Basso-Dixon diagrams

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Plan

- Basso-Dixon diagram in $d = 4$ and great determinant representation
  B.Basso, L.Dixon, *Gluing Ladder Feynman Diagrams into Fishnets,* Phys.Rev.Lett. 119 (2017) no.7, 071601.
- Graph building operator and commuting Q-operators
- Construction of eigenfunctions of the Q-operator for any d
  S.D., V.Kazakov, E.Olivucci, *Basso-Dixon Correlators in Two-Dimensional Fishnet CFT,* JHEP 1904 (2019) 032
  S.D., E.Olivucci, *Exactly solvable magnet of conformal spins in four dimensions,* Phys.Rev.Lett. 125 (2020) 3, 031603
  S.D., G.Ferrando, E.Olivucci, *Mirror channel eigenvectors of the d-dimensional fishnets,* e-Print: 2108.12620 [hep-th]

Based on the previous works S.D., G. Korchemsky, A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD,* Nucl.Phys. B617 (2001) 375-440
S.D., A.Manashov, *Iterative construction of eigenfunctions of the monodromy matrix for SL(2,C) magnet,* J.Phys. A47 (2014) 305204

The whole story was initiated by the works L.N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models,* JETP Lett. 59 (1994) 596

L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a completely integrable model,* Phys. Lett. B 342 (1995) 311

L.N. Lipatov, *Integrability of scattering amplitudes in N = 4 SUSY,* J. Phys. A 42 (2009) 304020

Plan

- Heavily based on uniqueness method

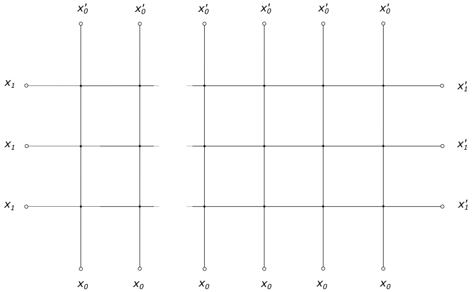
  D.Kazakov, *The method of uniqueness, a new powerful technique for multiloop calculations,* Phys.Lett.133B,406(1983)

  and in fact the representation of separated variables is relative of the Gegenbauer polynomial technique

  K. Chetyrkin, A. Kataev and F. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique,* Nucl. Phys. B 174, 345 (1980)

Basso-Dixon diagram

B.Basso, L.Dixon, *Gluing Ladder Feynman Diagrams into Fishnets,* Phys.Rev.Lett. 119 (2017) no.7, 071601.



N horizontal lines ($N = 3$ here), L vertical lines. Solid lines are the scalar propagators $1/(x-y)^2$ where x and y are the two endpoints of each segment. The boundary points are identified into four points (x_0, x_1, x_0', x_1').

$$I_{NL}(x_0, x_1, x_0', x_1') = \frac{(x_0 - x_0')^{-2(N+L)}}{(x_1 - x_0)^{2N}(x_1' - x_0)^{2N}} \left[\frac{(z\bar{z})^{\frac{1}{2}}}{z - \bar{z}}\right]^N I_{NL}(z, \bar{z})$$

$$u = \frac{x_{1'0}^2 x_{10'}^2}{x_{10}^2 x_{1'0'}^2} = z\bar{z} \quad ; \quad v = \frac{x_{11'}^2 x_{00'}^2}{x_{10}^2 x_{1'0'}^2} = (1-z)(1-\bar{z})\,,$$

Basso-Dixon great formula

Step one $\nu_k \in \mathbb{R}\,, \ell_k \in \mathbb{Z}\,, k = 1, \ldots, N \;\;, N < L + 1$

$$I_{NL}(z,\bar{z}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}} \int \frac{d^N \boldsymbol{\nu}}{(2\pi)^N N!}\, \mu(\boldsymbol{\nu},\boldsymbol{\ell}) \prod_{k=1}^{N} (\ell_k + 1)\; \frac{z^{i\nu_k + \frac{\ell_k+1}{2}}\, \bar{z}^{i\nu_k - \frac{\ell_k+1}{2}}}{\left(\frac{(\ell_k+1)^2}{4} + \nu_k^2\right)^{N+L}}$$

$$\mu(\boldsymbol{\nu},\boldsymbol{\ell}) \;=\; \prod_{1\le i<k\le N} \left((\nu_i - \nu_k)^2 + \frac{(\ell_i - \ell_k)^2}{4}\right) \left((\nu_i - \nu_k)^2 + \frac{(\ell_i + \ell_k + 2)^2}{4}\right)$$

