Amplitudes in fishnet theories in diverse dimensions

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hep-th
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ITEP, BLTP

N=4 SYM, integrability and fishnets

- **N=4 SYM - one may hope that this theory is exactly solvable.**
- **Physical content - resembles perturbative part of QCD (massless QED without running of the coupling). Tree amplitudes identical to QCD.**
- **The correlation functions in this theory can be studied in the weak and strong regimes (via AdS/CFT).**
- **The computation of anomalous dimensions of local operators in N=4 SYM in planar limit can be reduced to the problem of solving some integrable system.**
- **There are numerous results for perturbative expansions of amplitudes (S-matrix) and form factors/cor.functions with some results valid in all orders of PT (BDS ansatz for 4,5 points, collinear OPE).**
- **Can we utilise integrability to compute loop amplitudes ?**

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- **Can we utilise integrability to compute loop amplitudes ?**
- **Some simple sub-sector within N=4 SYM ?**
- **What about higher dimensions ?**

Where dose "fishnets" come from ?

We do not know (yet) but for D=4 we see that:

From V.Kazakov presentation at "Exactly Solvable Quantum Chains" International Institute of Physics, Natal, 25 Juin 2018

γ-twisted N=4 SYM and double scaling limit

- Lagrangian of N=4 SYM has $N_c \times N_c$ matrix gauge, scalar, and fermion fields:

$$\mathcal{L} = N_c \mathrm{tr}[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^\mu \phi_i^\dagger D_\mu \phi^i + i\bar\psi_A^{\dot\alpha} D_{\dot\alpha}^{\alpha} \psi_\alpha^A] + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = N_c g^2 \,\mathrm{tr}\left(\frac{1}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - e^{-i\epsilon^{ijk}\gamma_k}\phi_i^\dagger \phi_j^\dagger \phi^i \phi^j\right) +$$

$$N_c g\, \mathrm{tr}(-e^{-\frac{i}{2}\gamma_j^-}\bar\psi_j \phi^j \bar\psi_4 + e^{+\frac{i}{2}\gamma_j^-}\bar\psi_4 \phi^j \bar\psi_j + i\epsilon_{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\bar\psi^k \phi^i \bar\psi^j + \text{conjugate terms}).$$

$$\gamma_1^\pm = -\frac{\gamma_3 \pm \gamma_2}{2} \qquad \gamma_2^\pm = -\frac{\gamma_1 \pm \gamma_3}{2} \qquad \gamma_3^\pm = -\frac{\gamma_2 \pm \gamma_1}{2}$$

- γ-twisted N=4 SYM Lagrangian: product of matrix fields → star-product

$$A\,B \to A \star B \equiv q_{A,B}\,A\,B \quad \text{where} \quad q_{A,B} = e^{-\frac{i}{2}\epsilon^{mjk}\gamma_m J_j^A J_k^B} = (q_{B,A})^{-1}$$

Leigh, Strassler; Frolov, Tseytlin; Beisert, Roiban; Lunin, Maldacena

$J_1^A, J_2^A, J_3^A \in SO(6)$ - Cartan charges of R-symmetry; $\gamma_1, \gamma_2, \gamma_3$ - twists

- PSU(2,2|4)→SU(2,2) × U(1)^3 - breaks R-symmetry and all supersymmetry

- Double scaling limit: strong twist, weak coupling — Gurdogan, V.K. 2015

$$g \to 0, \qquad e^{-i\gamma_j/2} \to \infty, \qquad \xi_j = g\,e^{-i\gamma_j/2} \text{ – fixed}, \qquad (j = 1,2,3.)$$

$$\mathcal{L} = N_c \mathrm{tr}[-\frac{1}{2}\partial^\mu \phi_i^\dagger \partial_\mu \phi^i + i\bar\psi_A^{\dot\alpha}\partial_{\dot\alpha}^{\alpha}\psi_\alpha^A] + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = N_c \mathrm{tr}[\xi_1^2\,\phi_2^\dagger\phi_3^\dagger\phi_2\phi_3 + \xi_2^2\,\phi_3^\dagger\phi_1^\dagger\phi_3\phi_1 + \xi_3^2\,\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2 +$$

$$+ i\sqrt{\xi_2\xi_3}(\psi^3\phi^1\psi^2 + \bar\psi_3\phi_1^\dagger\bar\psi_2) + i\sqrt{\xi_1\xi_3}(\psi^1\phi^2\psi^3 + \bar\psi_1\phi_2^\dagger\bar\psi_3) + i\sqrt{\xi_1\xi_2}(\psi^2\phi^3\psi^1 + \bar\psi_2\phi_3^\dagger\bar\psi_1)].$$

$\mathcal{L}_{main}$ for D=4, ω=1

Why this D=6 fishnet theory ?

We will manly focus on d=4 and d=6 theories. Why ?
In short - D=6 dual conformal invariance and conjectures of

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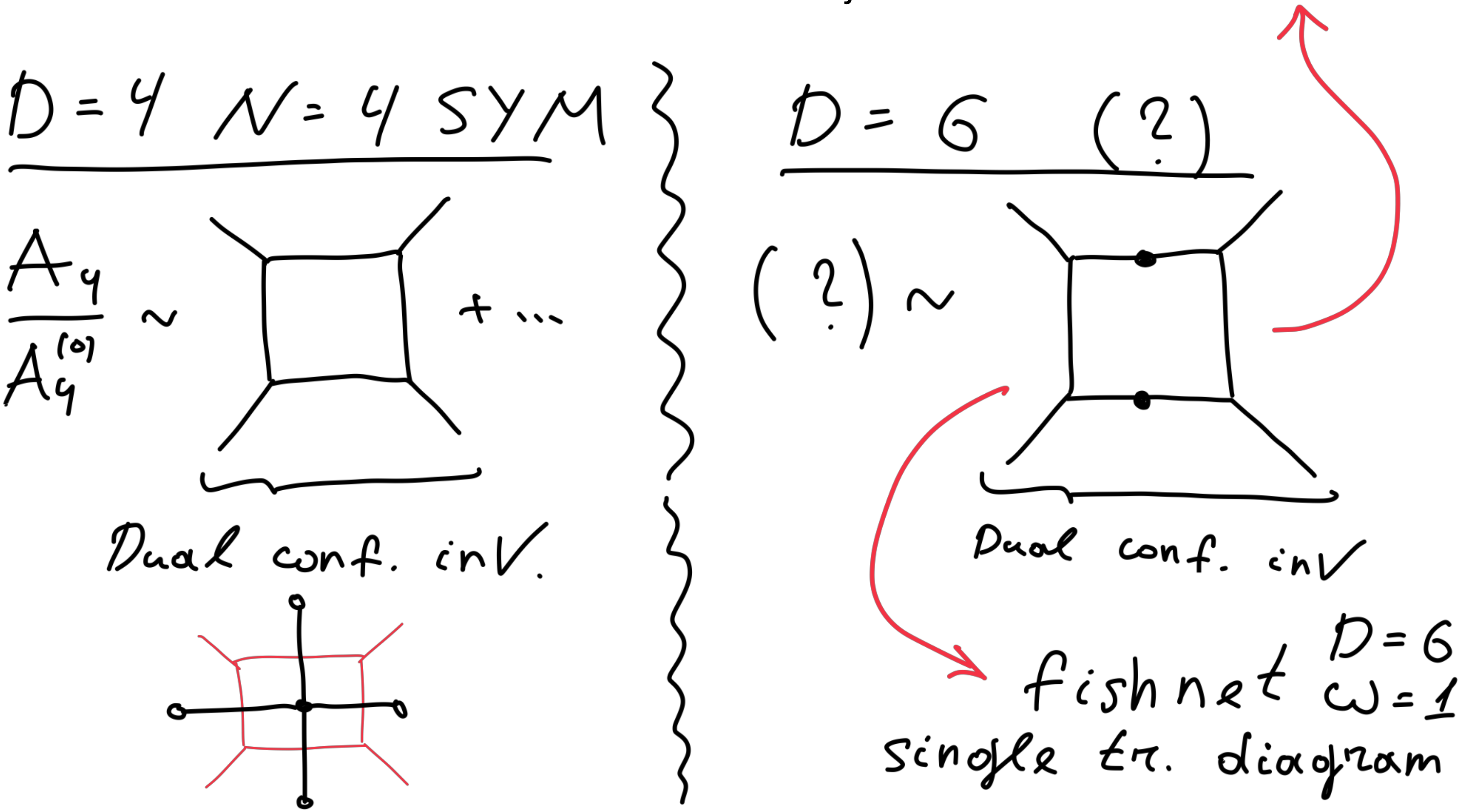
arXiv:1611.02179



"Fishnet" models.

In arbitrary dimension D one can consider the theory of pair of interacting scalar fields (ϕ_1, ϕ_2) given by the following Lagrangian

$$\mathcal{L}_{main} = N_c \, \mathrm{tr} \left(\phi_1^*(\partial^2)^\omega \phi_1 + \phi_2^*(\partial^2)^{D/2-\omega} \phi_2 + (4\pi)^{D/2} g^2 \, \phi_1^* \phi_2^* \phi_1 \phi_2 \right),$$

where ϕ_i transforms under adjoint representation of $SU(N_c)$ and parameter $\omega \in (0, \frac{D}{2})$. g is the dimensionless (for arbitrary D) coupling constant and large N_c limit is implemented.

$$\mathcal{L} = \mathcal{L}_{main} + \mathcal{L}_{c.t.}^{(D,\omega)}$$

Integrable model in any D!

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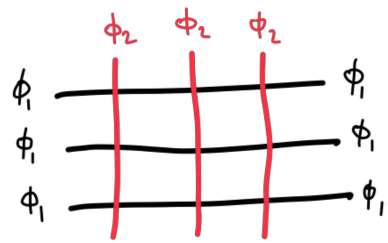
arXiv:1801.09844
etc.

Here:
$$\mathcal{L}_{c.t.}^{(D,\omega)} = \sum_i \alpha_i(g) \mathrm{tr}\left(\mathcal{O}_2^{(i)}\right) \mathrm{tr}\left(\tilde{\mathcal{O}}_2^{(i)}\right)$$

$\mathrm{Tr}(\phi_1 \phi_1)\,\mathrm{tr}(\phi_2^* \phi_2^*)$ etc.

Typical observables in such theories:

$$\langle \mathrm{tr}[\phi_1(x_1)\phi_1(x_2) \dots \phi_2(x_{11})\phi_2(x_{12})] \rangle =$$



A.B. Zamolodchikov .Phys. Lett. B, 97:63–66, 1980.

Integrability in "fishnet" models.

