



Heavy flavor corrections to DIS at 2- and 3-loop order

Advances in Quantum Field Theory, Dubna, RU

Johannes Blümlein | October 2021

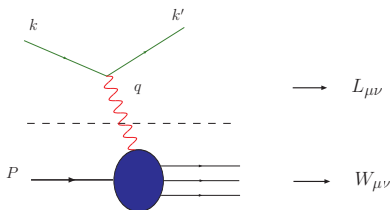
DESY

in collaboration with: J. Ablinger, A. Behring, A. De Freitas, M. Saragnese, C. Schneider, K. Schönwald

Outline



- 1 Introduction
- 2 Status of OME calculations
- 3 Polarized OMEs
- 4 2-mass contributions
- 5 Conclusions and Outlook



- Kinematic invariants:

$$Q^2 = -q^2, \quad x = \frac{Q^2}{2P \cdot q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | [J_\mu^{\text{em}}(\xi), J_\nu^{\text{em}}(\xi)] | P \rangle \\ &= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ &\quad + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho (q \cdot PS^\sigma - q \cdot SP^\sigma)}{(q \cdot P)^2} g_2(x, Q^2) \end{aligned}$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Why are Heavy Flavor Contributions important?

- They form a significant contribution to all structure functions particularly at small x and high Q^2 .
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b .
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(\text{exp}) \begin{matrix} +0.03 \\ -0.02 \end{matrix} (\text{scale}), \begin{matrix} +0.00 \\ -0.07 \end{matrix} (\text{thy})\text{GeV} \quad (\overline{\text{MS}}\text{-scheme})$$

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS & PS corrections are exact [J. Ablinger et al. (Nucl. Phys. B886 & B890 (2014))]

EIC: many more high precision data ahead for various detailed unpolarized and polarized precision measurements.

Factorization of the Structure Functions



At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{C_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

\otimes denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z).$$

The subsequent calculations are performed in Mellin space, where \otimes reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x).$$

Wilson coefficients:

$$C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right).$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) A_{ij} \left(\frac{m^2}{\mu^2}, N \right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the **light flavor Wilson coefficients** C and the **massive operator matrix elements (OMEs)** of local operators O_i between partonic states j

$$A_{ij} \left(\frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle.$$

→ additional **Feynman rules with local operator insertions** for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are **known up to NNLO** (for unpolarized scattering).

[Moch, Vermaseren, Vogt (Nucl.Phys.B (2005))]

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Unpolarized

Leading Order: [Witten (1976); Babcock, Sivers, Wolfram (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982)]

Next-to-Leading Order:

full m dependence (numeric) [Laenen, van Neerven, Riemersma, Smith (1993)]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven (1996)]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein (2007)]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein (2008, 2009)]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

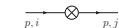
- Moments for F_2 : $N = 2 \dots 10(14)$ [Bierenbaum, Blümlein, Klein (2009)] using MATAD [Steinhauser (2000)]
- Contributions to transversity: $N = 1 \dots 13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments $N = 2, 4, 6$ [Blümlein, Wißbrock (2011)]

At 3-loop order for general values of N and extension to polarized scattering: Topic of this talk.

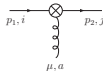
Calculation of the 3-loop operator matrix elements



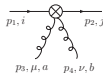
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



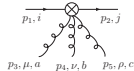
$$\delta^{ij} \Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}, \quad N \geq 1$$



$$g^2 t_{ji}^a \Delta^{\mu} \Delta^{\nu} \Delta \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$

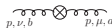


$$g^2 \Delta^{\mu} \Delta^{\nu} \Delta^{\rho} \Delta \gamma_{\pm} \sum_{j=0}^{N-3} \sum_{l=m-j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right], \quad N \geq 3$$

