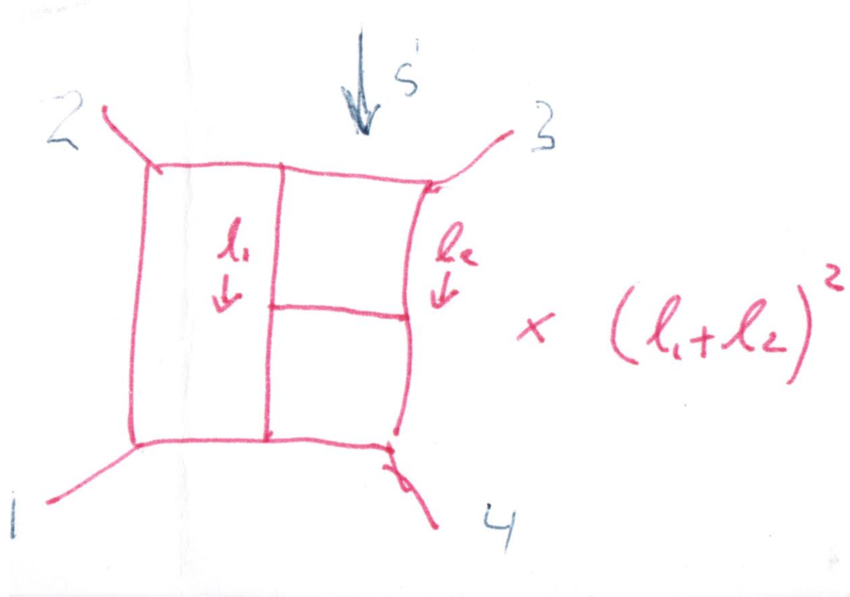
Scattering Iteratively with Volodya Smirnov



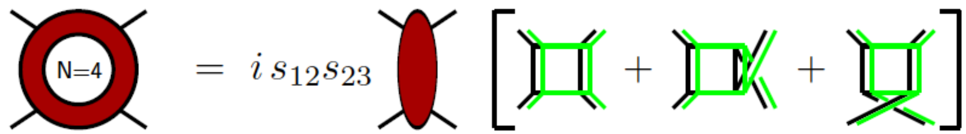
Lance Dixon (SLAC)

“Advances in Quantum Field Theory”, **7 @ 70**

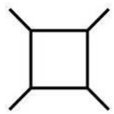
Dubna, October 12, 2021

Back around 2000:
Simplicity of N=4 SYM 4-point integrands

- 1 loop:

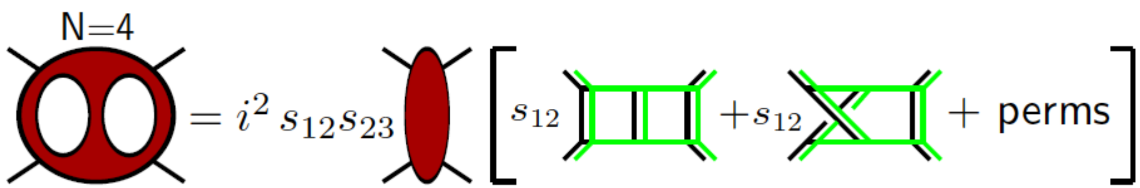


Green, Schwarz, Brink (1982)

where  $= \int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-k_1-k_2)^2(\ell_1+k_4)^2}$

 $= \delta^{ab}$  $= f^{abc}$

- 2 loops:

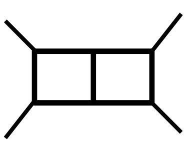


Bern, Rozowsky, Yan (1997); Bern, LD, Dunbar, Perelstein, Rozowsky (1998)

- Analogous computation in QCD not completed until 2001

Glover, Oleari, Tejeda-Yeomans (2001); Bern, De Freitas, LD (2002)

- Expansion around $D = 4 - 2\epsilon$ in QCD and N=4 needed **Volodya's integrals:**