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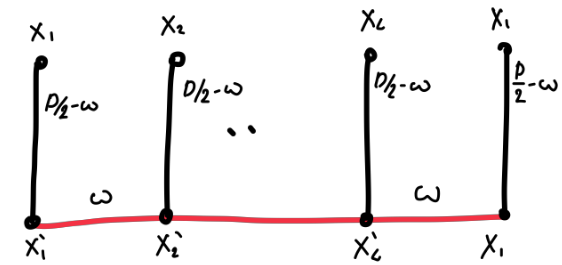
1711.04786

Why such models are integrable ?

$$\Phi^{(l)}(x_1,\ldots,x_L) = \mathcal{H}_L \Phi^{(l-2)}(x_1,\ldots,x_L) =$$

$$= \frac{1}{\pi^{DL/2}} \int \frac{d^D x_{1'} \ldots d^D x_{L'}\ \Phi^{(l')}(x_{1'},\ldots,x_{L'})}{|x_{11'}|^{D-2\omega}\ldots|x_{LL'}|^{D-2\omega}|x_{1'2'}|^{2\omega}\ldots|x_{L'1'}|^{2\omega}},$$

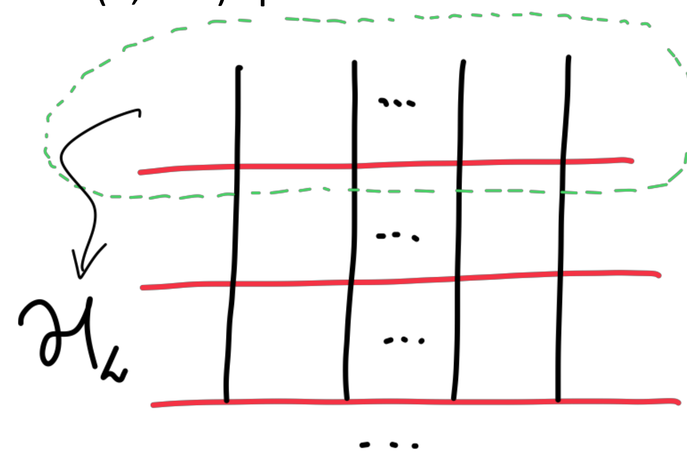


All diagrams can be represented as the consecutive action of the SO(1,D+1) spin chain Hamiltonian:

$$\mathbb{T}(u) = \mathrm{Tr}_0\left(R_{01}(u)\,R_{02}(u)\ldots R_{0L}(u)\right)$$

$$[R_{12}\Phi](x_1,x_2)(u) = c(u,D,\omega)\times$$

$$\times \int \frac{d^D x_{1'} d^D x_{2'}\ \Phi(x_{1'},x_{2'})}{(x_{12}^2)^{-u-\frac{D}{4}}(x_{21'}^2)^{\frac{D}{4}+u+\omega}(x_{12'}^2)^{\frac{3D}{4}+u-\omega}(x_{1'2'}^2)^{-u+\frac{D}{4}}}$$



$$\mathcal{H}_L = \pi^{-\frac{DL}{2}}\left[(4\pi^2)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)\right]^L \lim_{\epsilon\to 0}\epsilon^L\,\mathbb{T}\left(-\frac{D}{4}+\epsilon\right)$$

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arXiv:1801.09844
etc.

D=4 Example.

Single trace correlation functions are given by single fishnet diagram and are protected from quantum corrections.

Double trace correlation functions are none trivial and are given by infinite series of diagrams:

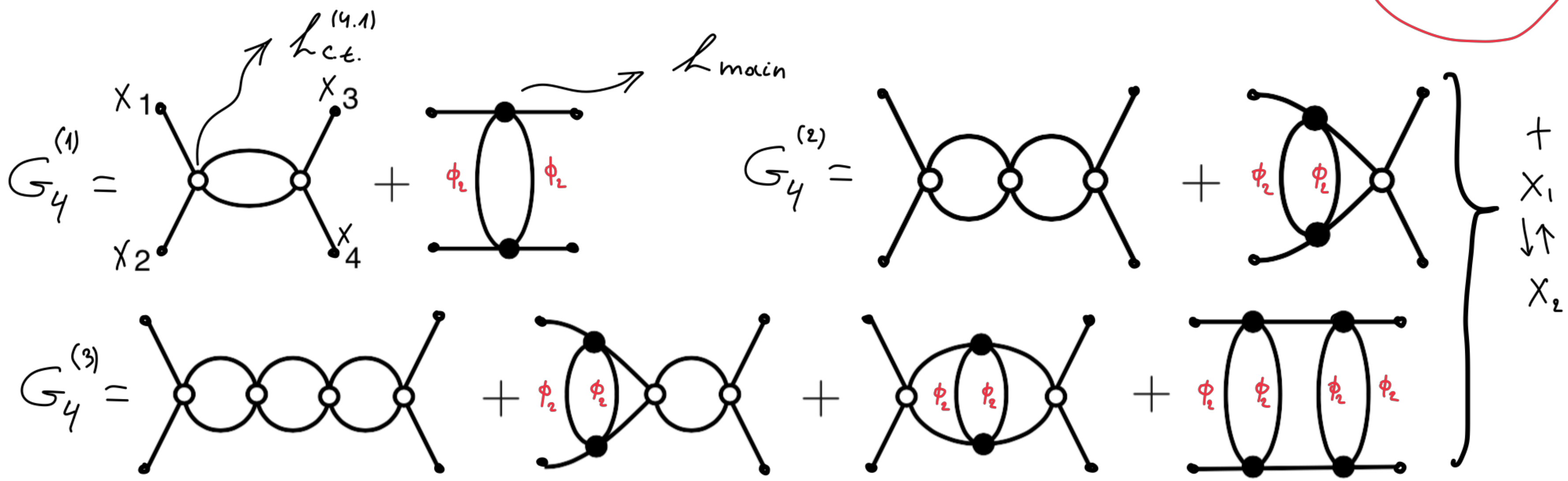
$$\begin{aligned}\mathcal{L}^{(4,1)}_{c.t.} &= (4\pi)^2\alpha_1(g)\left(\mathrm{tr}\left(\phi_1^*\phi_1^*\right)\mathrm{tr}\left(\phi_1\phi_1\right)+\mathrm{tr}\left(\phi_2^*\phi_2^*\right)\mathrm{tr}\left(\phi_2\phi_2\right)\right)\\ &+ (4\pi)^2\alpha_2(g)\left(\mathrm{tr}\left(\phi_1^*\phi_2^*\right)\mathrm{tr}\left(\phi_1\phi_2\right)+\mathrm{tr}\left(\phi_1\phi_2^*\right)\mathrm{tr}\left(\phi_1^*\phi_2\right)\right)\end{aligned}$$

$$\alpha_1(g) = \frac{ig^2}{2} + O(g^4)$$

$$\alpha_2(g) = g^2$$

hep-th **1711.04786**

$$G_4(x_1,\ldots,x_4) = \langle \mathrm{tr}\left(\phi_1(x_1)\phi_1(x_2)\right)\mathrm{tr}\left(\phi_1^*(x_3)\phi_1^*(x_4)\right)\rangle$$



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(*) $\alpha_1(g) = \dfrac{ig^2}{2} + O(g^4)$

$\alpha_2(g) = g^2$

$$G_4(x_1,\ldots,x_4) = \langle \mathrm{tr}\left(\phi_1(x_1)\phi_1(x_2)\right)\mathrm{tr}\left(\phi_1^*(x_3)\phi_1^*(x_4)\right)\rangle$$

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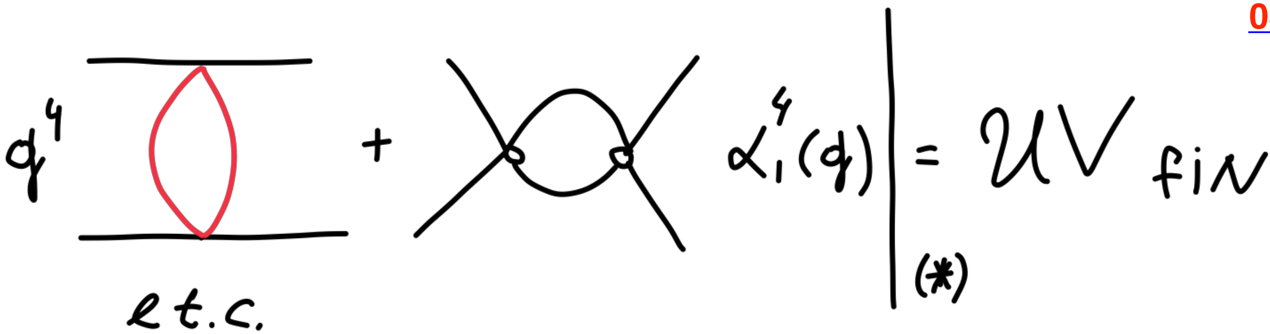
For appropriate choice of double trace coupling constants this theory is conformally invariant. One can argue that such choice is possible in all orders of PT.

hep-th **0805.2261**

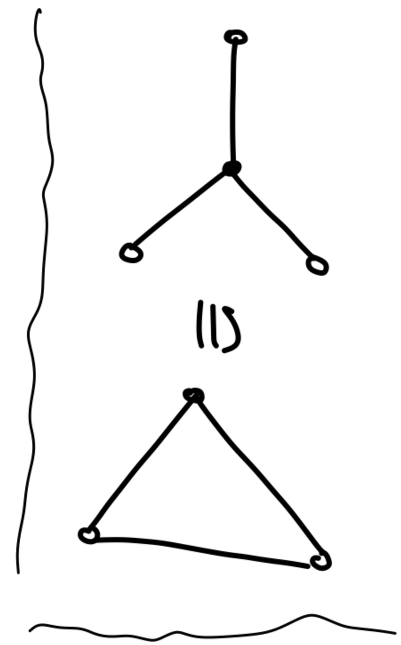
hep-th **0505099**



D=4 Example.

The sum of all such diagrams can be written as:

$$G = \frac{1}{(2\pi)^D} \int \frac{d^4x_{3'}d^4x_{4'}}{(x_{33'}^2 x_{44'}^2)^{\frac{D}{4}}} \langle x_1, x_2| \frac{1}{1 - \alpha^2\mathcal{V} - g^4\mathcal{H}_2} |x_3', x_4'\rangle$$



Where:

$$\mathcal{V}\,\Phi_{\Delta,S,n}(x_1,x_2) = \delta(\nu)\delta_{S,0}\Phi_{\Delta,S,n}(x_1,x_2),$$

$$\mathcal{H}\,\Phi_{\Delta,S,n}(x_1,x_2) = h_{\Delta,S}^{-1}\Phi_{\Delta,S,n}(x_1,x_2),$$

The eigenfunction of these operators is known:

$$\Phi_{\nu,J}(x_{10},x_{20}) = n_{\mu_1}\dots n_{\mu_J}\Phi_\nu^{\mu_1\dots\mu_J}(x_{10},x_{20})$$

$$= \frac{1}{x_{12}^2}\left(\frac{x_{12}^2}{x_{10}^2 x_{20}^2}\right)^{(\Delta-J)/2}\left(\frac{2(nx_{10})}{x_{10}^2} - \frac{2(nx_{20})}{x_{20}^2}\right)^J$$

hep-th **arXiv:1808.02688**

hep-th **arXiv:1801.09844**

$$\mathcal{V}\;\Phi(x_1,x_2) = \frac{2}{\pi^{\frac{D}{2}}}\int \frac{d^Dx_{1'}d^Dx_{2'}\,\delta^{(D)}(x_{1'2'})\,\Phi(x_{1'},x_{2'})}{|x_{11'}|^{D/2}|x_{22'}|^{D/2}}$$

D=4 Example.

This allows us to write:

$$G_4(x_1,\ldots,x_4) = \langle \mathrm{tr}\,(\phi_1(x_1)\phi_1(x_2))\,\mathrm{tr}\,(\phi_1^*(x_3)\phi_1^*(x_4))\rangle$$

$$G_4(x_1,\ldots,x_4) = \sum_{J=0}^{\infty}\int_{-\infty}^{+\infty} d\nu \frac{\mu(\nu,J)}{h(\nu,J)-g^4}\Pi_{\nu,J}(x_1,\ldots,x_4) + (x_1 \leftrightarrow x_2)$$

Where:

$$\mu(\nu,J) = 16\pi\frac{\nu^2(4\nu^2+(J+1)^2)(J+1)}{2^J}$$

$$h(\mu,J) = \left(\nu^2+\frac{J^2}{4}\right)\left(\nu^2+\frac{(J+2)^2}{4}\right)$$

$$\Pi_{\nu,J} = \int d^4x_0\, \Phi_{\nu}^{\mu_1\ldots\mu_J}(x_{10},x_{20})\Phi_{-\nu}^{\mu_1\ldots\mu_J}(x_{30},x_{40})$$

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arXiv:1808.02688

Amplitudes in “fishnet” models.

One can compute not only correlation functions but also scattering amplitudes in such theories:

$$A_4^{D=4} \sim \sum_{perm.} \left(N_c \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{D=4,sing.} + \mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4}) A_4^{D=4,doub.} \right)$$

const.

$$A_4^{D=4} = A_4^{D=4,u} + A_4^{D=4,t} = \int_{-\infty}^{+\infty} d\nu \sum_{J\geq 0}^{\infty} \frac{\mu(\nu,J)}{h(\nu,J) - g^4} \Omega_{\nu,J}(z) + (z \to -z)$$

Where:

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$$h(\mu,J) = \left(\nu^2 + \frac{J^2}{4}\right)\left(\nu^2 + \frac{(J+2)^2}{4}\right)$$

$$\Omega_{\nu,J}(z) = \frac{2^J}{\pi^2} \sinh^2(\pi\nu + i\pi J/2) \sum_{k=0}^{J} \frac{P_k(z)P_{J-k}(z)}{(J/2-k)^2 + \nu^2}$$

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arXiv:1812.06997

D=4 FishNet amplitudes.

