Multi-loop modularity

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I begin by citing notable work by Vladimir **Belokurov**, Konstantin **Chetyrkin**, Dmitry **Kazakov**, Nikolay **Krasnikov**, Anatoly **Radyushkin**, Vladimir **Smirnov** and Alexey **Vladimirov**, up to **5 loops**.

This leads to discussion of the **number content** of single-scale Feynman integrals, in **counterterms** and in $(g-2)$ for the **electron**.

In counterterms, multiple zeta values (**MZVs**) appear at **6 loops**. Multiple polylogarithms of **sixth roots** of unity appear at **7 loops**. **Modular forms** create obstructions at **8 loops**.

For the massive on-shell **sunrise diagrams** relevant to $(g-2)$, modular forms of **weight 4** and **level 6** appear at **4 loops**.

Modular forms of **levels 14 and 34** determine off-shell Feynman integrals that give the areas of **black holes** obtained by compactification of a 10-dimensional supergravity theory on **Calabi-Yau** three-folds with complex structure.

1 Some notable papers by *younger* physicists

I cite a dozen papers by the **seven seventy-year-olds**, which were written when they were **less than half their present age** and have influenced my own work.

[1] Ultraviolet asymptotics in presence of *non-abelian* gauge fields,
V.V. **Belokurov**, A.A. **Vladimirov**, D.I. **Kazakov**, A.A. Slavnov, D.V. Shirkov, Teor.Mat.Fiz. 19 (1974) 149-162. [$6\zeta_3$ appears at **3 loops**.]

[2] *Methods* of calculating many-loop diagrams and renormalization-group analysis of the ϕ^4 theory, A.A. **Vladimirov**, Teor.Mat.Fiz. 36 (1978) 732–737.
[$20\zeta_5$ appears at **4 loops**.]

[3] Finite energy *sum rules* for the cross-section of e^+e^- annihilation into hadrons in QCD, K.G. **Chetyrkin**, N.V. **Krasnikov**, A.N. Tavkhelidze, Phys.Lett.B 76 (1978) 83-84.

[4] Calculation of *critical exponents* by quantum field theory methods,
D.I. **Kazakov**, O.V. Tarasov, A.A. **Vladimirov**, Sov.Phys.JETP 50 (1979) 521.

[5] *New approach* to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x-space technique, K.G. **Chetyrkin**, A.L. Kataev, F.V. Tkachov, Nucl.Phys.B 174 (1980) 345-377.

[6] Integration by parts: the *algorithm* to calculate β-functions in 4 loops,
K.G. **Chetyrkin**, F.V.Tkachov, Nucl.Phys.B 192 (1981) 159-204.

[7] *Analytic* renormalization of massless theories, S.A. Anikin, V.A. **Smirnov**, Teor.Mat.Fiz. 51, (1982) 317-321.

[8] *Method* for computing higher gluonic power corrections to QCD charmonium sum rules, S.N. Nikolaev, A.V. **Radyushkin**, Phys.Lett.B 110 (1982) 476-480.

[9] Infrared R-operation and ultraviolet *counterterms* in the MS-scheme,
K.G. **Chetyrkin**, F.V.Tkachov, Phys.Lett.B 114 (1982) 340-344.

[10] *Five-loop* renormalization group calculations in the ϕ^4 theory,
K.G. **Chetyrkin**, S.G. Gorishny, S.A. Larin, F.V. Tkachov, Phys.Lett.B 132 (1983) 351-353. [Corrected by Verena Schulte-Frohlinde in 1995.]

[11] R^*-operation *corrected*, K.G. **Chetyrkin**, V.A. **Smirnov**, Phys.Lett.B 144 (1984) 419-424.

[12] Multiloop calculations: *method* of uniqueness and functional equations,
D.I. **Kazakov**, Teor.Mat.Fiz. 62 (1985) 84-89.
[The **5-loop zig-zag** diagram evaluates to $\frac{\mathbf{441}}{\mathbf{8}}\zeta_7$.]

2 What comes after $\frac{441}{8}\zeta_7$?

I studied 6-loop counterterms in 1985, determining

$$P_{6,1} = 168\zeta_9, \quad P_{6,2} = \frac{1063}{9}\zeta_9 + 8\zeta_3^3, \quad 16P_{6,3} + P_{6,4} = 1440\zeta_3\zeta_5$$

with Riemann zeta values $\zeta_a = \sum_{n>0} n^{-a}$. I had a strong intuition that $P_{6,3}$ and $P_{6,4}$ would involve ζ_8 and the **multiple zeta value** (MZV)

$$\zeta_{5,3} = \sum_{m>n>0} \frac{1}{m^5 n^3} = 0.037707672984847544011304782\ldots$$

but did not have enough digits for the periods to test this.