Basso-Dixon great formula

N. I. Usyukina and A. I. Davydychev, *Exact results for three and four point ladder diagrams with an arbitrary number of rungs,* Phys. Lett. B305, 136 (1993)

$$I_{1,p}(z,\bar{z}) = L_p(z,\bar{z}) = \sum_{j=0}^{p} \frac{(-1)^j (2p-j)!}{p! j! (p-j)!} \ln^j (z\bar{z}) \left(Li_{2p-j}(z) - Li_{2p-j}(\bar{z}) \right)$$

Step two

$$I_{NL}(z,\bar{z}) = \frac{\det M}{\prod_{k=1}^{N} (L-N+2k-2)!(L-N+2k-1)!}$$

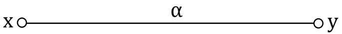
where M is a $N \times N$ Hankel matrix with ij element

$$M_{ij} = (L-N+i+j-2)!(L-N+i+j-1)! \times L_{L-N+i+j-1}(z,\bar{z})$$

B.Basso, L.Dixon, D.Kosower, A.Krajenbrink, De-liang Zhong, *Fishnet four-point integrals: integrable representations and thermodynamic limits,*
JHEP 07 (2021) 168

Simple diagrammatic rules

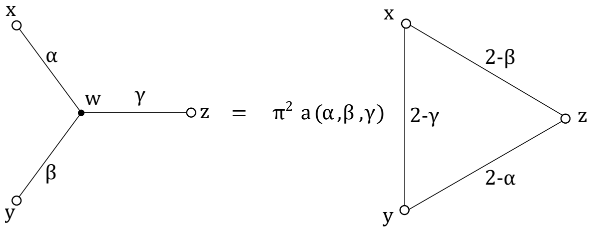
- The function $(x-y)^{-2\alpha} = ((x-y)^{\mu}(x-y)_{\mu})^{-\alpha}$ is represented by the line with index α connecting points x and y



- Star-triangle relation $\alpha+\beta+\gamma=4$

$$\int d^4 w \frac{1}{(x-w)^{2\alpha}(y-w)^{2\beta}(z-w)^{2\gamma}} = \frac{\pi^2 a(\alpha,\beta,\gamma)}{(y-z)^{2(2-\alpha)}(z-x)^{2(2-\beta)}(x-y)^{2(2-\gamma)}}$$

where $a(\alpha) = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}$ and $a(\alpha,\beta,\cdots,\gamma) = a(\alpha)\, a(\beta)\cdots a(\gamma)$.



Operator V_N

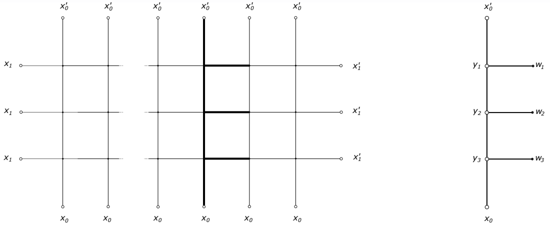
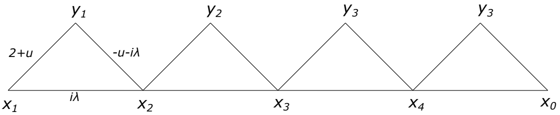
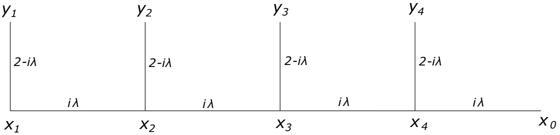


Diagram $\leftrightarrow V_N^{L+1}$





- The diagram for the Q-operator $Q_N(u)$: one-parametric family of commuting operators $[Q_N(u), Q_N(v)] = 0$.
- Reduction of the diagram for $u \to -i\lambda$ and $Q_N(u) \to V_N$.