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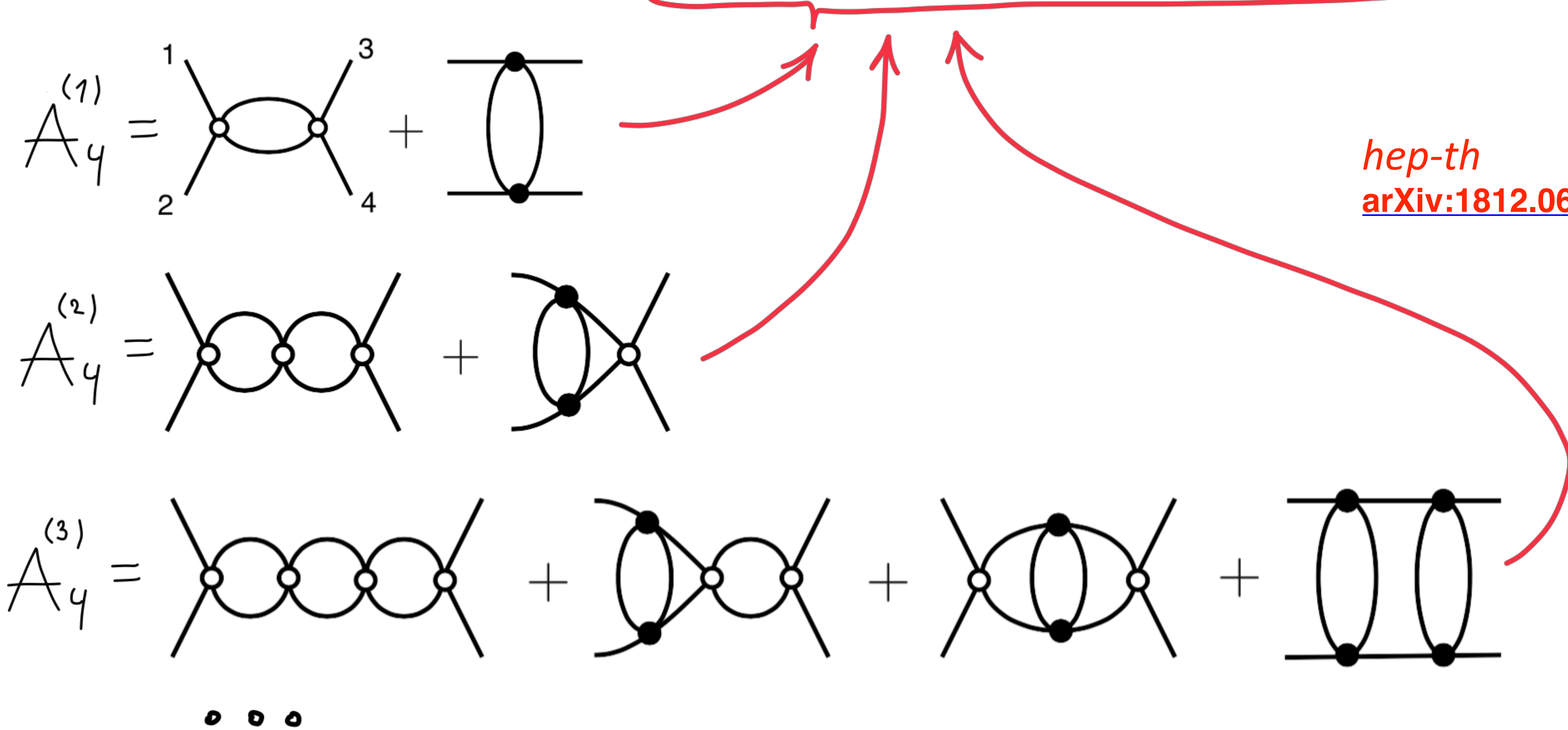

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arXiv:1812.06997

D=6 FishNet amplitudes.

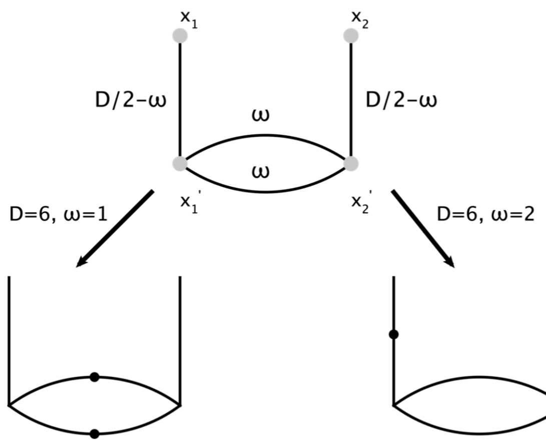
Let us consider another example of fishnet theory:

$D=6$
$\omega=1$
or
$D=6,\ \omega=2$

$$\mathcal{L}_{main} = N_c\ \mathrm{tr}\left(\ \phi^*(\partial^2)\phi + \chi^*(\partial^4)\chi + (4\pi)^3 g^2\ \phi^*\chi^*\phi\chi\ \right)$$

With:

$$\begin{aligned}\mathcal{L}^{(6,1)}_{c.t.}/(4\pi)^3 \;&=\; g\left(\mathrm{tr}\left(\phi\chi\right)\mathrm{tr}\left(\phi^*\chi^*\right) + \mathrm{tr}\left(\phi^*\chi\right)\mathrm{tr}\left(\phi\chi^*\right)\right)\\ &+\; \alpha_1(g)\mathrm{tr}\left(\chi\partial^2\chi\right)\mathrm{tr}\left(\chi^*\chi^*\right) + \alpha_2(g)\mathrm{tr}\left(\chi\partial^\mu\chi\right)\mathrm{tr}\left(\chi^*\partial_\mu\chi^*\right) + c.c.\end{aligned}$$



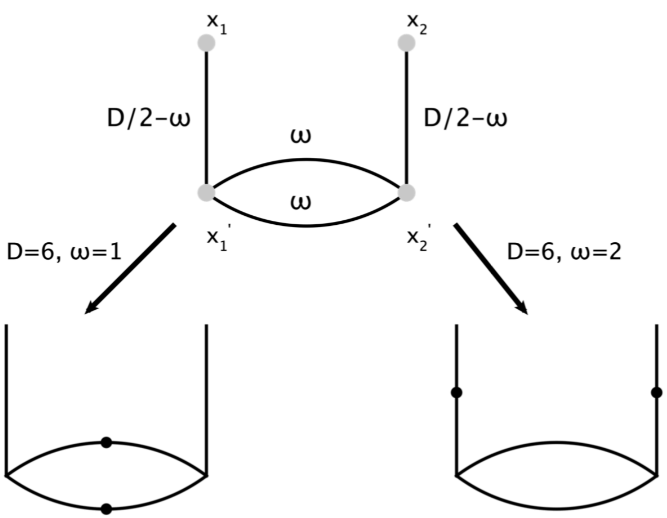
ϕ particles with $\frac{1}{p^2}$ prop.

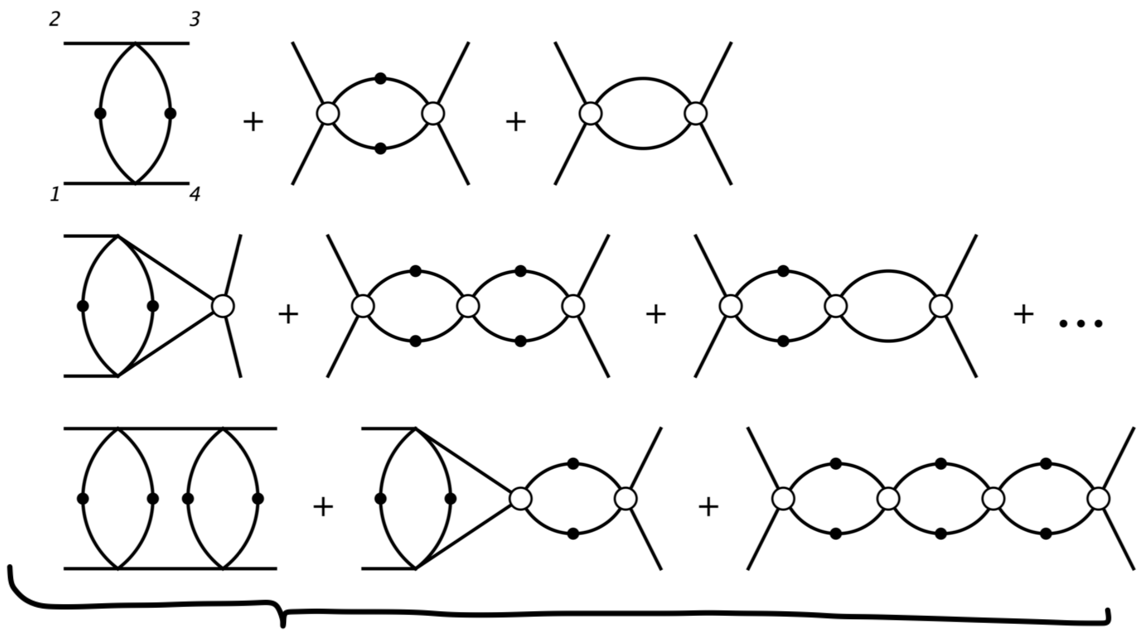
χ particles with $\frac{1}{p^4}$ prop.

$A_4\ \phi\chi \to \phi\chi = 0$

Let us consider another example of fishnet theory and 4-point double trace amplitudes there:

$$\mathcal{L}_{main} = N_c \ \mathrm{tr}\left(\ \phi^*(\partial^2)\phi + \chi^*(\partial^4)\chi + (4\pi)^3 g^2 \ \phi^*\chi^*\phi\chi \ \right)$$





$$A_{4,\ \phi\phi\to\phi\phi}^{(d.t.)}$$

D=6 FishNet amplitudes.