Later, Dirk **Kreimer** and I obtained $P_{6,3} = 256N_{3,5} + 72\zeta_3\zeta_5$ and $P_{6,4} = -4096N_{3,5} + 288\zeta_3\zeta_5$, with

$$N_{3,5} = \frac{27}{80}\zeta_{5,3} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8.$$

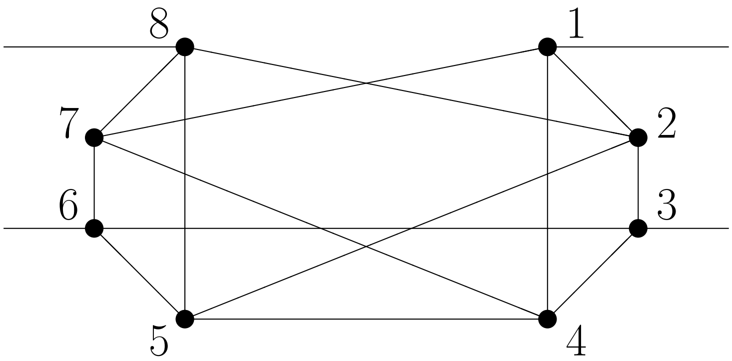
We found $\zeta_{3,5,3}$, with weight 11 and depth 3, in some 7-loop periods. Our conjecture for value of the L-loop **zig-zag** diagram was later proven by Francis **Brown** and Oliver **Schnetz**.

3 7-loop counterterms

I found empirical reductions to MZVs for a pair of 7-loop periods

$$
\begin{aligned}
P_{7,8} &= \frac{22383}{20}\zeta_{11} + \frac{4572}{5}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - 700\zeta_3^2\zeta_5 \\
&\quad + 1792\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right) \\
P_{7,9} &= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 \\
&\quad + 896\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right)
\end{aligned}
$$

that had been expected to involve alternating sums. These results were later proven, one by Erik **Panzer** and the other by Oliver **Schnetz**. They obtained complicated combinations of **alternating** sums which then gave my formulas by use of proven results in the MZV data-mine.



The period from this **7-loop** diagram is called $P_{7,11}$ in the census of Schnetz. All other periods up to 7 loops reduce to MZVs; only $P_{7,11}$ requires nested sums with **sixth roots of unity**. Panzer evaluated $\sqrt{3}P_{7,11}$ in terms of 4589 such sums, each of which he evaluated to 5000 digits. Then he found an empirical reduction to a 72-dimensional basis. The rational coefficient of π^{11} in his result was

$$C_{11} = -\frac{964259961464176555529722140887}{2733669078108291387021448260000}$$

whose **denominator** contains the large primes **50909** and **121577**.

I built a data-mine to enable substantial simplification of this result, with no prime greater than 3 appearing in the denominator of any rational coefficient.

4 Modular obstructions in 8-loop counterterms

There are **41 periods**, $P_{8,k}$, at **8 loops**, in the ϕ^4 census of Schnetz. Of these, 33 are known analytically. Moreover, 4 of the 8 undetermined cases are likely to be reducible to multiple polylogs of roots of unity. The **remaining 4 cases** are $P_{8,37}$, $P_{8,38}$ $P_{8,39}$ and $P_{8,41}$, for each of which there is an **obstruction**.

In the case of $P_{8,37}$, studied by Brown and Schnetz, the obstruction occurs after integrating over 12 of the 16 Schwinger parameters in the 15-dimensional projective integral. Then one hits a **denominator**

$$D(a,b,c,d) = b(a+c)(ac+bd) - ad(b+c)(c+d)$$

in the integrand for the remaining 3 integrations, over (a,b,c), with $d=1$. Counting zeros of D in finite fields, one may identity this obstruction with a **modular form** of **weight 3** and **level 7**, namely the **eta quotient**

$$f_{3,7}(\tau) = (\eta_1\eta_7)^3 = q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + 9q^9 - 6q^{11} + 21q^{14} + O(q^{16}),$$
$$\eta_n = q^{n/24}\prod_{k>0}(1-q^{nk}), \quad q = \exp(2\pi \mathrm{i}\tau).$$

Similarly, $P_{8,38}$ is obstructed by $(\eta_1\eta_5)^4$, $P_{8,39}$ by $(\eta_1\eta_8)^2\eta_2\eta_4$ and $P_{8,41}$ by $(\eta_1\eta_3)^6$.

