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Path Integrals in Quadratic Gravity

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1 Quadratic Gravity

is a straightforward generalization of Einstein General Relativity. It is given by the action

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 \,,$$

where

$$\mathcal{A}_0 = \Lambda \int d^4x \, \sqrt{-\mathcal{G}} \,,$$

$$\mathcal{A}_1 = -\frac{\kappa}{6} \int d^4x \, \sqrt{-\mathcal{G}} \, R \,, \qquad \kappa > 0 \,,$$

$$\mathcal{A}_2 \equiv \mathcal{A}_{2c_1} + \mathcal{A}_{2c_2} + \mathcal{A}_{2c_3} = \int d^4x \, \sqrt{-\mathcal{G}} \, \left(c_1 \, R^2 + c_2 \, R_{\mu\nu} R^{\mu\nu} + c_3 \, R_{\mu\nu\varrho\sigma} R^{\mu\nu\varrho\sigma} \right) \,.$$

In four-dimensional space-time, the Gauss-Bonnet Lagrangian

$$\sqrt{-\mathcal{G}} \, \left(R^2 - 4 \, R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\varrho\sigma} \, R^{\mu\nu\varrho\sigma} \right)$$

is the total derivative. If we choose the coefficients c in to be

$$c_1 = c_3 = -\frac{1}{2} c_2 = \frac{\lambda^2}{48} \,,$$

there is no a total derivative term in the normalized $\mathcal{A}_2$ action in this case.

We study the possibility to define path integrals in quadratic gravity.

We expect the solution to the problem to be more straightforward and rigorous than that in general relativity.

The point is, the Einstein action $\mathcal{A}_1$ is unbounded from below. Therefore,

$$\exp\{-\mathcal{A}_1\}$$

cannot be consider as a density of a measure of functional integration and the Euclidean path integrals of the form

$$\int \ldots \exp\{-\mathcal{A}_1\}\, d\mathcal{G}$$

are divergent.

2 Invariance of the action and gauge-fixing

Consider the quadratic gravity action in FLRW metric

$$ds^2 = N^2(t)\, dt^2 - a^2(t)\, d\vec{x}^2\,, \qquad N(t) > 0\,, \quad a(t) > 0\,.$$

In this case, action normalized to the unit space volume is written as

$$A \equiv \frac{\mathcal{A}}{V_3}$$

and the general coordinate invariance of the action is reduced to

its invariance under the group of diffeomorphisms ($\varphi \in Diff\,(\mathbf{R}^+)$) of the time coordinate t.

Define the action of the diffeomorphism φ on the functions $N(t)$ and $a(t)$ as follows:

$$\varphi\, N(t) = \left(\varphi^{-1}(t)\right)'\, N\left(\varphi^{-1}(t)\right)\,; \qquad \varphi\, a(t) = a\left(\varphi^{-1}(t)\right)\,.$$

Instead of the laps and the scale factors,
it is convenient to use the functions $f(t)$ and $h(t)$ defined by the following equations:

$$\left(f^{-1}(t)\right)' = \frac{N(t)}{a(t)}, \qquad f^{-1}(t) = \int\limits_0^t \frac{N(\tilde{t})}{a(\tilde{t})}\, d\tilde{t}, \quad f^{-1}(0) = 0;$$

$$h'(t) = N(t), \qquad h(t) = \int\limits_0^t N(\tilde{t})\, d\tilde{t}, \quad h(0) = 0,$$

with the transformation rules under the action of the diffeomorphism φ

$$(\varphi f)^{-1}(t) = \int\limits_0^t \frac{\varphi N(\bar{t})}{\varphi a(\bar{t})}\, d\bar{t} = \int\limits_0^{\varphi^{-1}(t)} \frac{N(\tilde{t})}{a(\tilde{t})}\, d\tilde{t} = f^{-1}\left(\varphi^{-1}(t)\right) \equiv \left(f^{-1} \circ \varphi^{-1}\right)(t);$$

$$(\varphi h)(t) = \int\limits_0^t \varphi N(\bar{t})\, d\bar{t} = \int\limits_0^{\varphi^{-1}(t)} N(\tilde{t})\, d\tilde{t} = h\left(\varphi^{-1}(t)\right) \equiv \left(h \circ \varphi^{-1}\right)(t).$$

From the above equation, we have

$$\varphi f = \varphi \circ f$$

The action is invariant under the group of diffeomorphisms:

$$A(f,\, h) = A\left(\varphi f,\, \varphi h\right) = A\left(\varphi \circ f,\, h \circ \varphi^{-1}\right) \,.$$

In particular, we can choose the diffeomorphism φ to be

$$\varphi = h\,,$$

therefore fixing the gauge.