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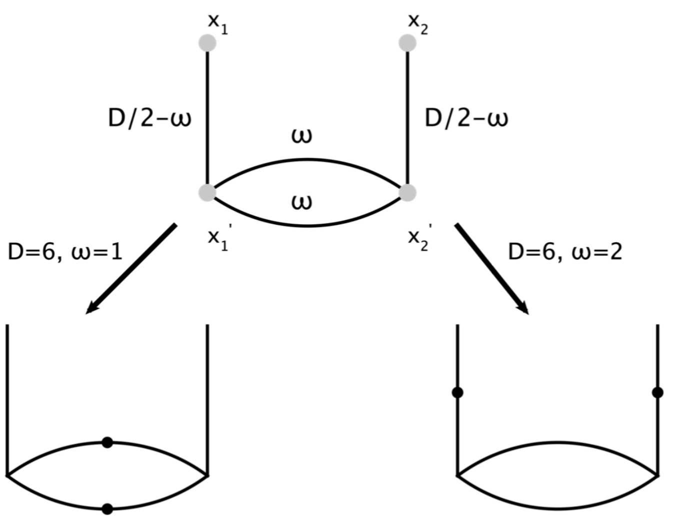
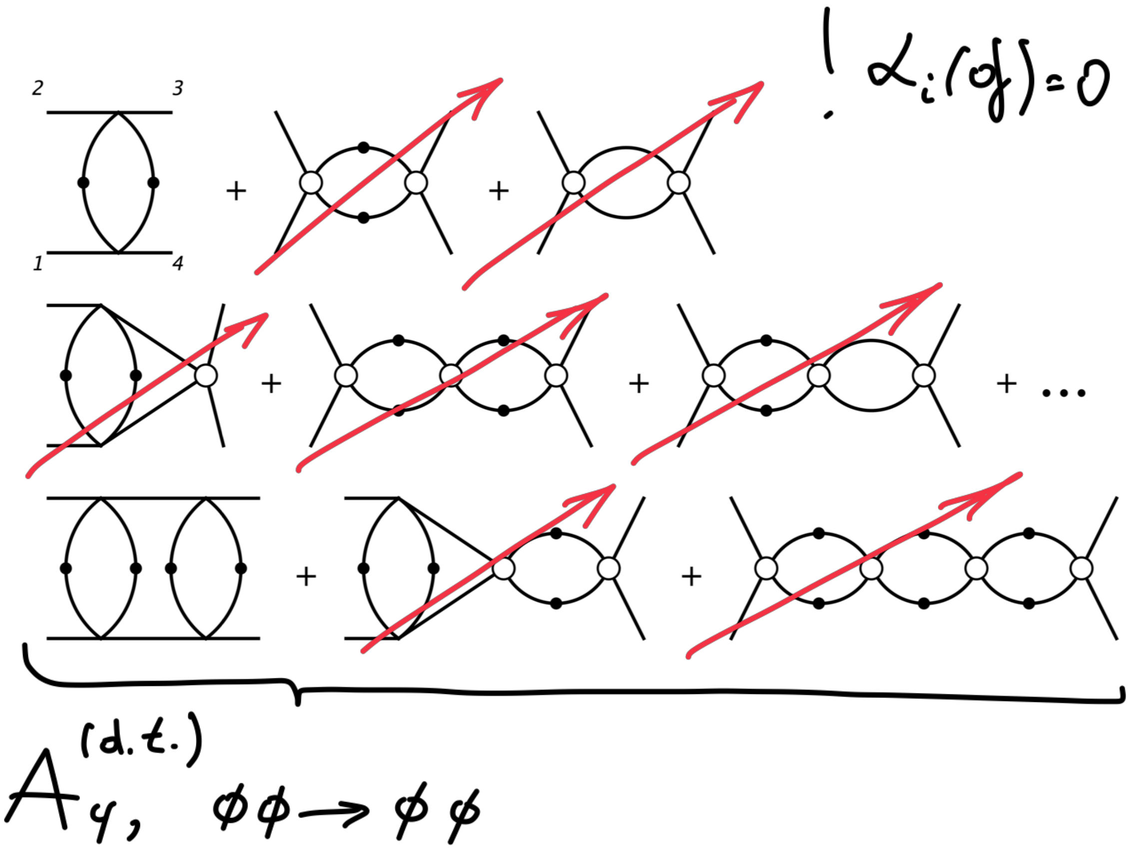
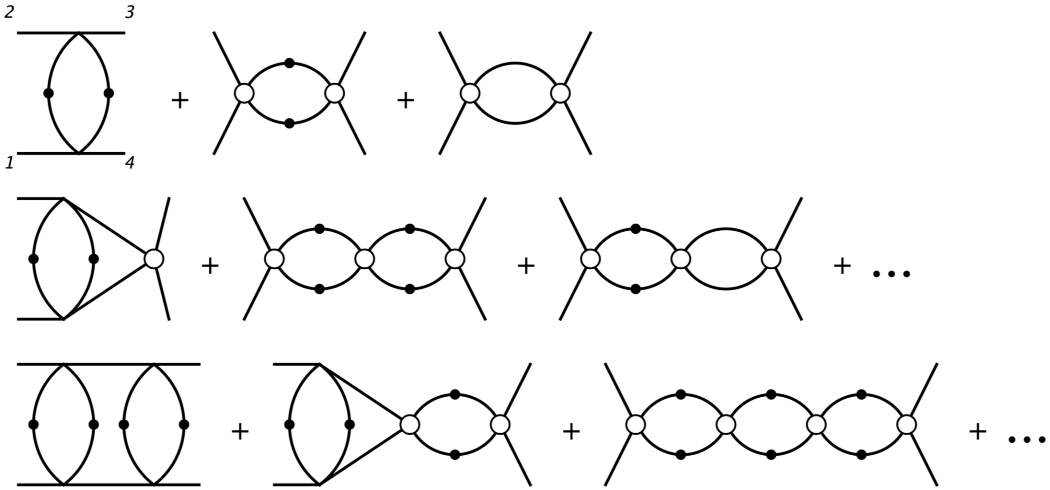




D=4 - D=6 correspondence



It is well known (? see for example V.Smironv book) that:

$$\frac{\partial}{\partial t} I_4(s,t;\ldots;\alpha_{v_1},\ldots,\alpha_{v_n};D) = I_4(s,t;\ldots;\alpha_{v_1}+1,\ldots,\alpha_{v_n}+1;D+2)$$

$$I_4(s,t;\ldots;\alpha_{v_1},\ldots,\alpha_{v_n};D) \sim \int_0^\infty \prod_{i=1}^{n+m} d\alpha_i \prod_{i=1}^{n+m} \alpha_i^{\nu_i-1} \frac{e^{i(\mathcal{V}_s s+\mathcal{V}_t t)/\mathcal{U}}}{\mathcal{U}^{d/2}}$$

D=4 - D=6 correspondance

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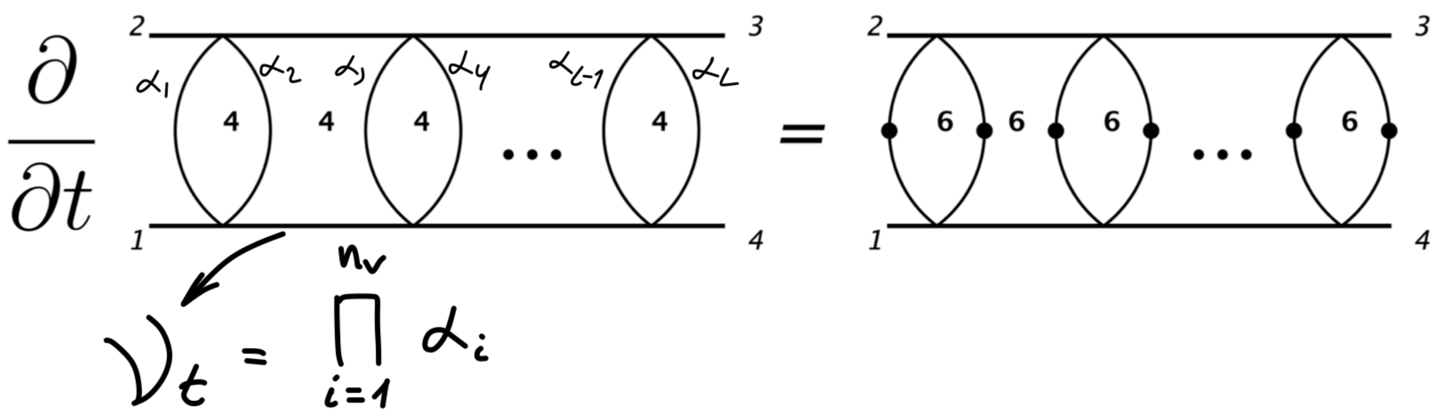
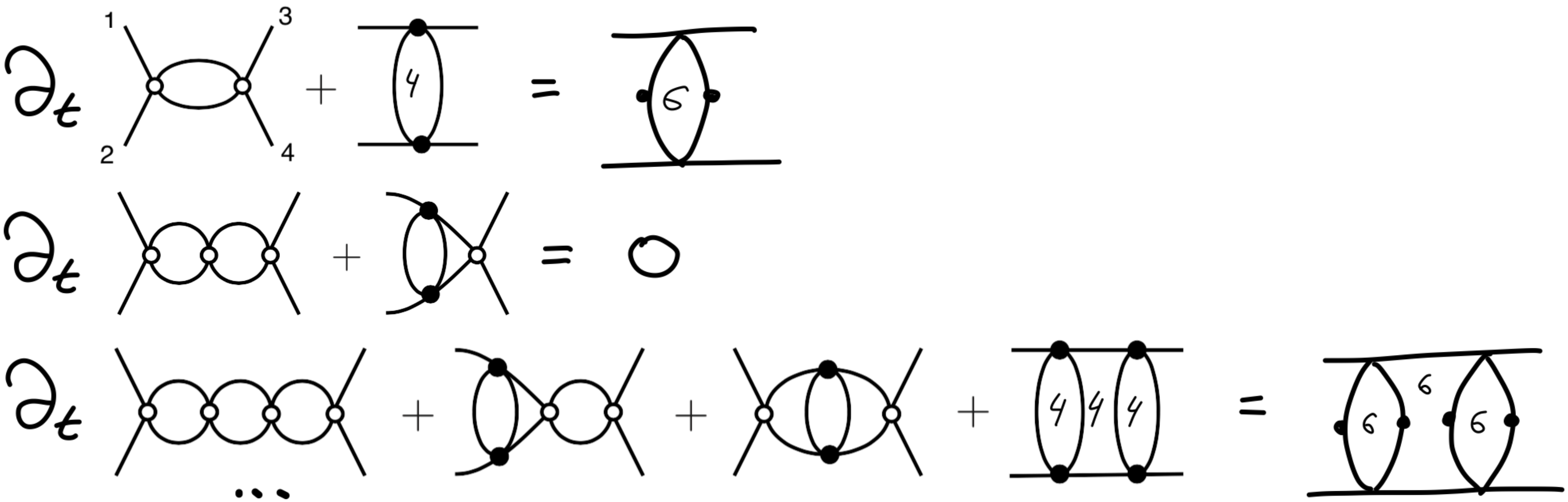
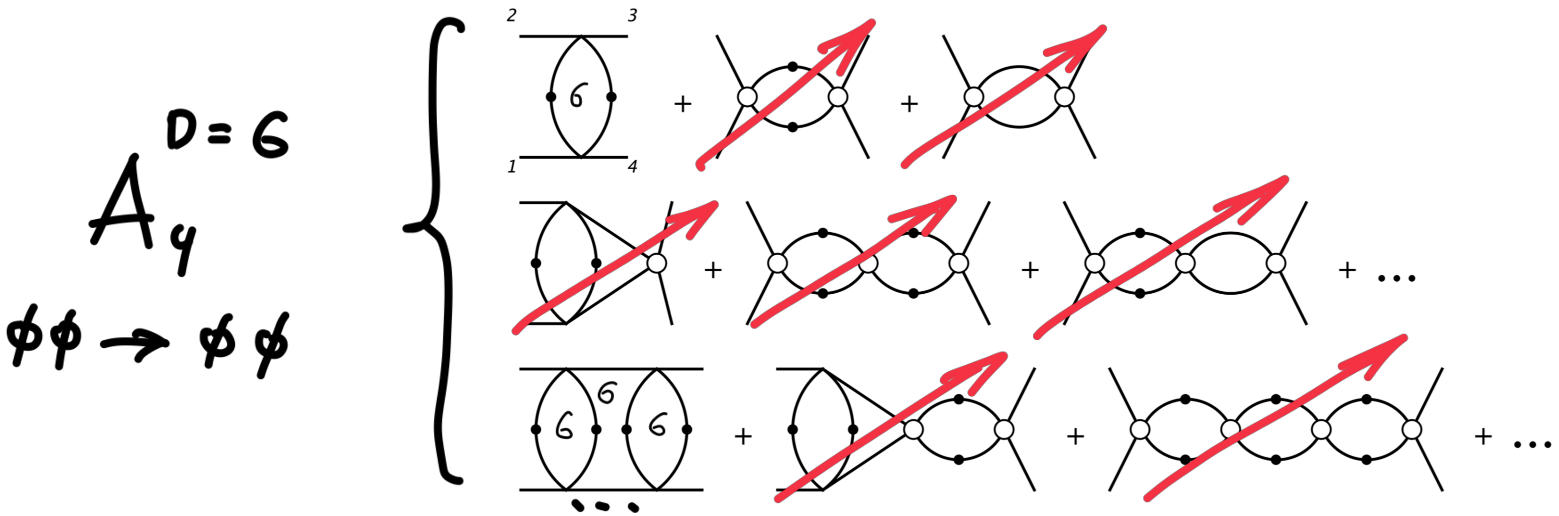
among other examples such relation imply that:



D=4 - D=6 correspondence



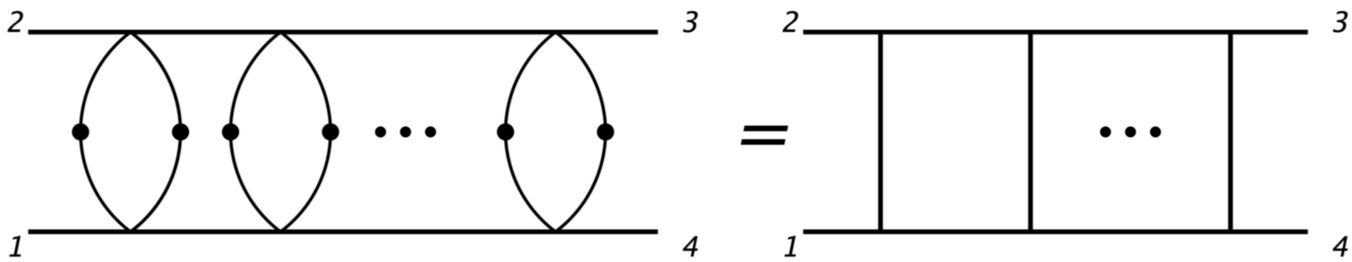


D=4 - D=6 correspondance

So we can conclude that:

$$A_4^{D=6}(s,z,g) = \frac{1}{s}\frac{\partial}{\partial z}A_4^{D=4}(z,g) = \frac{1}{s}\int_{-\infty}^{+\infty} d\nu \sum_{J\geq 0}^{\infty} \frac{\mu(\nu,J)}{h(\nu,J)-g^4}\frac{\partial\Omega_{\nu,J}(z)}{\partial z}$$

and because:



D=6 amplitude is **the generating function** of box ladder diagrams:

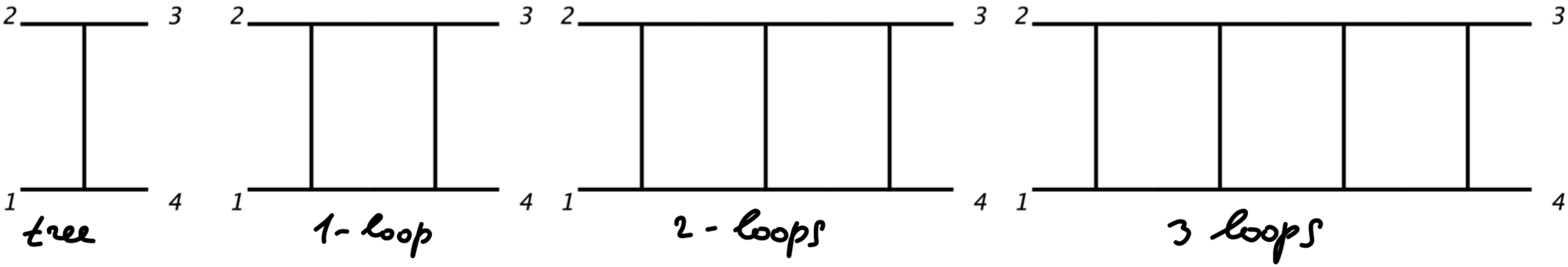
$$A_4^{D=6}(s,z,g) = A_4^{D=6,u}(s,z,g) + A_4^{D=6,t}(s,-z,g)$$

$$A_4^{D=6,u}(s,z,g) = \sum_{l=0}^{\infty}(g^4)^l\ B^{(l)}(s,u).$$

Amplitudes in D=6 fishnet theory. PT.

$$A_4^{D=6,u}(s,z,g) = \frac{1}{s}\left(g^4\ \mathcal{B}^{(0)}(x) + g^8\ \mathcal{B}^{(1)}(x) + \ldots\right),$$

math-ph
arXiv:1506.07243



$$B^{(l)}(s,t) = \frac{1}{s}\left(\frac{a^{(l)}(x)}{1+x} + \frac{b^{(l)}(x)}{x}\right) \qquad B^{(l)}(s,t) = \frac{\mathcal{B}^{(l)}(x)}{s}, \quad x = \frac{t}{s}$$

tree

$$a^{(0)}(x) = 0, \qquad b^{(0)}(x) = 1.$$

1-loop

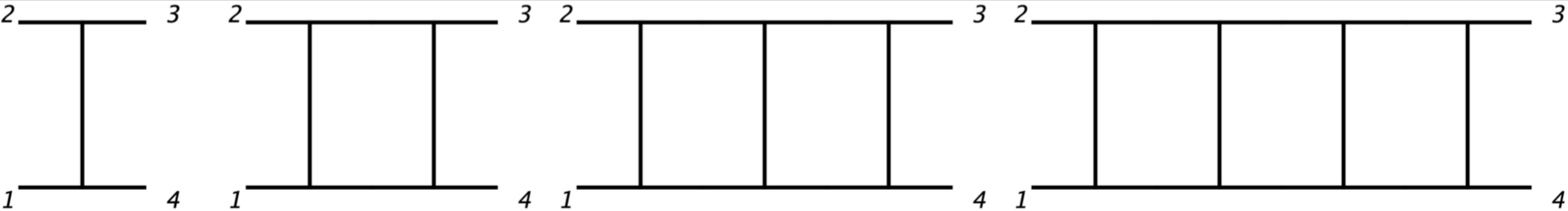
$$a^{(1)}(x) = H_{0,0} + \frac{\pi^2}{2}, \qquad b^{(1)}(x) = 0.$$

2-loops

$$a^{(2)}(x) = -\pi^2 H_{-1} + \frac{\pi^2}{3} H_0 - 2H_{-1,0,0} - 2\zeta_3,$$

$$b^{(2)}(x) = \pi^2 H_{-1,-1} - \frac{\pi^2}{3} H_{-1,0} + 2H_{-1,-1,0,0} + 2\zeta_3 H_{-1}$$

D=6 Box Ladders



$$B^{(l)}(s,t) = \frac{1}{s}\left(\frac{a^{(l)}(x)}{1+x} + \frac{b^{(l)}(x)}{x}\right)$$

$$\begin{cases} a^{(1)}(x) &= H_{0,0} + \frac{\pi^2}{2}, \\ b^{(1)}(x) &= 0. \end{cases}$$

$$\begin{cases} a^{(2)}(x) &= -\pi^2 H_{-1} + \frac{\pi^2}{3} H_0 - 2H_{-1,0,0} - 2\zeta_3, \\ b^{(2)}(x) &= \pi^2 H_{-1,-1} - \frac{\pi^2}{3} H_{-1,0} + 2H_{-1,-1,0,0} + 2\zeta_3 H_{-1}. \end{cases}$$

$$\begin{cases} a^{(3)}(x) &= \frac{\pi^2}{2} H_{-2,-2} - \frac{\pi^2(-15+2\pi^2)}{15} H_{-2} - \pi^2 H_{-1} + 2\pi^2 H_{-3,-1} - \frac{2\pi^2}{3} H_{-3,0} - \frac{2\pi^2}{3} H_{0,0} + \\ &+ \frac{6-\pi^2}{3} H_{-2,0,0} - 2H_{-1,0,0} + 4H_{-3,-1,0,0} + H_{-2,-2,0,0} + \frac{\pi^2 - 12\zeta_3}{3} H_0 + 4\zeta_3 H_{-3} - \\ &- 2\zeta_3 H_{-2,0} + \frac{-504\pi^4 + 31\pi^6 + 3780\zeta_3(-1+2\zeta_3)}{1890}, \\ b^{(3)}(x) &= -4\pi^2 H_{-2,-1} + \frac{4\pi^2}{3} H_{-2,0} - \pi^2 H_{-1,-2} + 3\pi^2 H_{-1,-1} + \frac{2\pi^2}{3} H_{-1,0,0} - 8H_{-2,-1,0,0} - \\ &- 2H_{-1,-2,0,0} + 6H_{-1,-1,0,0} - (\pi^2 - 4\zeta_3) H_{-1,0} - 8\zeta_3 H_{-2} + \frac{4\pi^2 + 90\zeta_3}{15} H_{-1}. \end{cases}$$

D=6 Box Ladders expansions:

Large z limit:

$$B^{(l)}(s,t) = \frac{\mathcal{B}^{(l)}(x)}{s}$$

$z = 1 + 2\frac{u}{s}$

$s + t + u = 0$

$u \gg s \Rightarrow z \gg 1$

$$\mathcal{B}^{(1)} = \underbrace{\frac{1}{2}L^2}_{LLA} + \ldots.$$

$L \equiv \log(z)$

$$\mathcal{B}^{(2)} = \underbrace{\frac{1}{12}L^4}_{LLA} + \underbrace{\frac{1}{3}L^3}_{NLA} + \underbrace{\frac{\pi^2}{3}L^2}_{NNLA} + \underbrace{\left(\frac{2\pi^2}{3} - 2\zeta_3\right)L}_{NNNLA} + \ldots$$

$$\mathcal{B}^{(3)} = \underbrace{\frac{1}{144}L^6}_{LLA} + \underbrace{\left(\frac{1}{12} + \frac{\pi^2}{3}\right)L^5}_{NLA} + \underbrace{\left(\frac{1}{3} - \frac{\pi^2}{16}\right)L^4}_{NNLA} + \underbrace{\left(\frac{1}{3} + \frac{\pi^2}{2} - \frac{\zeta_3}{3}\right)L^3}_{NNNLA} + \left(\frac{7\pi^2}{6} + \frac{71\pi^4}{720} - 2\zeta_3\right)L^2 + \left(\frac{2\pi^2}{3} + \frac{71\pi^4}{180} - 2\zeta_3 - \pi^2\zeta_3\right)L + \ldots$$

D=6 Box Ladders expansions:

<u>z=-1 limit:</u>

$z = 1 + 2\frac{u}{s}$

$\delta = +\frac{t}{s}$, $z = -1 - 2\delta$

$\delta \ll 1$

$$\mathrm{Re}\mathcal{B}^{(1)} = 0 - \frac{1}{2}\delta - \frac{1}{2}\delta^2 - \frac{11}{24}\delta^3 + O(\delta^4),$$
$$\mathrm{Im}\mathcal{B}^{(1)} = \pi + \frac{\pi}{2}\delta + \frac{\pi}{3}\delta^2 + \frac{\pi}{4}\delta^3 + O(\delta^4).$$

$$\mathrm{Re}\mathcal{B}^{(2)} = \left(\frac{-\pi^2}{3} + \frac{11\pi^4}{180}\right) + \left(-\frac{1}{2} - \frac{\pi^2}{2} + \frac{11}{180}\pi^4\right)\delta + O(\delta^2),$$
$$\mathrm{Im}\mathcal{B}^{(2)} = (2\pi - 2\pi\zeta_3) + \left(\frac{5\pi}{2} - 2\pi\zeta_3\right)\delta + O(\delta^2).$$

$$\mathrm{Re}\mathcal{B}^{(3)} = -\frac{\pi^2}{3} + \frac{11\pi^4}{60} - 6\zeta_3 - \pi^2\zeta_3 - \frac{15}{2}\zeta_5 + O(\delta),$$
$$\mathrm{Im}\mathcal{B}^{(3)} = 2\pi + \frac{\pi^2}{3} + \frac{7\pi^5}{360} + 6\pi\zeta_3 + O(\delta).$$

D=6 amplitude Regge Limit

Let us compare exact results with PT. Let us start with Regge limit:

$$A^{D=6,u}(s,z,g) = \frac{1}{s}\sum_{J=0}^{\infty}\sum_{i=1}^{2}\frac{2^{1-J}(J+1)((J+1)^2+4\nu_i^2)\nu_i}{(J+1)^2+4\nu_i+1}\frac{\partial\Omega_{J,\nu_i}(z)}{\partial z},$$

where

$$\nu_1 = -\frac{1}{2}\sqrt{-2-2J-J^2-2\sqrt{1+4g^4+2J+J^2}},$$
$$\nu_2 = -\frac{1}{2}\sqrt{-2-2J-J^2+2\sqrt{1+4g^4+2J+J^2}}.$$

J sum is divergent!