5 Periods and quasi-periods from Stephano Laporta

The **magnetic moment** of the electron, in Bohr magnetons, has electrodynamic contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L = 4$ loops by

$$\begin{aligned}
a_0 &= \quad 1 \qquad [\textbf{Dirac}, 1928] \\
a_1 &= \quad 0.5 \quad [\textbf{Schwinger}, 1947] \\
a_2 &= -0.328478965579193784582172816964892392411119298679 62\ldots \\
a_3 &= \quad 1.181241456587200006274753982212877853368789390932 13\ldots \\
a_4 &= -1.912245764926445574152647167439830054060873390658 72\ldots
\end{aligned}$$

In 1957, corrections by **Petermann** and **Sommerfield** resulted in

$$a_2 = \frac{197}{144} + \frac{\zeta_2}{2} + \frac{3\zeta_3 - 2\pi^2 \log 2}{4}\,.$$

In 1996, **Laporta** and **Remiddi** [hep-ph/9602417] gave us

$$\begin{aligned}
a_3 \ &= \ \frac{28259}{5184} + \frac{17101\zeta_2}{135} + \frac{139\zeta_3 - 596\pi^2 \log 2}{18} \\
&\quad - \frac{39\zeta_4 + 400U_{3,1}}{24} - \frac{215\zeta_5 - 166\zeta_3\zeta_2}{24}\,.
\end{aligned}$$

The 3-loop contribution contains a weight-4 depth-2 **polylogarithm**

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{\zeta_4}{2} + \frac{(\pi^2 - \log^2 2)\log^2 2}{12} - 2\sum_{n>0} \frac{1}{2^n n^4}$$

encountered in my study of **alternating** sums [arXiv:hep-th/9611004].

Equally fascinating is the **Bessel** moment B, at weight 4, in the breath-taking evaluation by **Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals.
U comes from 6 light-by-light integrals, still under investigation.

The weight-4 **non-polylogarithm** at 4 loops has $N = 6$ Bessel functions:

$$B = -\int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t) K_0^5(t) \mathrm{d}t.$$

5.1 Bessel moments and modular forms

Gauss noted on 30 May 1799 that the **lemniscate** constant

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \frac{\Gamma^2(\frac14)}{4\sqrt{2\pi}} = \frac{\pi/2}{\mathbf{agm}(1,\sqrt{2})} = \frac{\left(\sum_{n\in\mathbf{Z}}\exp(-\pi n^2)\right)^2\pi}{2\sqrt{2}}$$

is given by the reciprocal of an **arithmetic-geometric mean**. This is an example of the Chowla-Selberg formula (1949) at the **first** singular value. In 1939, **Watson** encountered the **sixth** singular value, in work on integrals from condensed matter physics. Here, $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{6}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{24})$ with $k=1,5,7,11$, as observed by **Glasser and Zucker** in 1977.

In 2007, I evaluated a **3-loop sunrise** integral at the **fifteenth** singular value, where $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{15})$ with $k=1,2,4,8$.

With $N=a+b$ **Bessel** functions and $c\geq 0$, I define **moments**

$$M(a,b,c)=\int_0^\infty I_0^a(t)K_0^b(t)t^c\mathrm{d}t$$

that converge for $b>a>0$. Then the 5-Bessel matrix is

$$\left[\begin{array}{cc} M(1,4,1) & M(1,4,3)\\ M(2,3,1) & M(2,3,3)\end{array}\right] = \left[\begin{array}{cc} \pi^2 C & \pi^2\left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right)\\ \frac{\sqrt{15}\pi}{2}C & \frac{\sqrt{15}\pi}{2}\left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right)\end{array}\right].$$

The **determinant** $2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C = \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{\mathbf{15}}\right).$$

The **L-series** for $N=5$ Bessel functions comes from a **modular form** of weight **3** and level **15** [arXiv:1604.03057]:

$$\begin{aligned}
f_{3,15}(\tau) &= (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n \\
L_5(s) &= \sum_{n>0}\frac{A_5(n)}{n^s} \quad \text{for } s>2 \\
L_5(1) &= \sum_{n>0}\frac{A_5(n)}{n}\left(2+\frac{\sqrt{\mathbf{15}}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{\mathbf{15}}}\right) \\
&= 5C = \frac{5}{\pi^2}\int_0^\infty I_0(t)K_0^4(t)t\mathrm{d}t\,.
\end{aligned}$$

