h\,\mathrm{e} = \mathrm{e},\quad \mathrm{e}^t\,h = \mathrm{e}^t\}$

$\dim_{\mathbb{C}} F = (N-1)^2$

Solution

$$W_{ij}(Q)=\frac{\prod_{a\neq i}^{N}(Q_j^{-1}-\omega Q_a^{-1})}{\prod_{a\neq j}^{N}(Q_j^{-1}-Q_a^{-1})}$$

Dimension of the reduced phase space

$$\underbrace{2N^2}_{\substack{\text{dim of}\\ \text{initial phase space}}}-\underbrace{(N^2-1)}_{\text{constraints}}-\underbrace{\dim_{\mathbb{C}}F}_{\text{stabilizer}}=2N^2-(N^2-1)-(N-1)^2=2N,$$

coordinates (P,Q)

Lax matrix

$$L=W(Q)P^{-1}$$

Lax of the hyperbolic RS model

$$L=\sum_{i,j=1}^{N}\frac{Q_i-\omega Q_i}{Q_i-\omega Q_j}\prod_{a\neq j}^{N}\frac{\omega Q_j-Q_a}{Q_j-Q_a}P_j^{-1}E_{ij}$$

Frobenius-invariant extension of L to the Heisenberg double

$$\boxed{L=T^{-1}BT\,,}$$

where $A=TQT^{-1}\,,T\,\mathrm{e}=\mathrm{e}$

$B=UP^{-1}T^{-1}$

Note that

$$L=WP^{-1}\,,\quad W=T^{-1}U$$

Poisson Reduction

Dirac brackets

$$\boxed{\{L_1, L_2\} = r_{12}L_1L_2 - L_1L_2\underline{r}_{12} + L_1\bar{r}_{21}L_2 - L_2\bar{r}_{12}L_1}$$

$$Q_{ij} = Q_i - Q_j$$

$$r = \sum_{i\neq j}^{N}\Big(\frac{Q_j}{Q_{ij}}E_{ii} - \frac{Q_i}{Q_{ij}}E_{ij}\Big)\otimes(E_{jj} - E_{ji})\,,$$

$$\bar{r} = \sum_{i\neq j}^{N}\frac{Q_i}{Q_{ij}}(E_{ii} - E_{ij})\otimes E_{jj}\,,$$

$$\underline{r} = \sum_{i\neq j}^{N}\frac{Q_i}{Q_{ij}}(E_{ij}\otimes E_{ji} - E_{ii}\otimes E_{jj})\,, \qquad \underline{r}_{12} = r_{12} + \bar{r}_{21} - \bar{r}_{12}$$

If we parametrise $P_i = e^{p_i}$ $Q_j = e^{q_j}$, then brackets between p_i and q_j are canonical

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \longleftarrow \text{CYBE}$$

Two more equations involving r and $\bar{r}$

$$[\bar{r}_{12}, \bar{r}_{13}] + \{\bar{r}_{12}, p_3\} - \{\bar{r}_{13}, p_2\} = 0\,,$$
$$[r_{12}, \bar{r}_{13}] + [r_{12}, \bar{r}_{23}] + [\bar{r}_{13}, \bar{r}_{23}] + \{r_{12}, p_3\} = 0$$

Gervais-Neveu-Felder equation

$$[\underline{r}_{12}, \underline{r}_{13}] + [\underline{r}_{12}, \underline{r}_{23}] + [\underline{r}_{13}, \underline{r}_{23}] + \{\underline{r}_{12}, p_3\} - \{\underline{r}_{13}, p_2\} + \{\underline{r}_{23}, p_1\} = 0$$

Quantities $I_k = \mathrm{Tr}L^k$ are in involution!

Quantum Heisenberg Double

$$\mathcal{R}_-^{-1} A_2 \mathcal{R}_+ A_1 = A_1 \mathcal{R}_-^{-1} A_2 \mathcal{R}_+ \,,$$
$$\mathcal{R}_-^{-1} B_2 \mathcal{R}_+ A_1 = A_1 \mathcal{R}_-^{-1} B_2 \mathcal{R}_- \,,$$
$$\mathcal{R}_+^{-1} A_2 \mathcal{R}_+ B_1 = B_1 \mathcal{R}_-^{-1} A_2 \mathcal{R}_+ \,,$$
$$\mathcal{R}_-^{-1} B_2 \mathcal{R}_+ B_1 = B_1 \mathcal{R}_-^{-1} B_2 \mathcal{R}_+ \,,$$

$$\mathcal{R}_{+12} = \mathcal{R}_{21}\,, \qquad \mathcal{R}_{-12} = \mathcal{R}_{12}^{-1}$$

$$\mathcal{R}_{+21}\mathcal{R}_{-12} = \mathbb{1}$$

$$\mathcal{R} = \sum_{i\neq j}^{n} E_{ii} \otimes E_{jj} + e^{\hbar/2} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + (e^{\hbar/2} - e^{-\hbar/2}) \sum_{i>j}^{n} E_{ij} \otimes E_{ji}$$