Now

$$A = A(f,\, h) = A\left(h \circ f,\, I\right) = A\left(g,\, I\right) \,,$$

$$g(\tau) = h\left(f(\tau)\right) \,,$$

and I is the identical function.

In the gauge chosen,

$$N(t) = 1\,, \qquad a(t) = g'\left(g^{-1}(t)\right) \,,$$

and the space-time metric is

$$ds^2 = dt^2 - \left(g'\left(g^{-1}(t)\right)\right)^2 \, d\vec{x}^2\,.$$

We call this gauge "cosmological gauge", and the time variable t "cosmological time".

Another gauge that can be obtained if we choose,

$$\varphi = f^{-1}\,,$$

It is the so-called "conformal gauge" where the space-time metric looks like

$$ds^2 = \left(g'(\tau)\right)^2\,\left[d\tau^2 - d\vec{x}^2\right]\,, \qquad N(\tau) = a(\tau) = g'(\tau)\,,$$

and the time variable τ is called "conformal time". It is related to the cosmological time t that is considered the true physical time variable in the obvious way

$$\tau = g^{-1}(t)\,, \qquad t = g(\tau)\,.$$

In the conformal gauge, the action is written as

$$A = A\,(I,\, g) = A_0\,(I,\, g) + A_1\,(I,\, g) + A_2\,(I,\, g)\ ,$$

$$A_0\,(I,\, g) = \Lambda \int \left(g'(\tau)\right)^4\, d\tau\,,$$

$$A_1\,(I,\, g) = -\kappa \int \left[\left(g''(\tau)\right)^2 - \frac{d}{d\tau}\left(g''(\tau)\, g'(\tau)\right)\right]\, d\tau\,,$$

$$A_2\,(I,\, g) = \frac{\lambda^2}{2} \int \left(\frac{g'''(\tau)}{g'(\tau)}\right)^2\, d\tau\,.$$

3 Classical solutions

The Euler-Lagrange equation

$$\left[\frac{\partial}{\partial g'} - \frac{d}{d\tau}\frac{\partial}{\partial g''} + \frac{d^2}{d\tau^2}\frac{\partial}{\partial g'''}\right]\mathcal{L} = 0$$

gives

$$2\Lambda\,\left(g'\right)^3 - \kappa\, g''' + \frac{\lambda^2}{2}\left[6\frac{\left(g''\right)^2 g'''}{\left(g'\right)^4} - 3\frac{\left(g'''\right)^2}{\left(g'\right)^3} - 4\frac{g''\, g^{(4)}}{\left(g'\right)^3} + \frac{g^{(5)}}{\left(g'\right)^2}\right] = 0\,.$$

For the solutions of the form

$$g'(\tau) = \sigma\,\tau^{\alpha}\,, \qquad \sigma = const\,,$$

the equation turns into

$$2\Lambda\,\sigma^4\,\tau^{3\alpha} - \kappa\,\sigma^2\,\alpha\,(\alpha-1)\,\tau^{\alpha-2} + 3\lambda^2\,\alpha\,(\alpha-1)\,(\alpha+1)\,\tau^{-\alpha-4} = 0\,.$$

We consider the solutions of the following types:

1. $\alpha = 0\,, \qquad \Lambda = 0\,.$ The solution describes universe with a constant scale factor a.
2. $\alpha = 1\,, \qquad \Lambda = 0\,.$

$$g'_{cl}(\tau) = \sigma\,\tau\,, \qquad g_{cl}(\tau) = \frac{1}{2}\sigma\,\tau^2\,, \qquad g_{cl}^{-1}(t) = \sqrt{\frac{2\,t}{\sigma}}\,,$$

the classical scale factor is

$$a_{cl}(t) = g'_{cl}\left(g_{cl}^{-1}(t)\right) = \sqrt{2\sigma\,t}\,.$$

It corresponds to the birth of the universe from a point $(a(0) = 0)$.