5.2 Periods and quasi-periods for the Laporta problem

Laporta's work engages the first row of the **6-Bessel determinant**

$$\det\left[\begin{array}{cc} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{array}\right] = \frac{5\zeta_4}{32}$$

associated to a **modular form** $f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2$ with weight **4** and level **6**. At top left we have $M(1,5,1)$, from the on-shell **4-loop sunrise** diagram, in two spacetime dimensions. Below it, $M(2,4,1)$ comes from **cutting** an internal line. The **second column** comes from **differentiating** the first, with respect to the external momentum, to produce **quasi-periods** associated with a **weakly holomorphic** modular form

$$\widehat{f}_{4,6}(\tau) = \mu f_{4,6}(\tau), \quad \mu = \frac{1}{32}\left(w+\frac{3}{w}\right)^4 - \frac{9}{16}\left(w+\frac{3}{w}\right)^2, \quad w = \frac{3\eta_3^4\eta_2^2}{\eta_1^4\eta_6^2}.$$

With $s = 1, 2$, I computed compute 10,000 digits of the **Eichler integrals**

$$\frac{\Omega_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} f_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy, \quad \frac{\widehat{\Omega}_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} \widehat{f}_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy.$$

The **linear relations** to the **periods** $\Omega_{1,2}$ and the **quasi-periods** $\widehat{\Omega}_{1,2}$ are

$$\frac{2}{\pi^2}\left[\begin{array}{cc} 4M(1,5,1) & \frac{36}{5}\left(M(1,5,1)+M(1,5,3)\right) \\ \frac{5}{3}M(2,4,1) & 3\left(M(2,4,1)+M(2,4,3)\right) \end{array}\right] = \left[\begin{array}{cc} -\Omega_2 & \widehat{\Omega}_2 \\ -\Omega_1 & \widehat{\Omega}_1 \end{array}\right].$$

The **intersection number** is the **determinant** of this matrix, namely $\frac{1}{12}$. David **Roberts** and I converted this into a **quadratic relation** between **hypergeometeric series**:

$$\begin{array}{rcl} F_a &=& {}_4F_3(\ 1/2,\ 2/3,\ 2/3,\ 5/6;\ 7/6,\ 7/6,\ 4/3;\ 1) \\ F_b &=& {}_4F_3(\ -1/2,\ 1/6,\ 1/3,\ 4/3;\ -1/6,\ 5/6,\ 5/3;\ 1) \\ F_c &=& {}_4F_3(\ 1/6,\ 1/3,\ 1/3,\ 1/2;\ 2/3,\ 5/6,\ 5/6;\ 1) \\ F_d &=& {}_4F_3(\ -7/6,\ -1/2,\ -1/3,\ 2/3;\ -5/6,\ 1/6,\ 1/3;\ 1) \end{array}$$

namely

$$7F_aF_b + 10F_cF_d = 40.$$

At **6 loops** we encounter periods of the **weight-6** modular form

$$f_{6,6}(\tau) = \frac{\eta_2^9\eta_3^9}{\eta_1^3\eta_6^3} + \frac{\eta_1^9\eta_6^9}{\eta_2^3\eta_3^3}.$$

6 Areas of black holes from modular Feynman integrals

In 2019, Philip **Candelas**, Xenia **de la Ossa**, Mohamed **Elmi** and Duco **van Straten** announced a remarkable discovery of *A One Parameter Family of Calabi-Yau Manifolds with Attractor Points of Rank Two* [arXiv:1912.06146].

They compactified a 10-dimensional **supergravity** theory on a **Calabi-Yau** three-fold with complex structure, to obtain 4-dimensional **black holes**, with event horizons whose **areas** are determined by their electric and magnetic charges and by ratios of **periods** of **modular forms** of weight 4 and **levels 14 or 34**.

Hearing of this on a visit to Oxford, in November 2019, I observed that their Calabi-Yau periods come from solutions to a **homogeneous** differential equation associated with **4 loop sunrise integrals**, namely

$$\begin{aligned} M_{m,n}(z) &= \int_0^\infty I_0(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+1}\mathrm{d}x \\ N_{m,n}(z) &= z\int_0^\infty I_1(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+2}\mathrm{d}x \end{aligned}$$

with $m \in \{0,1,2\}$, integers $n \geq 0$ and real $z^2 < (5-2m)^2$. The **uncut** diagram gives $M_{0,0}(z)$ and satisfies an **inhomogeneous** differential equation.