Diagonalization of the operator $V_N \leftrightarrow$ Transition to the representation of separated variables

E. K. Sklyanin, *Separation of variables - new trends,* Prog.Theor.Phys.Suppl. 118 (1995) 35-60

- Spectrum of operator V_N is continuous
- Eigenfunctions $\nu_k \in \mathbb{R}\,, \ell_k\,, m_k \in \mathbb{Z}_+$

$$\Psi_{\nu_1,\ell_1,m_1\,,\ldots\,,\nu_N,\ell_N,m_N}(x_1\,,\,\ldots\,,x_N) = |\nu_1,\ell_1,m_1\,,\,\ldots\,,\nu_N,\ell_N,m_N\rangle = |\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

$$V_N\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle = \lambda(\nu_1,\ell_1)\cdots\lambda(\nu_N,\ell_N)\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

One point example

$$[V_1\,\Phi]\,(x) \leftrightarrow (x-x_0)^{-2i\lambda}\int d^4w\,(w-x)^{-2(2-i\lambda)}\,\Phi(w)\,.$$

$$\Psi_{\nu,\ell,m}(x) \leftrightarrow \frac{(x-x_0)^{\mu_1\cdots\mu_\ell}}{(x-x_0)^{2(1+\frac{i\lambda}{2}+i\nu+\frac{\ell}{2})}}$$

$$\int d^dx\,\frac{x^{\mu_1\ldots\mu_\ell}}{x^{2(d/4+\ell/2+i\nu)}}\,\frac{x^{\nu_1\ldots\nu_{\ell'}}}{x^{2(d/4+\ell'/2-i\lambda)}} = c_\ell\delta_{\ell\ell'}\delta\left(\nu-\lambda\right)P^{\mu_1\ldots\mu_\ell}_{\nu_1\ldots\nu_\ell}$$

$$\sum_{\ell\geq 0}\frac{1}{c_\ell}\int_{\mathbb{R}}d\nu\,\frac{x^{\mu_1\ldots\mu_\ell}}{x^{2(d/4+\ell/2+i\nu)}}\,\frac{y^{\mu_1\ldots\mu_\ell}}{y^{2(d/4+\ell/2-i\nu)}} = \delta^{(d)}\left(x-y\right)\ \ ;\ \ c_\ell = \frac{\pi^{d/2+1}\ell!}{2^{\ell-1}\Gamma\left(d/2+\ell\right)},$$

Eigenfunctions

$$V_N\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle = \lambda(\nu_1,\ell_1)\cdots\lambda(\nu_N,\ell_N)\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

- Iterative construction

$$\Psi_{\nu_1,\ell_1,m_1\,,\dots,\nu_N,\ell_N,m_N}(x_1\,,\,\dots\,,x_N) = \Lambda^{(N)}_{\nu_N,\ell_N,m_N}\cdots\Lambda^{(2)}_{\nu_2,\ell_2,m_2}\Psi_{\nu_1,\ell_1,m_1}$$

- Main commutation relation

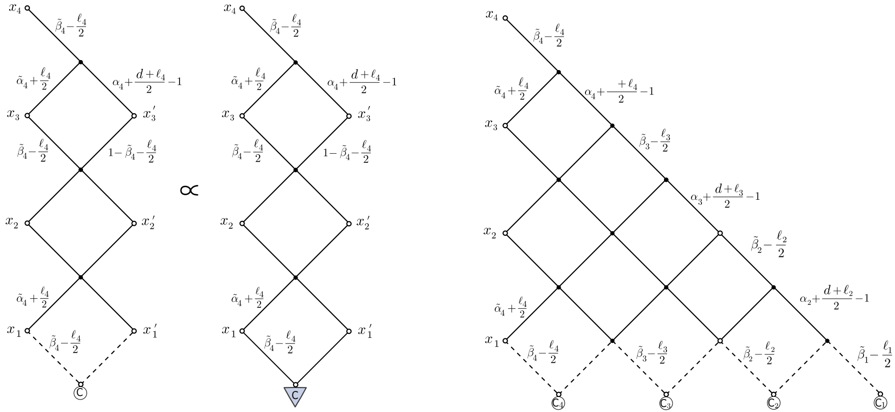
$$V_N\,\Lambda^{(N)}_{\nu_N,\ell_N,m_N} = \lambda(\nu_N,\ell_N)\,\Lambda^{(N)}_{\nu_N,\ell_N,m_N}\,V_{N-1}$$