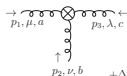


$$g^2 \Delta_{\mu} \Delta_{\nu} \Delta_{\rho} \Delta \gamma_{\pm} \sum_{j=0}^{N-4} \sum_{l=m-j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta \cdot p_2)^j (\Delta \cdot p_1)^{N-m-2} \left[(t^a t^b t^c)_{jil} (\Delta \cdot p_4 + \Delta \cdot p_5 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_5 + \Delta \cdot p_1)^{m-l-1} + (t^a t^c t^b)_{jil} (\Delta \cdot p_4 + \Delta \cdot p_5 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_4 + \Delta \cdot p_1)^{m-l-1} + (t^b t^c t^a)_{jil} (\Delta \cdot p_3 + \Delta \cdot p_5 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_5 + \Delta \cdot p_1)^{m-l-1} + (t^b t^a t^c)_{jil} (\Delta \cdot p_3 + \Delta \cdot p_5 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_3 + \Delta \cdot p_1)^{m-l-1} + (t^c t^a t^b)_{jil} (\Delta \cdot p_3 + \Delta \cdot p_4 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_4 + \Delta \cdot p_1)^{m-l-1} + (t^c t^b t^a)_{jil} (\Delta \cdot p_3 + \Delta \cdot p_4 + \Delta \cdot p_1)^{l-j-1} (\Delta \cdot p_3 + \Delta \cdot p_1)^{m-l-1} \right], \quad N \geq 4$$

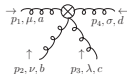
$$\gamma_+ = 1, \quad \gamma_- = \gamma_5.$$



$$\frac{1+i(-1)^N}{2} f^{abc} (\Delta \cdot p)^{N-2} \left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_{\mu} p_{\nu} + \Delta_{\nu} p_{\mu}) \Delta \cdot p + p^2 \Delta_{\mu} \Delta_{\nu} \right], \quad N \geq 2$$



$$-i g \frac{1+i(-1)^N}{2} f^{abc} \left(\left[(\Delta_{\nu} g_{\lambda\mu} - \Delta_{\lambda} g_{\mu\nu}) \Delta \cdot p_1 + \Delta_{\mu} (p_{1,\nu} \Delta_{\lambda} - p_{1,\lambda} \Delta_{\nu}) \right] (\Delta \cdot p_1)^{N-2} + \Delta_{\lambda} \left[\Delta \cdot p_1 p_{2,\mu} \Delta_{\nu} + \Delta \cdot p_2 p_{1,\nu} \Delta_{\mu} - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_{\mu} \Delta_{\nu} \right] \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} + \left\{ \begin{matrix} p_1 \cdot p_2 - p_2 \cdot p_1 \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \cdot p_2 - p_2 \cdot p_1 \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{matrix} \right\} \right), \quad N \geq 2$$



$$g^2 \frac{1+i(-1)^N}{2} \left(f^{abc} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) + f^{ace} f^{bdc} O_{\mu\nu\lambda\sigma}(p_1, p_3, p_2, p_4) + f^{abc} f^{bca} O_{\mu\nu\lambda\sigma}(p_1, p_4, p_2, p_3) \right), \quad O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_{\nu} \Delta_{\lambda} \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} + [p_{4,\mu} \Delta_{\sigma} - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} - [p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_{\mu} \Delta_{\sigma} - \Delta \cdot p_4 p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 p_{4,\mu} \Delta_{\sigma}] \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \right\} - \left\{ \begin{matrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{matrix} \right\} - \left\{ \begin{matrix} p_2 \leftrightarrow p_3 \\ \lambda \leftrightarrow \sigma \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{matrix} \right\}, \quad N \geq 2$$

- Resummation of operator insertion into propagator structure ($\Delta \cdot \Delta = 0$):

$$\sum_{N=0}^{\infty} t^N (\Delta \cdot k)^N = \frac{1}{1 - t \Delta \cdot k}$$

- Reduction to master integrals using IBP relations implemented in Reduze2[Manteuffel, Studerus (2012)].
- Solution of master integrals obtained by different methods:
 - direct integration using Mellin-Barnes and ${}_pF_q$ -techniques
 - differential equations in resummation variable t
- method of arbitrary high moments, i.e. reconstructing all- N solution from a large number of fixed moments

⇒ All these methods use the packages Sigma, EvaluateMultiSums [Schneider (2007-)] and HarmonicSums [Ablinger et al. (2010-)] which have been developed with these calculations.