Vs

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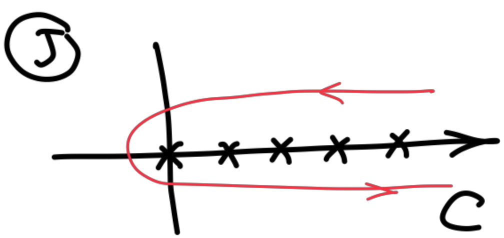
arXiv:1812.06997

$$A^{D=6,u}(s,z,g) = \frac{1}{s}\int_C \frac{dJ}{2i\sin(\pi J)}\int_{-\infty}^{+\infty} d\nu \frac{\mu(\nu,J)}{h(\nu,J)-g^4}\frac{\partial\Omega_{\nu,J}(z)}{\partial z}.$$



D=6 amplitude Regge Limit

$$A^{D=6,u}(s,z,g) = \frac{1}{s}\int_C \frac{dJ}{2i\sin(\pi J)}\int_{-\infty}^{+\infty} d\nu \frac{\mu(\nu,J)}{h(\nu,J)-g^4}\frac{\partial\Omega_{\nu,J}(z)}{\partial z}.$$

$z \gg 1$

combining this representation of the amplitude with

$$\Omega_{\nu,J}(z) = (-1)^J \frac{\sinh(2\pi\nu)}{2\pi\nu}\frac{\Gamma(J-2i\nu+1)\Gamma(J+2i\nu+1)}{\Gamma(J/2-i\nu+1)^2\Gamma(J/2+i\nu+1)^2} z^J + O(z^{J-2}).$$

one can obtain:

$$(*)\quad s\ A^{D=6,u}(s,z,g) \sim \int_{-g^2}^{g^2} d\nu \left(J_+F(\nu,J_+)z^{J_+-1} - J_-F(\nu,J_-)z^{J_--1}\right) + \dots,$$

All log enhanced terms

with

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$$J_\pm = -1+\sqrt{1-4\nu^2 \pm 4\sqrt{g^4-\nu^2}},$$

$$F(\nu,J) = \frac{\nu\sinh^2(2\pi\nu)\Gamma(J-2i\nu+2)\Gamma(J+2i\nu+2)}{\sin(\pi J)(J(J+2)+4\nu^2)\Gamma(J/2-i\nu+1)^2\Gamma(J/2+i\nu+1)^2}.$$

D=6 amplitude Regge Limit

$$s\ A^{D=6} \quad \sim \quad s\ A^{D=6,u} = \sum_l g^{4l} \mathcal{B}^{(l)}(x),$$

$$x\ \mathcal{B}^{(l)}(x) \quad = \quad a^{LLA}_{(l)} L^{2l} + a^{NLA}_{(l)} L^{2l-1} + a^{NNLA}_{(l)} L^{2l-2} + \ldots,$$

From (*) one can obtain that:

$$a^{LLA}_{(l)} = \frac{1}{l!(l+1)!},$$

$$a^{NLA}_{(l)} = \frac{2l(l-1)}{l!(l+1)!},$$

$$a^{NNLA}_{(l)} = \frac{2l(l-1)(l+2) + \pi^2(l+1)}{l!(l+2)!},$$

$$a^{NNNLA}_{(l)} = \frac{2l\,(l(l+1)\,(2l^2+2l+3\pi^2-13) + 6(l-4)\zeta_3 + 18)}{3l!(l+2)!}.$$

Note that these results are valid in arbitrary order of PT !

Let us compare them with direct PT computations:

$$a_{(l)}^{LLA} = \frac{1}{l!(l+1)!}$$

$$\mathcal{B}^{(1)} = \underbrace{\frac{1}{2}L^2}_{LLA} + \ldots.$$

$$a_{(l)}^{NNLA} = \frac{2l(l-1)(l+2) + \pi^2(l+1)}{l!(l+2)!}$$

$$\mathcal{B}^{(2)} = \underbrace{\frac{1}{12}L^4}_{LLA} + \underbrace{\frac{1}{3}L^3}_{NLA} + \underbrace{\frac{\pi^2}{3}L^2}_{NNLA} + \underbrace{\left(\frac{2\pi^2}{3} - 2\zeta_3\right)L}_{NNNLA} + \ldots$$

$$a_{(l)}^{NLA} = \frac{2l(l-1)}{l!(l+1)!}$$

$$\mathcal{B}^{(3)} = \underbrace{\frac{1}{144}L^6}_{LLA} + \underbrace{\left(\frac{1}{12} + \frac{\pi^2}{3}\right)L^5}_{NLA} + \underbrace{\left(\frac{1}{3} - \frac{\pi^2}{16}\right)L^4}_{NNLA} + \underbrace{\left(\frac{1}{3} + \frac{\pi^2}{2} - \frac{\zeta_3}{3}\right)L^3}_{NNNLA} + \left(\frac{7\pi^2}{6} + \frac{71\pi^4}{720} - 2\zeta_3\right)L^2 + \left(\frac{2\pi^2}{3} + \frac{71\pi^4}{180} - 2\zeta_3 - \pi^2\zeta_3\right)L + \ldots$$

$$a_{(l)}^{NNNLA} = \frac{2l\left(l(l+1)\left(2l^2+2l+3\pi^2-13\right)+6(l-4)\zeta_3+18\right)}{3l!(l+2)!}$$

D=6 amplitude z=-1 expansion

$$A^{D=6,u}(s,z,g)=\frac{1}{s}\sum_{J=0}^{\infty}\sum_{i=1}^{2}\frac{2^{1-J}(J+1)((J+1)^2+4\nu_i^2)\nu_i}{(J+1)^2+4\nu_i+1}\frac{\partial\Omega_{J,\nu_i}(z)}{\partial z},$$

where

$$\nu_1 = -\frac{1}{2}\sqrt{-2-2J-J^2-2\sqrt{1+4g^4+2J+J^2}},$$
$$\nu_2 = -\frac{1}{2}\sqrt{-2-2J-J^2+2\sqrt{1+4g^4+2J+J^2}}.$$

$$\Omega_{\nu,J}(z)=\frac{2^J}{\pi^2}\frac{\sinh^2(\pi\nu+i\pi J/2)}{(2\pi i)^2}\int\limits_{[1,z]}\int\limits_{[1,z]}dt_1dt_2\frac{(t_2^2-1)^J}{2^J(t_1-z)(t_2-z)^{J+1}}\Sigma\left(\mathcal{Z},\nu,J\right),$$

where

$$\Sigma\left(\mathcal{Z},\nu,J\right)=-\frac{i}{2\nu}\left(\Phi\left(\mathcal{Z},1,-\frac{J}{2}-i\nu\right)-\Phi\left(\mathcal{Z},1,-\frac{J}{2}+i\nu\right)+\right.$$
$$\left.+\mathcal{Z}^{1+J}\left(\Phi\left(\mathcal{Z},1,1+\frac{J}{2}+i\nu\right)-\Phi\left(\mathcal{Z},1,1+\frac{J}{2}-i\nu\right)\right)\right),$$

and

Lerch Zeta function

$$\mathcal{Z}\equiv\frac{(t_2^2-1)(t_2-z)}{(t_1^2-1)(t_1-z)}.$$

D=6 amplitude z=-1 expansion

$$\Phi(1,n+1,z) = \frac{(-1)^{s+1}}{\Gamma(n+1)}\Psi^{(n)}(z).$$

$$\Omega_{\nu,J}(z=-1) = i2^J \frac{\sinh^2(\pi\nu + i\pi J/2)}{2\pi^2\,\nu}\left(\Psi^{(0)}\left(-1-\frac{J}{2}-i\nu\right) - \Psi^{(0)}\left(\frac{J}{2}-i\nu\right) - \right.$$
$$\left. - \Psi^{(0)}\left(-1-\frac{J}{2}+i\nu\right) + \Psi^{(0)}\left(\frac{J}{2}+i\nu\right)\right),$$

$z = 1 + 2\frac{u}{s} = -1 - 2\frac{t}{s}$

$y = -2\delta, \quad \delta = \frac{t}{s}$

$$\Omega_{\nu,J}(\overbrace{-1+y}) = \sum_{n=0} y^n\ \Omega^{(n)}_{\nu,J}(-1)$$

$$\Omega^{(n)}_{\nu,J}(-1) = \frac{i2^J\sinh^2\left(\pi\nu + i\pi\frac{J}{2}\right)}{2\pi^2\,\nu}P_1^{(n)}(J,\nu) + P_2^{(n)}(J,\nu)\Omega_{\nu,J}(-1),$$

D=6 amplitude z=-1 expansion

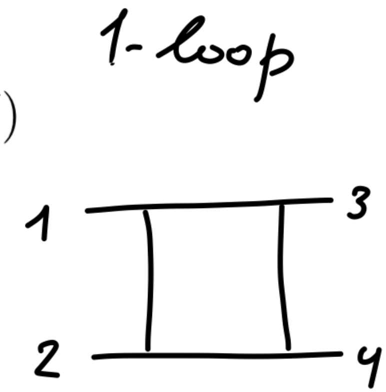
$$\begin{cases} \mathrm{Re}A_4^{D=6,u}(s,-1+y,g) = \sum_{k=0}^{\infty} \frac{c^{(k)}(g)}{s}\, y^k, \\ c^{(k)}(g) \sim \sum_{J=1}^{\infty}\sum_{i=1}^{2} \frac{2^{1-J}(J+1)((J+1)^2+4\nu_i^2)\nu_i}{(J+1)^2+4\nu_i+1}\Omega^{(k)}_{J,\nu_i}(-1) \end{cases}$$

$$\mathrm{Re}A_4^{D=6,u}(s,-1+y,g) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{c^{(k,l)}}{s}\, g^{4(l+1)l} y^k,$$

$$c^{(k,l)} = \sum_{J=1}^{\infty}(-1)^J \sum_{n} \mathrm{Rational\ function}^{(n)}(J)\times S_n(J)$$

D=6 amplitude z=-1 expansion

$$c^{(k,l)} = \sum_{J=1}^{\infty} (-1)^J \sum_n \text{Rational function}^{(n)}(J) \times S_n(J)$$



One loop example:

$$c[0,1] = 4*1*\left(\frac{i\,(2+k+2k^2)}{4k^2\,(2+k)} - \frac{i\,(1+4k+2k^2)\,\text{HarmonicNumber}[k]}{2k\,(1+k)\,(2+k)}\right)\Big/ I;$$