- Symmetry $\leftrightarrow$ Faddeev-Zamolodchikov algebra

$$\Lambda^{(N)}_{\nu,\ell,m}\,\Lambda^{(N-1)}_{\nu',\ell',m'} = R^{nn'}_{mm'}(\nu-\nu')\,\Lambda^{(N)}_{\nu',\ell',n'}\,\Lambda^{(N-1)}_{\nu,\ell,n}$$

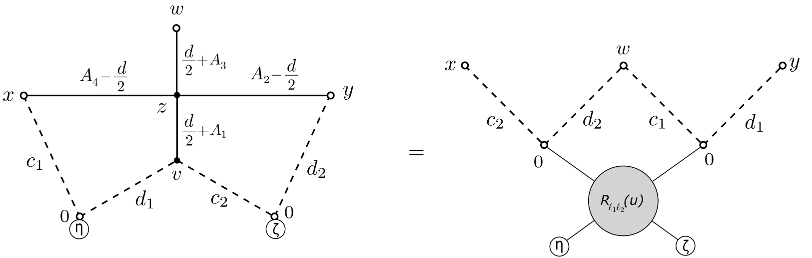
where $R(\nu-\nu')$ – R-matrix, solution of the Yang-Baxter equation

Eigenfunctions



Layer operators and construction of eigenfunction.

Eigenfunctions

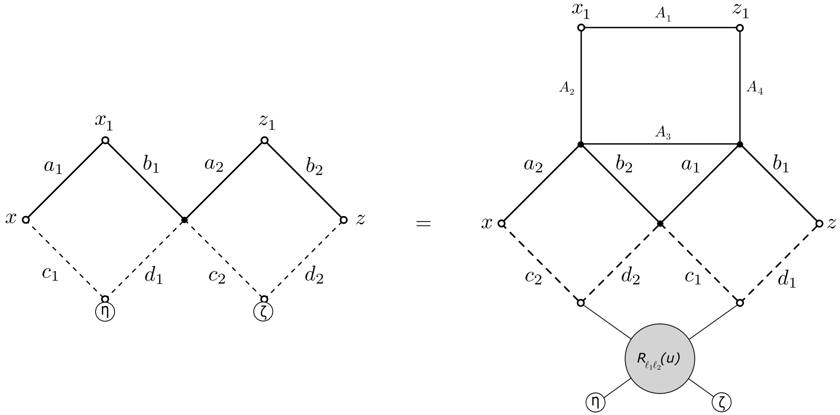


For simplicity $x_0 \to 0$

$$\int \mathrm{d}^d z \mathrm{d}^d v \frac{(z-w)^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)} \left[\zeta\cdot\left(\frac{y}{y^2}-\frac{v}{v^2}\right)\right]^{\ell_1} \left[\eta\cdot\left(\frac{x}{x^2}-\frac{v}{v^2}\right)\right]^{\ell_2}}{(z-x)^{2\left(iu+\frac{\ell_{21}}{2}\right)}(z-y)^{2\left(iu+\frac{\ell_{12}}{2}\right)}(z-v)^{2\left(d-1+\frac{\ell_1+\ell_2}{2}-iu\right)} v^{2\left(1-\frac{\ell_1+\ell_2}{2}+iu\right)}}$$

$$\leftrightarrow \frac{w^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)}}{x^{2\left(iu+\frac{\ell_{21}}{2}\right)} y^{2\left(iu+\frac{\ell_{12}}{2}\right)}} \left[R_{\ell_1,\ell_2}(u)\zeta^{\otimes \ell_1}\otimes\eta^{\otimes\ell_2}\right]\cdot\left[\left(\frac{x}{x^2}-\frac{w}{w^2}\right)^{\otimes\ell_1}\otimes\left(\frac{y}{y^2}-\frac{w}{w^2}\right)^{\otimes\ell_2}\right]$$