The Wilson Coefficients at large Q^2



$$L_{q,(2,L)}^{NS}(N_F + 1) = a_s^2 [A_{qq,Q}^{(2),NS}(N_F + 1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),NS}(N_F)] + a_s^3 [A_{qq,Q}^{(3),NS}(N_F + 1)\delta_2 + A_{qq,Q}^{(2),NS}(N_F + 1)C_{q,(2,L)}^{(1),NS}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),NS}(N_F)]$$

$$L_{q,(2,L)}^{PS}(N_F + 1) = a_s^3 [A_{qq,Q}^{(3),PS}(N_F + 1)\delta_2 + N_F A_{gg,Q}^{(2),NS}(N_F) \tilde{C}_{g,(2,L)}^{(1),NS}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{(3),PS}(N_F)]$$

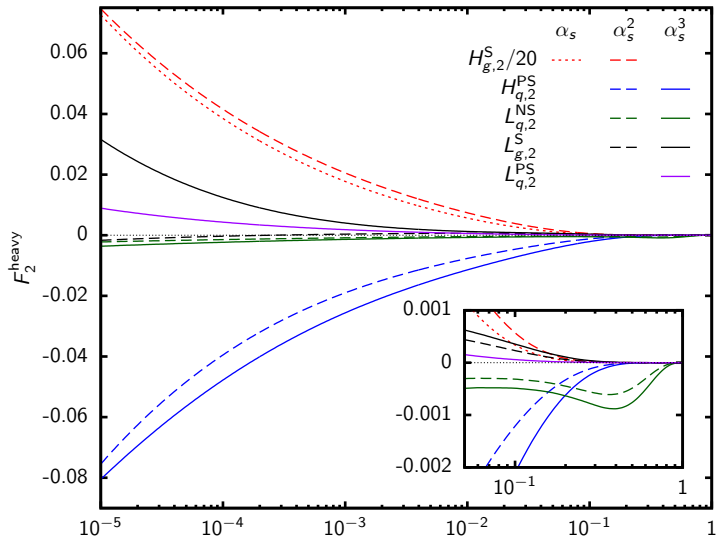
$$L_{g,(2,L)}^S(N_F + 1) = a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + a_s^3 [A_{qq,Q}^{(3)}(N_F + 1)\delta_2 + A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),PS}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{(3)}(N_F)]$$

$$H_{q,(2,L)}^{PS}(N_F + 1) = a_s^2 [A_{Qq}^{(2),PS}(N_F + 1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),PS}(N_F + 1)] + a_s^3 [A_{Qq}^{(3),PS}(N_F + 1)\delta_2 + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(1,L)}^{(2)}(N_F + 1) + A_{Qq}^{(2),PS}(N_F + 1) \tilde{C}_{q,(2,L)}^{(1),NS}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(3),PS}(N_F + 1)]$$

$$H_{g,(2,L)}^S(N_F + 1) = a_s [A_{Qg}^{(1)}(N_F + 1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)] + a_s^2 [A_{Qg}^{(2)}(N_F + 1)\delta_2 + A_{Qg}^{(1)}(N_F + 1) \tilde{C}_{q,(2,L)}^{(1)}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1)] + a_s^3 [A_{Qg}^{(3)}(N_F + 1)\delta_2 + A_{Qg}^{(2)}(N_F + 1) \tilde{C}_{q,(2,L)}^{(1)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \tilde{C}_{q,(2,L)}^{(2),S}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1)]$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

Heavy Flavor contribution to F_2



The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

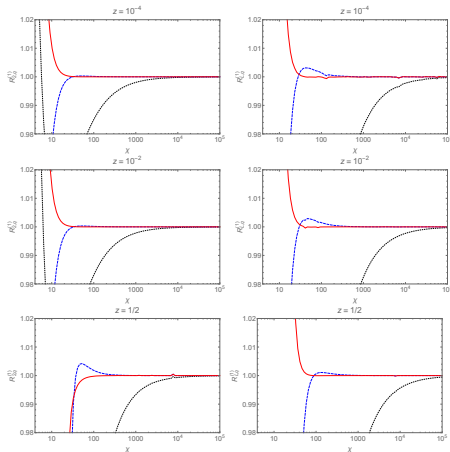


Figure 1: The ratios $R_{2,q}^{(1)}$ (left) and $R_{L,q}^{(1)}$ (right) as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines: $O((m^2/Q^2)^2)$ improved.