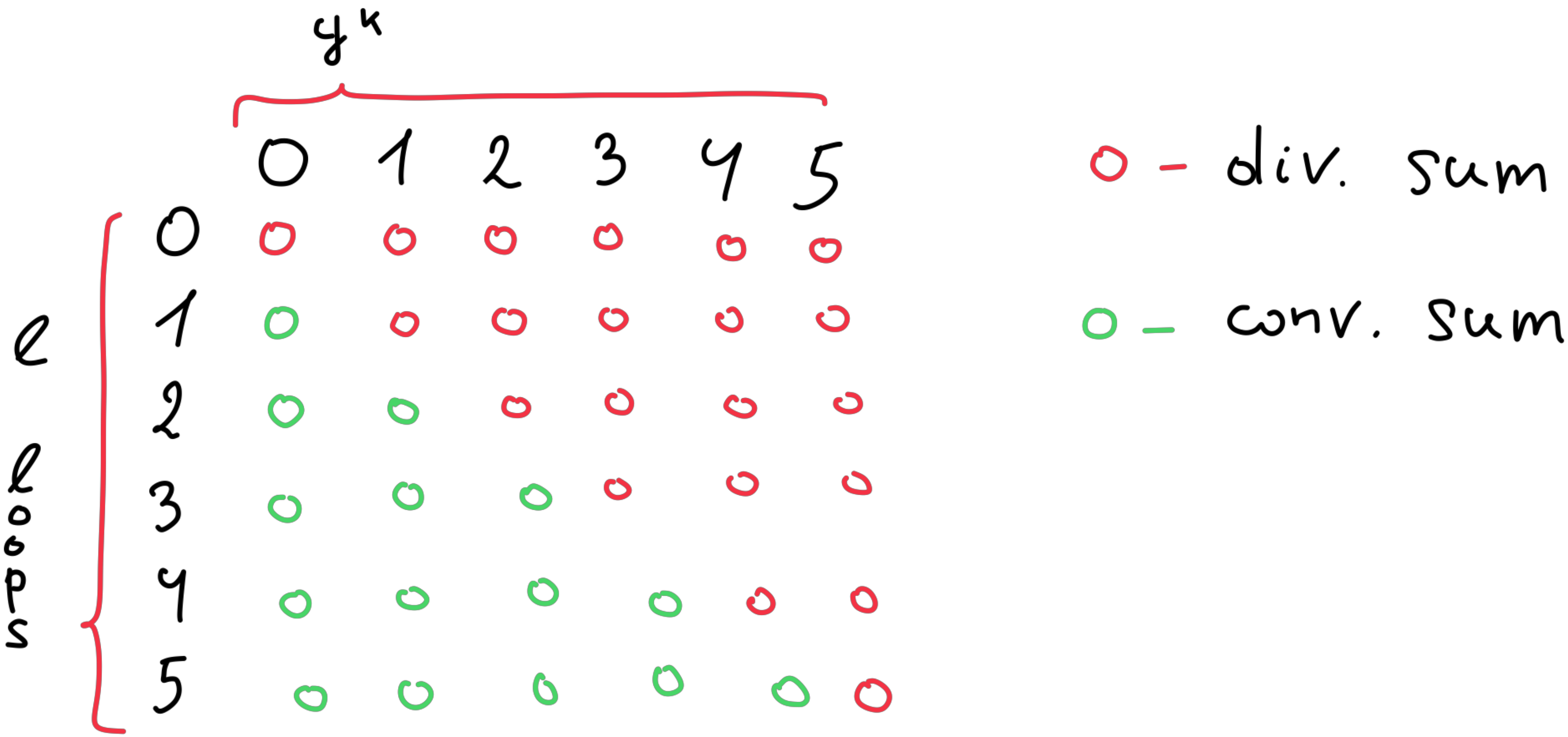
$$c[1,1] = 4*2*\left(\frac{i\,(-4+2k-k^2+9k^3+4k^4)}{32k^2\,(2+k)} - \frac{i\,(-1-2k+3k^2+4k^3+k^4)\,\text{HarmonicNumber}[k]}{8k\,(1+k)\,(2+k)}\right)\Big/ I;$$

$$c[2,1] = 4*3*\left(\frac{i\,(-1+k)\,(-72-8k^2+102k^3+81k^4+16k^5)}{1728k^2\,(2+k)} - \frac{i\,(-1+k)\,(-12-34k-k^2+27k^3+14k^4+2k^5)\,\text{HarmonicNumber}[k]}{288k\,(1+k)\,(2+k)}\right)\Big/ I;$$

$$c[3,1] = 4*4*\left(\frac{i\,(2-3k+k^2)\,(-432-72k+87k^2+571k^3+516k^4+169k^5+19k^6)}{55296k^2\,(2+k)} - \frac{i\,(2-3k+k^2)\,(-36-117k-59k^2+40k^3+40k^4+11k^5+k^6)\,\text{HarmonicNumber}[k]}{4608k\,(1+k)\,(2+k)}\right)\Big/ I;$$

D=6 amplitude z=-1 expansion. Divergent Vs Finite

$$c^{(k,l)} = \sum_{J=1}^{\infty} (-1)^J \sum_n \text{Rational function}^{(n)}(J) \times S_n(J)$$



D=6 amplitude z=-1 expansion. Conjecture. All divergent coefficients can be reconstructed.

Using Abel summation method (method of analytical continuation) one can evaluate all sums:

$$c^{(k),reg}(g,\tau) = \sum_{J=1}^{\infty} \tau^J \sum_{i=1}^{2} \frac{2^{1-J}(J+1)((J+1)^2+4\nu_i^2)\nu_i}{(J+1)^2+4\nu_i+1}\Omega^{(k)}_{J,\nu_i}(-1).$$

HPLs $(1+\tau)$

Then one can:

$$c^{(k),reg}(g,\tau) = \sum_{l=0}^{\infty} c^{(k,l),reg}(\tau)\; g^{4(l+1)}$$

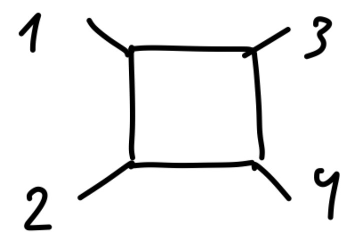
$$c^{(k,l)} = \lim_{\tau\to 1-0} c^{(k,l),reg}(\tau).$$

The results are surprising! We can reproduce all PT results:

D=6 amplitude z=-1 expansion

For example:

$$c^{(4,1),reg} = \sum_{J=1}^{\infty}(-\tau)^J\left(\frac{(-2+J)(-1+J)(3+J)(4+J)(-3-8J+4J^3+J^4)}{36(1+J)(2+J)J}S_1(J)+\right.$$
$$\left.+\ \frac{(-2+J)(-1+J)(-432-72J+87J^2+571J^3+516J^4+169J^5+19J^6)}{432J^2(2+J)}\right),$$
$$c^{(4,1)} = c^{(4,1),reg}(\tau=1) = \frac{11}{24}$$

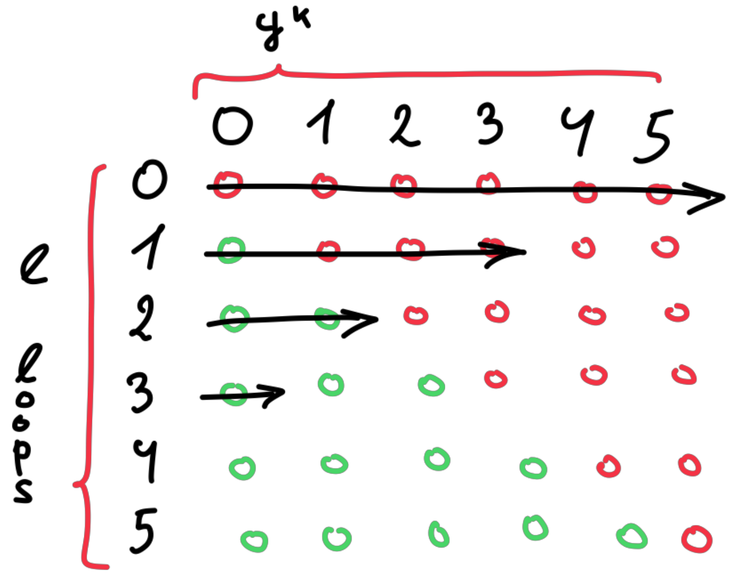


Compare with:

$$\mathrm{Re}\mathcal{B}^{(1)} = 0 - \frac{1}{2}\delta - \frac{1}{2}\delta^2 - \frac{11}{24}\delta^3 + O(\delta^4),$$
$$\mathrm{Im}\mathcal{B}^{(1)} = \pi + \frac{\pi}{2}\delta + \frac{\pi}{3}\delta^2 + \frac{\pi}{4}\delta^3 + O(\delta^4).$$

$$\frac{t}{s} = \delta = \frac{y}{2}$$

All results for the real parts are reproduced using Abel summation method:



$$\mathrm{Re}\mathcal{B}^{(1)} = 0 - \frac{1}{2}\delta - \frac{1}{2}\delta^2 - \frac{11}{24}\delta^3 + O(\delta^4),$$
$$\mathrm{Im}\mathcal{B}^{(1)} = \pi + \frac{\pi}{2}\delta + \frac{\pi}{3}\delta^2 + \frac{\pi}{4}\delta^3 + O(\delta^4).$$

$$\mathrm{Re}\mathcal{B}^{(2)} = \left(\frac{-\pi^2}{3} + \frac{11\pi^4}{180}\right) + \left(-\frac{1}{2} - \frac{\pi^2}{2} + \frac{11}{180}\pi^4\right)\delta + O(\delta^2),$$
$$\mathrm{Im}\mathcal{B}^{(2)} = (2\pi - 2\pi\zeta_3) + \left(\frac{5\pi}{2} - 2\pi\zeta_3\right)\delta + O(\delta^2).$$

$$\mathrm{Re}\mathcal{B}^{(3)} = -\frac{\pi^2}{3} + \frac{11\pi^4}{60} - 6\zeta_3 - \pi^2\zeta_3 - \frac{15}{2}\zeta_5 + O(\delta),$$
$$\mathrm{Im}\mathcal{B}^{(3)} = 2\pi + \frac{\pi^2}{3} + \frac{7\pi^5}{360} + 6\pi\zeta_3 + O(\delta).$$

z=1 point and E.Panzer HyperInt results

t-channel Boxes

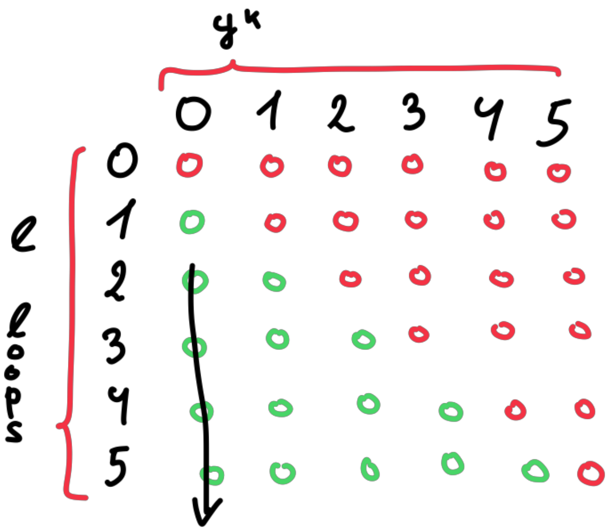
math-ph
arXiv:1506.07243

$$\sum_{l=2}^{\infty} \mathcal{B}^{(l)}(0)\ g^{4l} = \sum_{l=2}^{\infty} a^{(0,l)}\ g^{4l} \sim \frac{\partial}{\partial g^8} \int_{-\infty}^{+\infty} d\nu \sum_{J \geq 0}^{\infty} (-1)^J \frac{\mu(\nu, J)}{h(\nu, J) - g^4} \frac{\partial \Omega_{\nu,J}(-1)}{\partial z}.$$

Where first pair of coefficients can be evaluated:

$$\begin{cases} a^{(0,2)} = 2\zeta_2, \\ a^{(0,3)} = 4\zeta_3^2 + \frac{124}{35}\zeta_2^3 - 8\zeta_3 - 6\zeta_2. \\ \ldots \end{cases}$$

In agreement with E.Panzer results.



One more (finite) sum:

$$
\begin{aligned}
c^{(2,2)} = &\sum_{J=1}^{\infty}(-1)^J \left(\frac{(-J^4-4J^3+J^2+10J+4)}{J^2(J+1)^2(J+2)^2}S_2(J)+\right.\\
+ &\frac{2\left(2J^8+16J^7+43J^6+34J^5-31J^4-52J^3-2J^2+20J+8\right)}{J^3(J+1)^3(J+2)^3}S_1(J)-\\
- &\frac{2\left(12+\pi^2\right)J^{10}+\left(210+19\pi^2\right)J^9+\left(726+71\pi^2\right)J^8}{6J^4(J+1)^4(J+2)^3}-\\
- &\frac{\left(1221+125\pi^2\right)J^7+\left(861+79\pi^2\right)J^6-8\left(138+17\pi^2\right)J^4}{6J^4(J+1)^4(J+2)^3}-\\
- &\left.\frac{-2\left(171+32\pi^2\right)J^5-4\left(153+20\pi^2\right)J^3-8\left(2\pi^2-27\right)J^2+336J+96}{6J^4(J+1)^4(J+2)^3}\right)\\
= &\frac{1}{2}+\frac{\pi^2}{2}-\frac{11\pi^4}{180}.
\end{aligned}
$$

Possible bootstrap for D=6 boxes ?