Eigenfunctions



Interchange relation $\leftrightarrow$ symmetry of eigenfunctions $\leftrightarrow$ Faddeev-Zamolodchikov algebra

Representation of separated variables

- Orthogonality

$$\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m} | \boldsymbol{\nu'}, \boldsymbol{\ell'}, \boldsymbol{m'} \rangle = \mu^{-1}(\boldsymbol{\nu}, \boldsymbol{\ell})\, \delta(\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m} | \boldsymbol{\nu'}, \boldsymbol{\ell'}, \boldsymbol{m'})$$

$$\Lambda^{(N)\dagger}_{\nu',\ell',m'}\, \Lambda^{(N)}_{\nu,\ell,m} \quad = \quad \frac{R^{nm'}_{mn'}(\nu-\nu')\, \Lambda^{(N-1)}_{\nu,\ell,n}\, \Lambda^{(N-1)\dagger}_{\nu',\ell',n'}}{\left[(\nu-\nu')^2 + \frac{(\ell-\ell')^2}{4}\right] \left[(\nu-\nu')^2 + \frac{(d-2+\ell+\ell')^2}{4}\right]}$$

- Completeness

$$\sum_{\boldsymbol{\ell}, \boldsymbol{m}} \int d^N \boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle \langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}| = \mathbb{1}$$

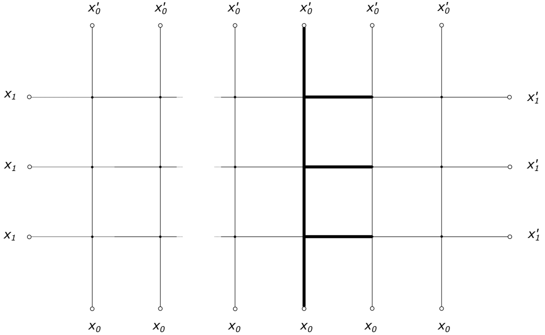
$$\mu(\boldsymbol{\nu}, \boldsymbol{\ell}) = \prod_{1\le i<k\le N} \left((\nu_i-\nu_k)^2 + \frac{(\ell_i-\ell_k)^2}{4} \right) \left((\nu_i-\nu_k)^2 + \frac{(\ell_i+\ell_k+d-2)^2}{4} \right)$$

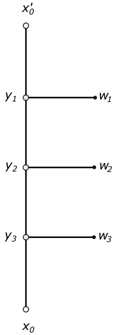
- Spectral decomposition

$$V^{L+1} = \sum_{\boldsymbol{\ell}, \boldsymbol{m}} \int d^N \boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, \lambda^{L+1}(\nu_1, \ell_1) \cdots \lambda^{L+1}(\nu_N, \ell_N)\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle \langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}|$$

Representation of separated variables

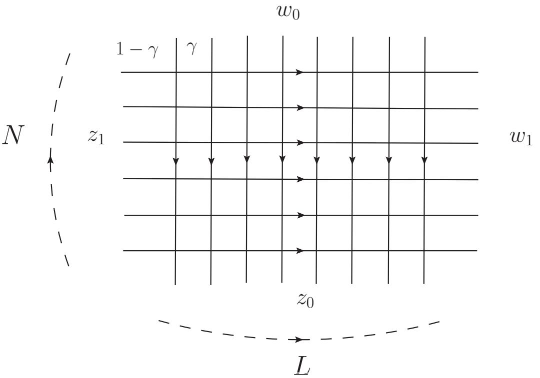
$$V^{L+1}\left(x_1\,,\ldots\,,x_N\,,x_1'\,,\ldots\,,x_N'\right)=\sum_{\boldsymbol{\ell},\boldsymbol{m}}\int d^N\boldsymbol{\nu}\,\mu(\boldsymbol{\nu},\boldsymbol{\ell})\,\lambda^{L+1}(\nu_1,\ell_1)\cdots\lambda^{L+1}(\nu_N,\ell_N)$$
$$\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\ldots\,,x_N\right)\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1'\,,\ldots\,,x_N'\right)$$