The **external mass** is z. At $z=1$ we obtain Laporta's **on-shell** periods, for the **magnetic moment of the electron** at 4 loops, coming from the modular form $f_{4,6}(\tau)=(\eta_1\eta_2\eta_3\eta_6)^2$ with **level 6**. With mass $z=\sqrt{\mathbf{17}}-4$, I obtained **level 34** periods. At the **space-like** point $z=\sqrt{-\mathbf{7}}$, I obtained **level 14** periods.

Candelas et al. were unable to identify all of the 16 Calabi-Yau periods. At each of the levels 14 and 34, I found that are given by 8 Feynman integrals, satisfying two **quadratic relations**. These 8 integrals determine a pair of **periods** and a pair of **quasi-periods** at each of the weights 2 and 4.

Here I indicate the situation at **level 14**, where I identified

$$f_{4,14}(\tau)=\frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2}-4(\eta_1\eta_2\eta_7\eta_{14})^2+\frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}$$

as the relevant modular form of **weight 4**. Its **periods** are **critical values** of the L-function $L(f_{4,14},s)=((2\pi)^s/\Gamma(s))\int_0^\infty f_{4,14}(\mathrm{i}y)y^{s-1}\mathrm{d}y$, with

$$\begin{aligned}
L(f_{4,14},3) &= M_{1,0}(\sqrt{-7})=\int_0^\infty J_0(\sqrt{7}x)I_0(x)K_0^4(x)x\mathrm{d}x=\frac{\pi^2}{7}L(f_{4,14},1)\\
\tfrac{1}{2}L(f_{4,14},2) &= M_{2,0}(\sqrt{-7})=\int_0^\infty J_0(\sqrt{7}x)I_0^2(x)K_0^3(x)x\mathrm{d}x.
\end{aligned}$$

There is also a modular form of **weight 2** to consider, $f_{2,14}(\tau)=\eta_1\eta_2\eta_7\eta_{14}$. This provides a modular parametrization of a **quartic elliptic curve**, namely

$$\begin{aligned} d^2 &= (1+h)(1+8h)(1+5h+8h^2), \\ h &= \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7}\right)^3 = q+3q^2+6q^3+13q^4+O(q^5), \\ d &= \frac{q}{f_{2,14}}\frac{\mathrm{d}h}{\mathrm{d}q} = 1+7q+27q^2+92q^3+259q^4+O(q^5). \end{aligned}$$

Kevin **Acres** and I determined a **weakly holomorphic** form that gives the **weight-4 quasi-periods**. The space of cuspforms is **4-dimensional** and we had to solve a $\mathbf{4}\times\mathbf{10}$ matrix problem, for weakly holomorphic forms obtained by multiplying $f_{2,14}^2$ by polynomials that are **linear** in d and **quartic** in h.

At **level 34**, Feynman integrals determine the **area** of the event horizon of a **black hole** with charges specified by (k,ℓ) studied by Candelas at al., namely

$$A=\mathbf{34}\pi\left(\frac{k^2}{v}+\ell^2 v\right),\quad v=4\pi\frac{M_{2,0}(z)}{M_{1,0}(z)}=4\pi\frac{\int_0^\infty I_0(zx)I_0^2(x)K_0^3(x)x\mathrm{d}x}{\int_0^\infty I_0(zx)I_0(x)K_0^4(x)x\mathrm{d}x}$$

where $z=\sqrt{\mathbf{17}}-4$ is the **external mass** in the **4-loop sunrise diagram**, with two internal propagators cut in the numerator of v and one in the denominator. The **level 14** case corresponds to the **space-like** momentum $z=\sqrt{-\mathbf{7}}$.

Summary

1. Zeta values suffice for counterterms up to 5 loops. MZVs appear at 6 loops.
2. Multiple polylogs of sixth roots of unity appear at 7 loops.
3. Modular forms obstruct reductions to polylogs at 8 loops.
4. Sunrise diagrams involve modular forms at 3, 4 and 6 loops.
5. Off-shell massive 4-loop diagrams involve modular forms whose periods determine areas of black holes in compactified supergravity.

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