3. $\alpha = -1\,, \qquad \sigma^2 = \frac{\kappa}{\Lambda}\,.$

$$g'_{cl}(\tau) = \sigma\,\tau^{-1}.$$

The conformal time τ is related to the cosmological time t by the equation

$$t = g_{cl}(\tau) = \sigma \int\limits_{\tau_0}^{\tau} \tau^{-1} = \sigma\,\ln\left|\frac{\tau}{\tau_0}\right|.$$

Consider the conformal time in the region $-\infty < \tau < 0\,,$
and assume the constants σ and τ_0 to be negative:

$$-\infty < \tau_0 < \tau < 0\,, \qquad \sigma < 0\,.$$

Note that $t > 0$ in this case.
Now the classical scale factor is

$$a_{cl}(t) = g'_{cl}\left(g_{cl}^{-1}(t)\right) = \sigma\left(\tau_0\,\exp\{\frac{t}{\sigma}\}\right)^{-1} = \frac{\sigma}{\tau_0}\,\exp\{-\frac{t}{\sigma}\} = \left|\frac{\sigma}{\tau_0}\right|\,\exp\{\frac{t}{|\sigma|}\}.$$

4 Path integrals measure

Now we can consider path integrals in the theory

$$\int F(g)\,\exp\{-A(g)\}\,dg$$

as the integrals of the form

$$\int F(g)\,\exp\{-A_1(g)\}\,\mu(dg)\,,$$

over the functional measure

$$\mu_\lambda(dg) = \exp\left\{-\frac{\lambda^2}{2}\int\left(\frac{g'''(\tau)}{g'(\tau)}\right)^2 d\tau\right\}\,dg\,.$$

If we substitute

$$q(\tau) = \frac{g''(\tau)}{g'(\tau)}\,,$$

we can rewrite the integral in the exponent in the measure density as

$$-\frac{\lambda^2}{2}\int\left(\frac{g'''(\tau)}{g'(\tau)}\right)^2 d\tau$$

$$= -\frac{\lambda^2}{2}\int\left[(q'(\tau))^2 + 2q'(\tau)q^2(\tau) + q^4(\tau)\right]\,d\tau = -\frac{\lambda^2}{2}\int (p'(\tau))^2\,d\tau\,,$$

where p is given by the nonlinear nonlocal substitution

$$p(\tau) = q(\tau) + \int_0^\tau q^2(\tau_1)\,d\tau_1\,.$$

The space of integration over the variable q is different from that over the variable p.

While the paths $p(\tau)$ form the space of all continuous functions on the interval $[0,\, T]$, the paths $q(\tau)$ are continuous almost at all points of the interval but may have singularities of the form

$$q(\tau) \sim \frac{1}{\tau - \tau_j^*}$$

at a finite number of points of the finite interval.

Nevertheless, we prove the one-to-one correspondence between the function $g(\tau)$ and the Wiener variable $p(\tau)$. It should be stressed that, in spite of the singular character of the functions $\xi(\tau)$, $\eta(\tau)$, the function $g(\tau)$ is continuous.

Now the measure $\mu_\lambda(dg)$ written in terms of $p(\tau)$ is the Wiener measure $w_{\frac{1}{\lambda}}(dp)$.

The function $g(\tau)$ is written as

$$g(\tau) = \sigma \int\limits_0^\tau |\eta(\bar\tau)| \, \exp\left\{ \int\limits_0^{\bar\tau} p(\tau_1)\, \left[1 - p(\tau_1)\, \eta(\tau_1)\right] d\tau_1 \right\} d\bar\tau \, .$$

$$q(\tau) = p(\tau) - \frac{1}{\eta(\tau)}\,,$$

$$\eta'(\tau) = -1 + 2p(\tau)\eta(\tau) - p^2(\tau)\eta^2(\tau)\,.$$

5 First order perturbative correction to the scale factor

The value of the scale factor at the moment of the cosmological time t is the functional on the space of functions g