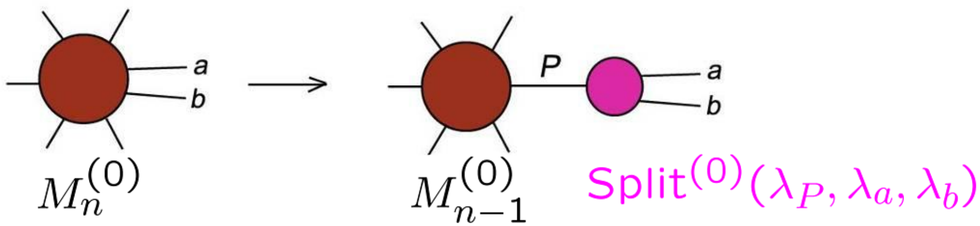
V. Smirnov, hep-ph/9905323; V. Smirnov, Veretin, hep-ph/9907385

N=4 SYM simplicity for $n > 4$ gluons?
→ inspect two-loop collinear limits, $n \rightarrow n$-1

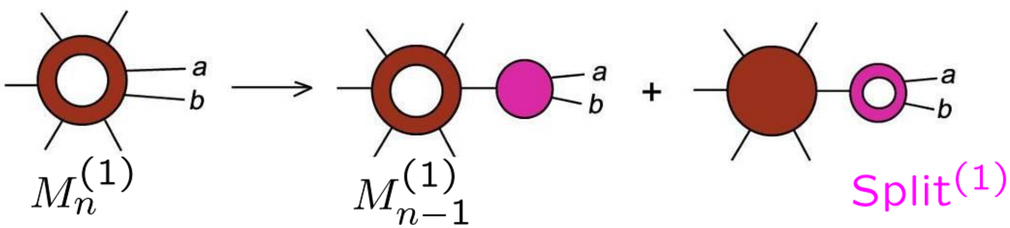
- Study limits as 2 momenta become **collinear:**

$$k_a \rightarrow z k_P$$
$$k_b \rightarrow (1-z) k_P$$

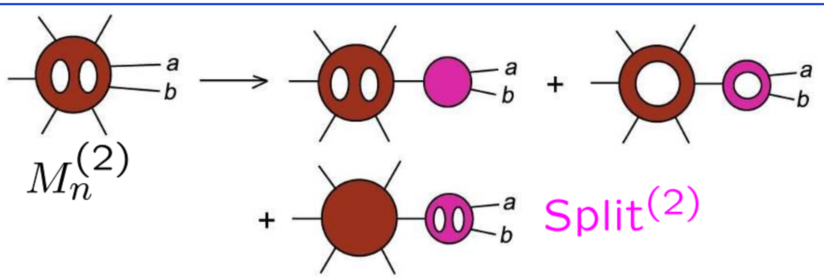
- Tree amplitude behavior:



- One-loop behavior:



- Two-loop behavior:



Two-loop splitting amplitude iteration for planar N=4 SYM and an amplitude conjecture

• In N=4 SYM, all helicity configurations are equivalent, can write

$$\text{Split}^{(l)}(\lambda_P, \lambda_a, \lambda_b) = r_S^{(l)}(z, s_{ab}, \epsilon) \times \text{Split}^{(0)}(\lambda_P, \lambda_a, \lambda_b)$$

• Found that two-loop splitting amplitude obeys:

Anastasiou, Bern, LD, Kosower, hep-th/0309040

$$r_S^{(2)}(\epsilon) \;=\; \frac{1}{2}\Big[r_S^{(1)}(\epsilon)\Big]^2 + f^{(2)}(\epsilon) r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