Different convergence range for F_2 and F_L w.r.t Q^2 at $O(\alpha_s^2)$.

The variable flavor number scheme



- Matching conditions for parton distribution functions:

$$f_k(N_F + 2) + \bar{f}_k(N_F + 2) = A_{qq,Q}^{\text{NS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot [f_k(N_F) + \bar{f}_k(N_F)] + \frac{1}{N_F} A_{qq,Q}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) \\ + \frac{1}{N_F} A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

$$f_Q(N_F + 2) + \bar{f}_Q(N_F + 2) = A_{Qq}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F),$$

$$\Sigma(N_F + 2) = \left[A_{qq,Q}^{\text{NS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qq,Q}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qq}^{\text{PS}} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot \Sigma(N_F) \\ + \left[A_{qg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qg} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot G(N_F),$$

$$G(N_F + 2) = A_{gq,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{gg,Q} \left(N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F).$$

- Polarized OMEs for heavy flavor production at 2-loop order have been calculated before.
[Buza et al (1996), Klein (2009), Hasselhuhn (2013)]
- Calculation of OMEs relied on tensor-decomposition in order to arrive at Larin scheme, i.e.

$$\gamma_5 \rightarrow \frac{i}{24} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma.$$

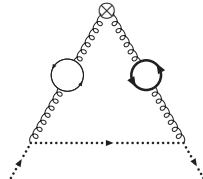
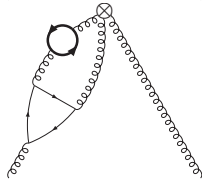
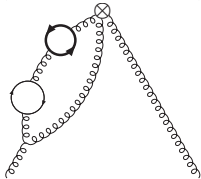
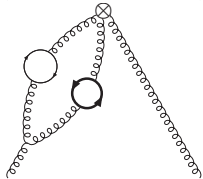
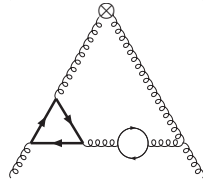
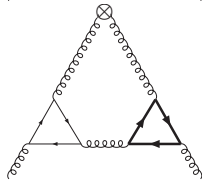
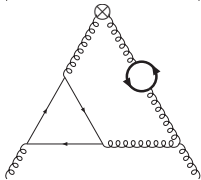
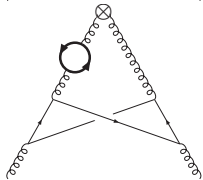
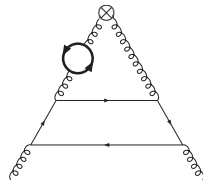
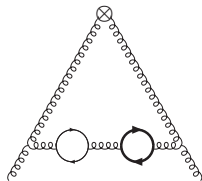
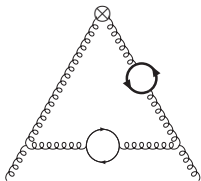
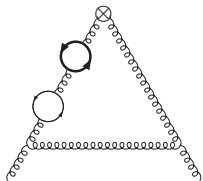
- We found out that a change of projector can accomplish the same:

$$\epsilon_{\mu\nu\rho\sigma} \text{tr} [\not{p} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma G_q] \rightarrow \epsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} [\not{p} \gamma^\mu \gamma^\nu G_q]$$

⇒ this allows to apply all technologies for the unpolarized OMEs directly to the polarized ones, i.e.