$$a_g(t) = F(g) = g'\left(g^{-1}(t)\right) .$$

The classical solution

$$a_{cl}(t) = g'_{cl}\left(g_{cl}^{-1}(t)\right) = \sqrt{2\sigma\, t}\, .$$

Now we define the scale factor averaged over the space of functions g as

$$< a(t) >_g = \mathcal{Z}^{-1} \int g'\left(g^{-1}(t)\right)$$

$$\times \exp\left\{ \kappa \int\limits_0^{g^{-1}(t)} \left(g''(\tau_1)\right)^2 d\tau_1 - \kappa g''(g^{-1}(t))\, g'(g^{-1}(t)) \right\} \mu_\lambda(dg) \tag{5.1}$$

with the normalizing factor

$$\mathcal{Z} = \int \exp\left\{ \kappa \int\limits_0^{g^{-1}(t)} \left(g''(\tau_1)\right)^2 d\tau_1 - \kappa g''(g^{-1}(t))\, g'(g^{-1}(t)) \right\} \mu_\lambda(dg)\, .$$

Our ansatz consists in cutting off the upper limit in integrals for the action term A_1 .

For the action term A_2 entering the measure density, the cut off occurs automatically due to the properties of the (Wiener) measure.

The ansatz ensures the causality of the theory.

To calculate the first nontrivial perturbative correction to $a_{cl}(t)$, we represent the function $g(\tau)$ in terms of the function $p(\tau)$, and expand the integrand in the path integral up to the terms $O(p^2)$.

Now the first factor in the integrand is written as

$$g'\left(g^{-1}(t)\right) = g'_{cl}\left(g_{cl}^{-1}(t)\right) + \sigma\, X + \sigma\, Y = \sqrt{2\sigma\, t} + \sigma\, X + \sigma\, Y\,,$$

where X contains the terms of the order p^1 and Y contains the terms of the order p^2 .

After the cancellation of the same terms in the nominator and the denominator, the second factor $\exp\{-A_1\}$ gives

$$\left(1 + E + O\left(p^2\right)\right)\ .$$

Thus

$$< a(t) >= \sqrt{2\sigma\, t} + \sigma \int\ \{\, X\ \times\ E + Y\,\}\ w_{\frac{1}{\lambda}}(dp)\,.$$

$$(\ \sim\ 50\ \int\ )$$

Then we change the order of the ordinary and path integration and use the following simple rules for Wiener integration:

$$\int p(\tau_1)\, w_{\frac{1}{\lambda}}(dp) = 0\,, \qquad \int p(\tau_1)\, p(\tau_2)\, w_{\frac{1}{\lambda}}(dp) = \frac{1}{\lambda^2}\ \min\{\tau_1\,,\, \tau_2\}\ ,$$

with the result

$$< a(t) >= \sqrt{2\sigma\, t}\ \left\{1 + \frac{1}{\lambda^2}\ \left[-\frac{59}{63}\left(\frac{2t}{\sigma}\right)^{\frac{3}{2}} + \frac{11}{120}\,\kappa\,\sigma^2\left(\frac{2t}{\sigma}\right)^2\right]\right\}\ .$$

6 Conclusion

- We reformulate quadratic gravity in FLRW metric in terms of the new dynamical variables and study the invariance of the theory under the group of diffeomorphisms of the time coordinate.

- After fixing the gauge we find classical solutions of the different types.

One classical solution gives the power behavior of the scale factor $(\sim \sqrt{t})$,

and the other leads to the exponential behavior of the scale factor $(\sim \exp\{ct\})$.

- We construct the path integrals measure that

appears to be equivalent to the Wiener measure,

and calculate the first nontrivial perturbative correction to the averaged scale factor.

The experience gained in quantum field theory prevents us from considering first order perturbative results too seriously without an analysis of the other terms. So, further studies in this directions are needed.

The measure is quasi-invariant under the action of the group of diffeomorphisms $Diff^3$ acting as a composition from the left

$$\chi g = \chi \circ g \, .$$

That is,

$$\mu_\chi(dg) \equiv \mu\left(d\left(\chi g\right)\right) = \mathcal{P}_\chi(g)\, \mu(dg)\, ,$$

We believe that quasi-invariance of the measure and the explicit form of the Radon-Nikodim derivative will be helpful.