Semenov-Tian-Shansky,'92

Quantum RS model should be obtained from the double by "quantum" Poisson reduction

This can be done for the rational case where the appropriate algebra is the quantum cotangent bundle

Frolov & G.A., '98

Assume that the matrices R and $\bar{R}$ for the hyperbolic RS model satisfy the system of equations

$$\boxed{R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}}$$

$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$

$p_j \to P_j = e^{\hbar \frac{\partial}{\partial q_j}}$

$$\begin{aligned} R_{12}\bar{R}_{13}\bar{R}_{23} &= \bar{R}_{23}\bar{R}_{13}P_3R_{12}P_3^{-1}, \\ \bar{R}_{12}P_2\bar{R}_{13}P_2^{-1} &= \bar{R}_{13}P_3\bar{R}_{12}P_3^{-1}. \end{aligned}$$

$[\bar{r}_{12}, \bar{r}_{13}] + \{\bar{r}_{12}, p_3\} - \{\bar{r}_{13}, p_2\} = 0,$
$[r_{12}, \bar{r}_{13}] + [r_{12}, \bar{r}_{23}] + [\bar{r}_{13}, \bar{r}_{23}] + \{r_{12}, p_3\} = 0$

Since $r^2 = -r$, for R we can take $\quad R = \exp \hbar r$

$$r = \sum_{i \neq j}^{N} \left(\frac{Q_j}{Q_{ij}} E_{ii} - \frac{Q_i}{Q_{ij}} E_{ij} \right) \otimes (E_{jj} - E_{ji})$$

$$R_+ = \mathbb{1} + (1-q) \sum_{i \neq j}^{N} \left(\frac{Q_j}{Q_{ij}} E_{ii} - \frac{Q_i}{Q_{ij}} E_{ij} \right) \otimes (E_{jj} - E_{ji}) \qquad R_+ = R$$

$$R_- = \mathbb{1} - (1-q^{-1}) \sum_{i \neq j}^{N} (E_{ii} - E_{ij}) \otimes \left(\frac{Q_i}{Q_{ij}} E_{jj} - \frac{Q_j}{Q_{ij}} E_{ji} \right)$$

$R_{+21}R_{-12} = \mathbb{1} \leftarrow$ precisely in the same way as their non-dynamical counterparts.

$$R_{\pm 12}\bar{R}_{13}\bar{R}_{23} = \bar{R}_{23}\bar{R}_{13}P_3R_{\pm 12}P_3^{-1}$$

$$\bar{r} = \sum_{i \neq j}^{N} \frac{Q_i}{Q_{ij}} (E_{ii} - E_{ij}) \otimes E_{jj}$$

$$\bar{R} = \mathbb{1} - \sum_{i \neq j}^{N} \frac{qQ_i - Q_i}{qQ_i - Q_j} (E_{ii} - E_{ij}) \otimes E_{jj}$$

Introduce $$\underline{R}_{12} = \bar{R}_{12}^{-1} R_{12} \bar{R}_{21} \,,$$ quantisation of $\underline{r}$

$$\underline{R}_{+} = \mathbb{1} + (1-q) \sum_{i \neq j}^{N} \frac{Q_i}{Q_{ij}} (E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj}) \,,$$
$$\underline{R}_{-} = \mathbb{1} - (1-q^{-1}) \sum_{i \neq j}^{N} \frac{Q_j}{Q_{ij}} (E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj})$$

These matrices satisfy the Gervais-Neveu-Felder equation

$$\underline{R}_{\pm 12} P_2^{-1} \underline{R}_{\pm 13} P_2 \underline{R}_{\pm 23} = P_1^{-1} \underline{R}_{\pm 23} P_1 \underline{R}_{\pm 13} P_3^{-1} \underline{R}_{\pm 12} P_3$$