If the summation conjecture is correct then on can bootstrap D=6 boxes (i.e. fishnet amplitudes):

unknown coefficients (real numbers)

$$\mathcal{B}^{(l)}(-1+y) = \frac{\sum_{\vec{a}} C^{(1)}_{\vec{a}} H_{a_1,\dots a_{w_2 l}}(-1+y)}{y} + \frac{\sum_{\vec{a}} C^{(2)}_{\vec{a}} H_{a_1,\dots a_{w_2 l}}(-1+y)}{-1+y}$$

Compare with:

$$\mathrm{Re} A_4^{D=6,u}(s,-1+y,g) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{c^{(k,l)}}{s} g^{4(l+1)l} y^k,$$

$C_{\vec{\alpha}}^{(1,2)}$

Additional expansions (like Regge limit) will help

Strong coupling limit in D=4 and D=6

$$A^{D=4,u}(z,g) = \frac{1}{2i}\int_C \frac{d(gJ)}{\sin(\pi gJ)}\int_{-\infty}^{+\infty} d\nu \frac{\mu(\nu,gJ)}{h(\nu,gJ)-g^4}\Omega_{\nu,gJ}(z),$$

J→

g J.

hep-th

arXiv: 1808.02688

$$g \mapsto \frac{1}{g}, \; g \to 0$$

$$\Phi(z,1,a) = \frac{1}{1-z}\frac{1}{a} + O(a^{-2}).$$

$$z \ll \exp\left[\frac{\sqrt{2}\pi}{\sqrt{g}}\right], \; z > 1$$

$$A_4^{D=4,u}(z,1/g) = \frac{1}{2\pi i(\sqrt{z^2-1})}\int\limits_C \frac{d(J/g)}{\sin(\pi J/g)} J\sqrt{4-J^2}\exp\left(\frac{\pi\sqrt{4-J^2}}{g} + (1+\frac{J}{g})\mathbf{L}\right) + \ldots$$

where $\ldots$ corresponds to all terms suppressed by powers of g or $\exp(-1/g)$ and

$$\mathbf{L} \equiv \log\left(z + \sqrt{z^2-1}\right).$$

Strong coupling limit in D=4 and D=6

1)
$$A_4^{D=4,u}(z,1/g) = g^{-1/2}\frac{4\pi\,\pi^{1/2}\,\mathbf{L}\exp\left(\frac{2}{g}\sqrt{\pi^2+\mathbf{L}^2}\right)}{i\sqrt{z^2-1}\,(\pi^2+\mathbf{L}^2)^{7/4}\sin\left(\frac{2\pi\mathbf{L}}{g\sqrt{\pi^2+\mathbf{L}^2}}\right)}+\ldots,$$

if $z \gg \exp\left[\frac{\sqrt{2}\pi}{\sqrt{g}}\right] \gg 1$?

2)
$$A_4^{D=4,u}(z,1/g) \sim \frac{z^{2/g-1}}{\log^{3/2}(z)}+\ldots,$$

$$\begin{aligned} J_R^{D=4}(g) &= \sqrt{1+4g^2}-1, \\ J_R^{D=6}(g) &= \sqrt{1+4g^2}-2, \end{aligned}$$

if $z=1$?

3)
$$A_4^{D=4,u}(z=\pm 1,1/g) = \sqrt{g}\frac{8\pi}{2\pi^3 i}\exp\left(2\pi/g\right)+\ldots.$$

$$A_4^{D=6} = \frac{\partial A_4^{D=4}}{s\,\partial z}$$

Conclusions

- **We have simple enough family CFT's in D=4 and D=6.**
- **Within such theories correction functions and amplitudes can be exactly evaluated.**
- **More loops&legs for D=6 and D=6 fishnets. n=6 amplitude = double tase contributions to NMHV6 in N=4 SYM.**
- **Dual gravitational description for such models.**
- **Origins of D=6 fishnet theory ? (N=(2,0) SYM , some N=(1,0) D=6 CFT ??)**

Lerch Zeta function

Let us briefly discuss definition and main properties of Lerch transcendent zeta functions. The Lerch transcendent, can be defined as the following series:

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

where: $a \neq 0, -1, -2, \ldots$, $|z| < 1$ or $|z| = 1$ and $\mathrm{Re}\, s > 1$. Other values of z can be also considered by means of analytical continuation. This can be done via contour integral of the following form:

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt \frac{t^{s-1}\exp(-at)}{1 - z\exp(-t)},$$

where $\mathrm{Re}\, s > 0$, $\mathrm{Re}\, a > 0$ and $z \in \mathbb{C}/[1,+\infty)$.

$$\Phi(z,s,a) = \frac{1}{1-z}\frac{1}{a^s} + \sum_{n=1}^{N-1} \frac{(-1)^n \mathrm{Li}_{-n}(z)}{n!}\frac{(s)_n}{a^{n+s}} + O(a^{-N+s}),$$

where $\mathrm{Arg}\, a < \pi$, $s \in \mathbb{C}$, $z \in \mathcal{C}_a$, $\mathcal{C}_a = \mathbb{C}/[1,+\infty)$ if $\mathrm{Re}\, a > 0$ or $\mathcal{C}_a = |z| < 1$ if $\mathrm{Re}\, a < 0$

D=6 amplitude z=-1 expansion

$$\begin{cases} \Omega_{\nu,J}(-1+y) = \sum_{n=0} y^n \, \Omega^{(n)}_{\nu,J}(-1) \\ \Omega^{(n)}_{\nu,J}(-1) = \frac{i2^J \sinh^2\left(\pi\nu + i\pi\frac{J}{2}\right)}{2\pi^2 \, \nu} P_1^{(n)}(J,\nu) + P_2^{(n)}(J,\nu)\Omega_{\nu,J}(-1), \end{cases}$$

$$\begin{cases} \mathrm{Re}A_4^{D=6,u}(s,-1+y,g) = \sum_{k=0}^{\infty} \frac{c^{(k)}(g)}{s} \, y^k, \\ c^{(k)}(g) \sim \sum_{J=1}^{\infty}\sum_{i=1}^{2} \frac{2^{1-J}(J+1)((J+1)^2+4\nu_i^2)\nu_i}{(J+1)^2+4\nu_i+1}\Omega^{(k)}_{J,\nu_i}(-1) \end{cases}$$

$$\begin{aligned} \Omega_{\nu,J}(z=-1) \;&=\; i2^J \, \frac{\sinh^2(\pi\nu + i\pi J/2)}{2\pi^2 \, \nu}\left(\Psi^{(0)}\left(-1-\frac{J}{2}-i\nu\right) - \Psi^{(0)}\left(\frac{J}{2}-i\nu\right) - \right. \\ &- \left. \Psi^{(0)}\left(-1-\frac{J}{2}+i\nu\right) + \Psi^{(0)}\left(\frac{J}{2}+i\nu\right)\right), \end{aligned}$$

D=6 amplitude z=-1 expansion

```
P1[2] = 2 * ν * (I * SS) ^ (-1) * (Ω[2] /. Hto0)
```

$$-\frac{3}{16}\,i\,(1+J)\,\nu\,\left(J\,(2+J)-4\,\left(1+\nu^2\right)\right)$$

```
P2[2] = ((Ω[2] - P1[2] * I * SS / (2 * ν)) // Simplify) / Ω[0]
```

$$\frac{1}{128}\left(12\,J^3+3\,J^4+J^2\,\left(4-8\,\nu^2\right)-16\,J\,\left(1+\nu^2\right)+48\,\left(\nu^2+\nu^4\right)\right)$$

```
(**)
```

```
P1[3] = 2 * ν * (I * SS) ^ (-1) * (Ω[3] /. Hto0)
```

$$-\frac{i\,(1+J)\,\nu\,\left(528+J\,(2+J)\,(-176+15\,J\,(2+J))+768\,\nu^2-56\,J\,(2+J)\,\nu^2+240\,\nu^4\right)}{1728}$$

```
P2[3] = ((Ω[3] - P1[3] * I * SS / (2 * ν)) // Simplify) / Ω[0]
```

$$\frac{30\,J^5+5\,J^6+J^4\,\left(8-12\,\nu^2\right)-24\,J^3\,\left(7+2\,\nu^2\right)+96\,J\,\left(2+3\,\nu^2+\nu^4\right)+16\,J^2\,\left(-7+6\,\nu^2+3\,\nu^4\right)-64\,\nu^2\,\left(11+16\,\nu^2+5\,\nu^4\right)}{4608}$$

```
(**)
```

```
P1[4] = 2 * ν * (I * SS) ^ (-1) * (Ω[4] /. Hto0)
```

$$-\frac{1}{442\,368}\,5\,i\,(1+J)\,\nu\,\Big(J\,(2+J)\,(4320+J\,(2+J)\,(-556+21\,J\,(2+J)))-76\,J\,(2+J)\,(-28+J\,(2+J))\,\nu^2+16\,(-600+19\,J\,(2+J))\,\nu^4-1344\,\nu^6-192\,\left(60+103\,\nu^2\right)\Big)$$

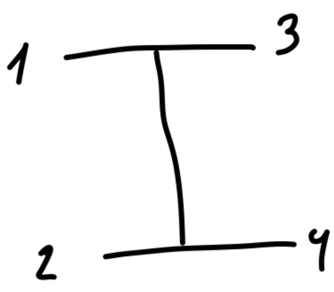
```
P2[4] = ((Ω[4] - P1[4] * I * SS / (2 * ν)) // Simplify) / Ω[0]
```

$$\frac{1}{1\,179\,648}\Big(280\,J^7+35\,J^8-40\,J^6\,\left(1+2\,\nu^2\right)-160\,J^5\,\left(26+3\,\nu^2\right)+128\,J^3\,\left(131+60\,\nu^2+9\,\nu^4\right)+16\,J^4\,\left(-253+70\,\nu^2+18\,\nu^4\right)-512\,J\,\left(36+65\,\nu^2+34\,\nu^4+5\,\nu^6\right)+1280\,\nu^2\,\left(60+103\,\nu^2+50\,\nu^4+7\,\nu^6\right)-128\,J^2\,\left(-114+65\,\nu^2+59\,\nu^4+10\,\nu^6\right)\Big)$$

D=6 amplitude z=-1 expansion

$$c^{(k,l)} = \sum_{J=1}^{\infty} (-1)^J \sum_{n} \text{Rational function}^{(n)}(J) \times S_n(J),$$

tree level:



c[0, 0] = 4 * 1 * $\frac{1}{4}$ (1 + 2 k) // FullSimplify

1 + 2 k

c[1, 0] = 4 * 2 * $\frac{1}{32}$ (1 + 2 k) $\left(-2 + k + k^2\right)$ // FullSimplify

$\frac{1}{4}$ (1 + 2 k) $\left(-2 + k + k^2\right)$

c[2, 0] = 4 * 3 * $\frac{1}{576}$ (-2 + k) (-1 + k) (2 + k) (3 + k) (1 + 2 k) // FullSimplify

$\frac{1}{48}$ (-2 + k) (-1 + k) (2 + k) (3 + k) (1 + 2 k)

c[3, 0] = 4 * 4 * $\frac{(-3 + k)\ (-2 + k)\ (-1 + k)\ (2 + k)\ (3 + k)\ (4 + k)\ (1 + 2 k)}{18432}$ // FullSimplify

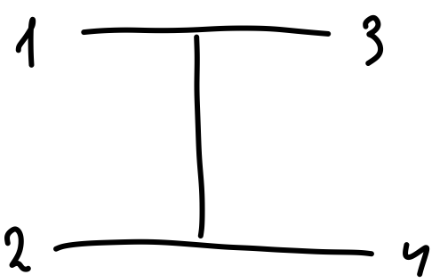
$\frac{(-3 + k)\ (-2 + k)\ (-1 + k)\ (2 + k)\ (3 + k)\ (4 + k)\ (1 + 2 k)}{1152}$

D=6 amplitude z=-1 expansion

$$c^{(k,0),reg} = \sum_{J=1}^{\infty}(-\tau)^J \left(\frac{(J-k)\ldots(J-1)(J+k+1)\ldots(J+2)(1+2J)}{(k+1)!k!}\right)$$

$$c^{(k,0)} = c^{(k,0),reg}(\tau=1) = \frac{(-1)^k}{2^k},$$

$$\sum_{k=0}^{\infty} c^{(k,0)} y^k$$



$$= \frac{1}{u} = \frac{-1}{s+t} = \frac{-1}{s}\frac{1}{1+\frac{t}{s}} = \frac{1}{s}\frac{2}{2+y}$$