- 1 the terms $\sim T_F$ of the polarized 3-loop **anomalous dimensions** from a massive calculation
[Behring et al. (Nucl. Phys. B948 (2020))]
- 2 $A_{qq,Q}^{(3),PS}$, $A_{Qq}^{(3),PS}$, $A_{gq,Q}^{(3)}$ (single and 2-mass contributions)
[Ablinger et al. (Nucl. Phys. B952,953,955 (2020)), Behring et al. (Nucl. Phys. B964 (2021))]

2-mass contributions



2-mass contributions



$$A_{qq,Q}^{(3),NS}, A_{gg,Q}^{(3)}$$

Harmonic Sums

[Vermaseren '98; Blümlein, Kurth '98]

$$\sum_{i=1}^N \frac{1}{i^\beta} \sum_{j=1}^i \frac{1}{j}$$

HPLs

[Remiddi, Vermaseren '99]

$$\int_0^x \frac{d\tau_1}{1+\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{1-\tau_2}$$

$$A_{gg,Q}^{(3)}$$

Generalized harmonic and binomial sums

[Ablinger, Blümlein, Schneider '13]

[Ablinger, Blümlein, Raab, Schneider '14]

$$\sum_{i=1}^N \frac{4^i (1-\eta)^{-i}}{i \binom{2i}{i}} \sum_{j=1}^i \frac{(1-\eta)^j}{j^2}$$

Iterated integrals over root and η valued letters

[Ablinger, Blümlein, Raab, Schneider '14]

$$\int_0^x d\tau_1 \frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2}$$

$$A_{Qq}^{(3),PS}$$

—

Iterated integrals over root valued letters with restricted support

$$\theta(x - \eta_+) \int_0^{x(1-x)/\eta} d\tau \frac{\sqrt{1-4\tau}}{\tau}$$

Results: $A_{gg,Q}^{(3)}$

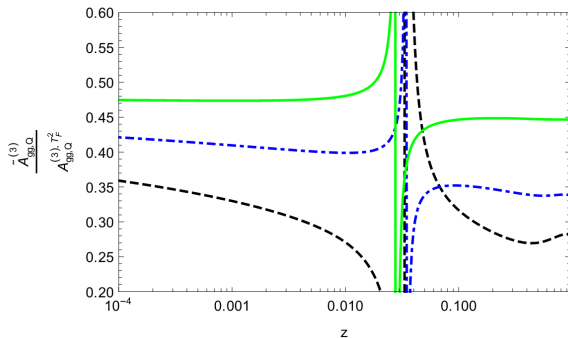
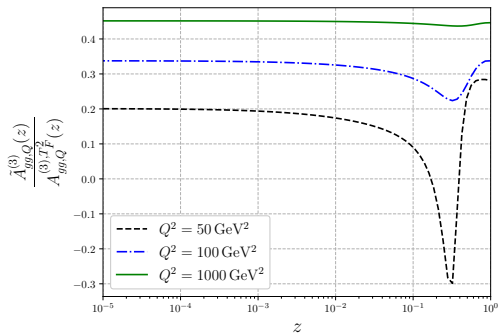


$$\begin{aligned}
 \tilde{a}_{gg,Q}^{(3)}(N) &= \frac{1}{2} \left(1 + (-1)^N\right) \left\{ T_F^3 \left\{ \frac{32}{3} (L_1^3 + L_2^3) + \frac{64}{3} L_1 L_2 (L_1 + L_2) + 32 \zeta_2 (L_1 + L_2) + \frac{128}{9} \zeta_3 \right\} \right. \\
 &+ C_F T_F^2 \left\{ \dots + 32 \left(H_0^2(\eta) - \frac{1}{3} S_2 \right) S_1 + \frac{128}{3} S_{2,1} - \frac{64}{3} S_{1,1,1} \left(\frac{1}{1-\eta}, 1-\eta, 1, N \right) \right. \\
 &\quad - \frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)} \left(\frac{\eta}{1-\eta} \right)^N \left[H_0^2(\eta) \right. \\
 &\quad \left. \left. - 2H_0(\eta) S_1 \left(\frac{\eta-1}{\eta}, N \right) - 2S_2 \left(\frac{\eta-1}{\eta}, N \right) + 2S_{1,1} \left(\frac{\eta-1}{\eta}, 1, N \right) \right] + \dots \right\} \\
 &+ C_A T_F^2 \left\{ \dots + \left[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} \right. \right. \\
 &\quad + \frac{8P_{37}H_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)} + \frac{2P_{23}H_0^2(\eta)}{9\eta(N-1)N(N+1)^2} + \frac{32}{27} H_0^3(\eta) + \frac{128}{9} H_{0,0,1}(\eta) \\
 &\quad \left. \left. + \frac{64}{9} H_0^2(\eta) H_1(\eta) - \frac{128}{9} H_0(\eta) H_{0,1}(\eta) \right] S_1 \right. \\
 &\quad \left. + \frac{2^{-1-2N} P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \binom{2N}{N} \sum_{i=1}^N \frac{4^i \left(\frac{\eta}{\eta-1} \right)^i}{i \binom{2i}{i}} \left\{ \frac{1}{2} H_0^2(\eta) \right. \right. \\
 &\quad \left. \left. S_{1,1} \left(\frac{\eta-1}{\eta}, 1, i \right) \right\} + \dots \right\}
 \end{aligned}$$