Quantum L-operator algebra

$$\boxed{\begin{aligned} R_{+12} L_2 \bar{R}_{12}^{-1} L_1 &= L_1 \bar{R}_{21}^{-1} L_2 \underline{R}_{+12} \,, \\ R_{-12} L_2 \bar{R}_{12}^{-1} L_1 &= L_1 \bar{R}_{21}^{-1} L_2 \underline{R}_{-12} \,. \end{aligned}}$$

quantisation of $\{L_1, L_2\}$

Quantum L-operator

$$L = \sum_{i,j=1}^{N} \frac{Q_i - \omega Q_i}{Q_i - \omega Q_j} b_j \top_j E_{ij} \,, \qquad b_j = \prod_{a \neq j}^{N} \frac{\omega Q_j - Q_a}{Q_j - Q_a} \,,$$

where $\omega = e^{-\gamma}$ and $\top_j$ is the operator $\top_j = e^{-\hbar \frac{\partial}{\partial q_j}}$. On smooth functions $f(Q_1, \ldots, Q_N)$ it acts as

$$(\top_j f)(Q_1, \ldots, Q_N) = f(Q_1, \ldots, q Q_j, \ldots Q_N) \,.$$

Quantum trace formulae

$$I_k^{\pm} = \mathrm{Tr}_{12}\left(C_{12}^{t_2} L_1 \boxed{\bar{R}_{21}^{t_2} R_{\pm 12}^{t_2}} L_1 \dots L_1 \bar{R}_{21}^{t_2} R_{\pm 12}^{t_2} L_1\right)$$

quantisation of $I_k = \mathrm{Tr} L^k$

generalize the formula for the rational case

Frolov & G.A., '98

the pad

$$\bar{R}_{21}^{t_2} R_{+\,12}^{t_2} = \mathbb{1} + (1-q)\sum_{i,j} \frac{Q_i}{Q_i - qQ_j} E_{ij} \otimes (E_{ij} - E_{jj}),$$

$$\bar{R}_{21}^{t_2} R_{-\,12}^{t_2} = \mathbb{1} + (1-q)\sum_{i,j} \left[\frac{Q_j}{Q_i - qQ_j} E_{ij} \otimes (E_{ij} - E_{jj}) + \frac{1}{q}(E_{ii} - E_{ij}) \otimes E_{jj}\right]$$

Commutative family of *Macdonald operators* (two-parametric ω, q finite-difference operators)

$$\mathcal{S}_k = \omega^{\frac{1}{2}k(k-1)} \sum_{\substack{J\subset\{1,\dots,N\}\\ |J|=k}} \prod_{\substack{i\in J\\ j\notin J}} \frac{\omega Q_i - Q_j}{Q_i - Q_j} \prod_{i\in J} \top_i$$

Generating function

$$:\det(L-\zeta\mathbb{1}): = \sum_{k=0}^{N} (-\zeta)^{N-k}\mathcal{S}_k\,, \qquad \mathcal{S}_0 = 1$$

Determinant formulae

$$[k]_q = \sum_{n=0}^{k-1} q^n = \frac{1-q^k}{1-q}$$

q-number $[k]_q$

$$q = e^{-\hbar}$$

$$\mathcal{S}_k = \frac{1}{[k!]_{q^{\pm1}}}\begin{vmatrix} I_1^{\pm} & [k-1]_{q^{\pm1}} & 0 & \cdots & 0 \\ I_2^{\pm} & I_1^{\pm} & [k-2]_{q^{\pm1}} & \cdots & 0 \\ \vdots & \vdots & \cdot & \cdots & \vdots \\ I_{k-1}^{\pm} & I_{k-2}^{\pm} & \cdot & \cdots & [1]_{q^{\pm1}} \\ I_k^{\pm} & I_{k-1}^{\pm} & \cdot & \cdots & I_1^{\pm} \end{vmatrix}$$

$$I_k^{\pm} = \begin{vmatrix} \mathcal{S}_1 & 1 & 0 & \cdots & 0 \\ [2]_{q^{\pm1}}\mathcal{S}_2 & \mathcal{S}_1 & 1 & 0 & \cdots \\ \vdots & \vdots & \cdots & \cdots & 1 \\ [k]_{q^{\pm1}}\mathcal{S}_k & \mathcal{S}_{k-1} & \mathcal{S}_{k-2} & \cdots & \mathcal{S}_1 \end{vmatrix}$$