- For the functions $\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\ldots\,,x_N\right)\leftrightarrow$ amputation of the left horizontal lines and then reduction $x_k\to x_1$
- For the functions $\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1'\,,\ldots\,,x_N'\right)\leftrightarrow$ reduction $x_k'\to x_1'$

Two-dimensional Basso-Dixon diagram



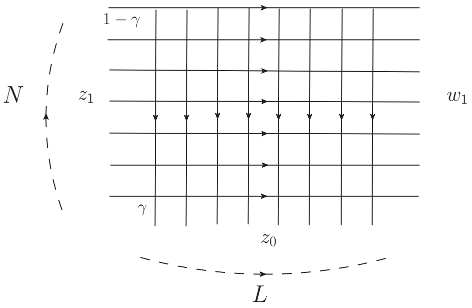
$$\eta = \frac{z_0 - w_1}{z_0 - z_1} \frac{w_0 - z_1}{w_0 - w_1}$$

$$z \xrightarrow{\alpha} w \quad = \frac{1}{[w-z]^{\alpha}}$$

The propagator in $d = 2$ is given by the following expression ($\alpha - \bar{\alpha} \in Z$)

$$\frac{1}{[z-w]^{\alpha}} \equiv \frac{1}{(z-w)^{\alpha}(\overline{z}-\overline{w})^{\bar{\alpha}}} = \frac{(\overline{z}-\overline{w})^{\alpha-\bar{\alpha}}}{|z-w|^{2\alpha}}$$

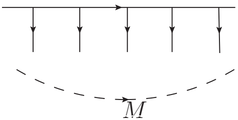
Two-dimensional Basso-Dixon diagram



$\eta = \frac{z_0-w_1}{z_0-z_1}\frac{w_0-z_1}{w_0-w_1} \to \eta = \frac{z_0-w_1}{z_0-z_1}$

$w_0 \to \infty$

$I_{M+1}(\eta,\bar{\eta})$



$$I_{L,N}(\eta,\bar{\eta}) \leftrightarrow \underset{1\leq j,k\leq N}{Det}\left[(\eta\partial_\eta)^{i-1}(\bar{\eta}\partial_{\bar{\eta}})^{k-1} I_{N+L}(\eta,\bar{\eta})\right]$$

Two-dimensional Basso-Dixon diagram

Step one

$$I_{L,N}(\eta,\bar\eta) \leftrightarrow \int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} \left([\eta]^{ix_k}\, \lambda^{N+L}(x_k)\right)$$

where

$$ix = \frac{n}{2} + i\nu \ , \ \int \mathcal{D}x = \sum_{n\in Z} \int_{-\infty}^{+\infty} d\nu$$

Step two – Determinant representation

$$\int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} [\eta]^{ix_k}\, \lambda^{N+L}(x_k) = N!\, Det\, M$$

$$M_{ik} = \int \mathcal{D}x\ x^{i-1}\bar{x}^{j-1}[\eta]^{ix}\, \lambda^{N+L}(x) = (\eta\partial_\eta)^{i-1}(\bar\eta\partial_{\bar\eta})^{k-1} \int \mathcal{D}x\, [\eta]^{ix}\, \lambda^{N+L}(x)$$
$$i,k = 1\,,\ldots\,,N$$

Conclusions

- Calculation of diagrams $\leftrightarrow$ Sklyanin SOV method from the theory of integrable spin chains
- Search of simpler proof of the zig-zag conjecture

  D.J. Broadhurst, D. Kreimer, *Knots and numbers in ϕ^4 theory to 7 loops and beyond,* Int. J. Mod. Phys. C 6, 519 (1995)

  F.C.S. Brown, O. Schnetz, *Single-valued multiple polylogarithms and a proof of the zig-zag conjecture,* Jour.of Numb. Theory 148, 478-506 (2015)