Results: $A_{gg,Q}^{(3)}$



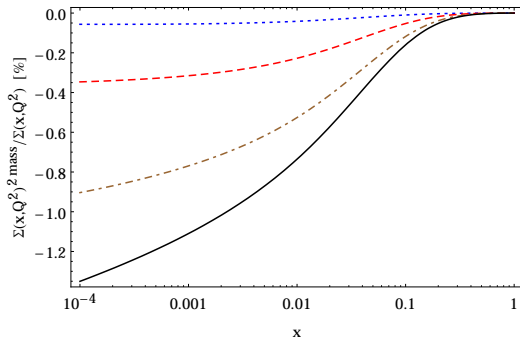
The two mass contributions over the whole T_F^2 - contributions to the OME $A_{gg,Q}^{(3)}$:



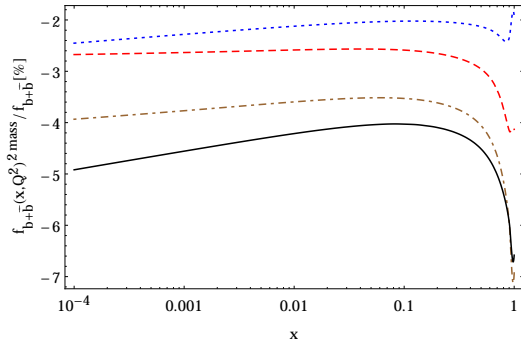
The variable flavor number scheme at NLO



$$\Sigma(x, Q^2)^{2\text{-mass}} / \Sigma(x, Q^2)$$



$$f_{b+\bar{b}}(x, Q^2)^{2\text{-mass}} / f_{b+\bar{b}}(x, Q^2)$$



- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV²; Dotted line: $Q^2 = 30$ GeV²; Dashed line: $Q^2 = 100$ GeV²; Dash-dotted line: $Q^2 = 1000$ GeV²; Full line: $Q^2 = 10000$ GeV².
- For the PDFs the NNLO variant of ABMP16 with $N_f = 3$ flavors was used. [Alekhin et al., Phys. Rev. D 96 \(2017\) 1](#)



Conclusions

- The 2-loop contributions for the pure-singlet Wilson coefficients have been calculated for general kinematics.
- All operator matrix elements except of $A_{Qg}^{(3)}$ have been calculated for general values of N . $A_{Qg}^{(3)}$ is in progress.
- Calculation of 2-mass contributions to all operator matrix elements except of $A_{Qg}^{(3)}$ for general values of N . $A_{Qg}^{(3)}$ is in progress.
- Extension to the polarized case without the need of tensor-decomposition.
- The technologies have also been applied to QED ISR calculations for e^+e^- annihilation at higher order of massive leptons, including the electron.
- There were several new mathematica developments concerning massive Feynman integral calculations during the last 2 years.

Outlook

- Study of the VFNS in the polarized case.
- High accuracy reconstruction of x -space expression of $A_{Qg}^{(3)}$ from the large number of moments already constructed. The analytic solution requires more mathematical effort. The technology is currently implemented.
- 2-mass contributions to $A_{Qg}^{(3)}$ in an expansion in the mass ratio m_c/m_b .