Baxterisation

Quantum baxterised r-matrices

$q = e^{-\hbar}$

Quantum L-operator

Original quantum L-operator

$$L(\lambda) = L - \frac{\omega\, e^{\hbar/2}}{\lambda} \underbrace{Q^{-1} L Q}_{\text{gauge equivalent}}$$

Quantum L-operator algebra

$$\underline{R}_{12}(\lambda,\mu) = \bar{R}_{12}^{-1}(\lambda) R_{12}(\lambda,\mu) \bar{R}_{21}(\mu)$$

$$\boxed{R_{12}(\lambda,\mu) L_2(\mu) \bar{R}_{12}^{-1}(\lambda) L_1(\lambda) = L_1(\lambda) \bar{R}_{21}^{-1}(\mu) L_2(\mu) \underline{R}_{12}(\lambda,\mu)}$$

$$R(\lambda,\mu) = \frac{\lambda e^{\hbar/2} R_+ - \mu e^{-\hbar/2} R_-}{\lambda-\mu} - \frac{e^{\hbar/2}-e^{-\hbar/2}}{e^{\hbar/2}\lambda - 1} X_{12} + \frac{e^{\hbar/2}-e^{-\hbar/2}}{e^{-\hbar/2}\mu - 1} X_{21}$$

$$\bar{R}(\lambda) = \bar{R} - \frac{e^{\hbar}-1}{e^{\hbar/2}\lambda - 1} X_{12}\,.$$

$$X = \sum_{i,j=1}^{N} E_{ij} \otimes E_{jj}$$

Compatibility

$$R_{12}(\lambda,\mu) R_{13}(q\lambda, q\tau) R_{23}(\mu,\tau) = R_{23}(q\mu, q\tau) R_{13}(\lambda,\tau) R_{12}(q\lambda, q\mu)$$

quantum shifted Yang-Baxter equation
Frolov, Chekhov & G.A., '97

$$R_{12}(\lambda,\mu) \bar{R}_{13}(q\lambda) \bar{R}_{23}(\mu) = \bar{R}_{23}(q\mu) \bar{R}_{13}(\lambda) P_3 R_{12}(q\lambda, q\mu) P_3^{-1}$$

$$\bar{R}_{12}(\lambda) P_2 \bar{R}_{13}(q\lambda) P_2^{-1} = \bar{R}_{13}(\lambda) P_3 \bar{R}_{12}(q\lambda) P_3^{-1}$$

Semi-classical expansion

$$R(\lambda,\mu) = \mathbb{1} + \hbar r(\lambda,\mu) + o(\hbar)\,, \quad \bar{R}(\lambda) = \mathbb{1} + \hbar r(\lambda) + o(\hbar)$$

$$r_{12}(\lambda,\mu) = \frac{\lambda r_{12} + \mu r_{21}}{\lambda-\mu} + \frac{\sigma_{12}}{\lambda-1} - \frac{\sigma_{21}}{\mu-1} + \left(\frac{1}{2}\frac{\lambda+\mu}{\lambda-\mu} - \frac{1}{\lambda-1} + \frac{1}{\mu-1}\right) \mathbb{1}\otimes\mathbb{1}$$

$$\bar{r}_{12}(\lambda) = \bar{r}_{12} + \frac{\sigma_{12}}{\lambda-1} - \frac{\mathbb{1}\otimes\mathbb{1}}{\lambda-1}$$

Commutative integrals

Theorem

$$\mathrm{Tr}L(\lambda)\,\mathrm{Tr}L(\mu) = \mathrm{Tr}L(\mu)\,\mathrm{Tr}L(\lambda)$$

proof in Klabbers, Olivucci and G.A. 1902.06755 [hep-th]

Quantum spectral curve

$$:\det(L(\lambda)-\zeta\mathbb{1}):=\sum_{k=0}^{N}(-\zeta)^{N-k}\mathcal{S}_k(\lambda)$$

$$\mathcal{S}_k(\lambda) \;=\; \lambda^{-k}(\lambda-\omega^k\,e^{-\hbar/2})(\lambda-e^{-\hbar/2})^{k-1}\,\mathcal{S}_k$$

Macdonald operators

Future Directions

Poisson reduction procedure is a key tool !

★ Quantum Poisson reduction of the Heisenberg double

★ Extension to the Ruijsenaars-Schneider models with spin

Olivucci & G.A., '20
Chalykh & Fairon '20
Fairon, Feher & Marshall '20

Thank You!