where $f^{(2)}(\epsilon) = -\zeta_2 - \epsilon\,\zeta_3 - \epsilon^2\,\zeta_4$

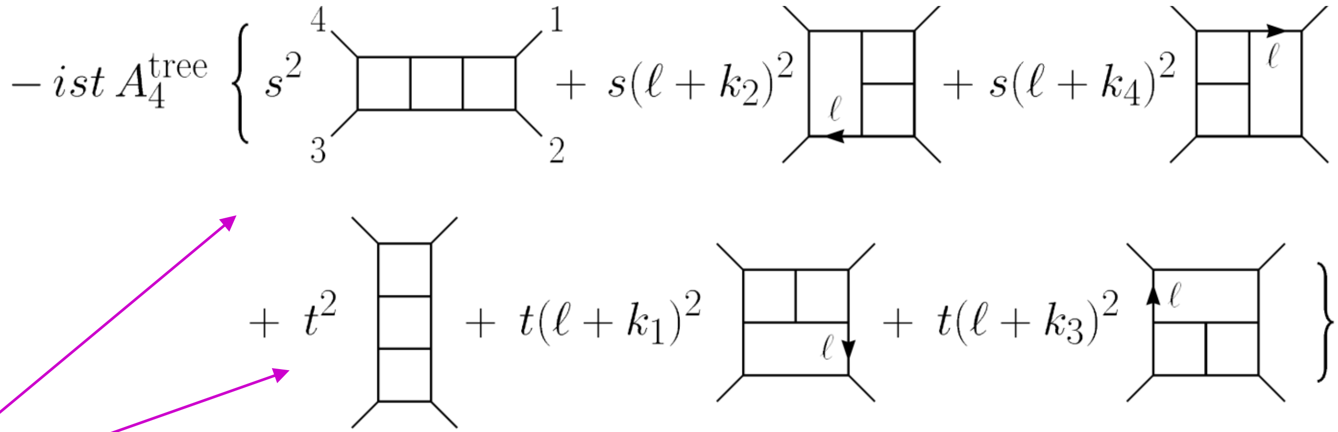
consistent with *n*-point amplitude ABDK ansatz (2-loop only)

$$\mathcal{M}_n^{(2)}(\epsilon) \;=\; \frac{1}{2}\Big[M_n^{(1)}(\epsilon)\Big]^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)}$$

constant from 4-point amplitude and Volodya's planar double box

Needed 3-loop data to see more of the pattern!

- Planar integrands known from Bern, Rozowsky, Yan (1997)
- Needed $D = 4 - 2\epsilon$ expansion of integrals



First half of the job:
V. Smirnov,
hep-ph/0305142

N·H
ELSEVIER
Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
Physics Letters B 567 (2003) 193–199
PHYSICS LETTERS B
www.elsevier.com/locate/npe

Analytical result for dimensionally regularized massless on-shell planar triple box

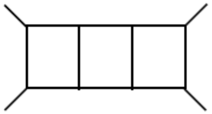
V.A. Smirnov

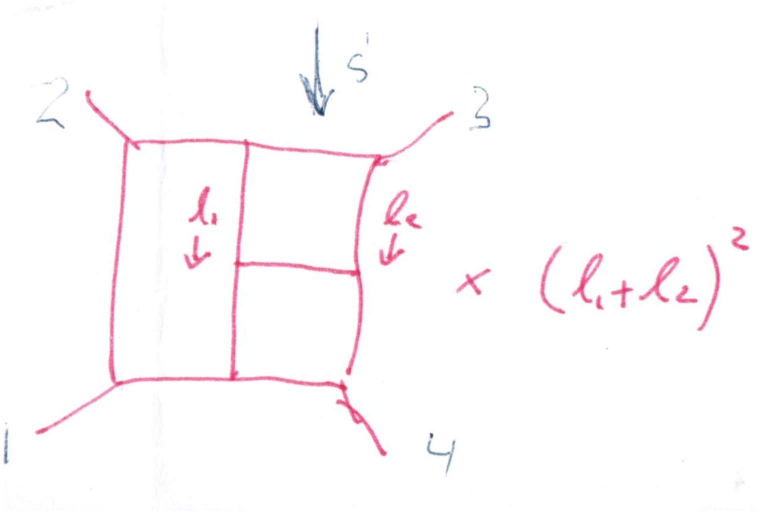
Nuclear Physics Institute of Moscow State University, Moscow 119992, Russia

Received 12 June 2003; received in revised form 17 June 2003; accepted 18 June 2003

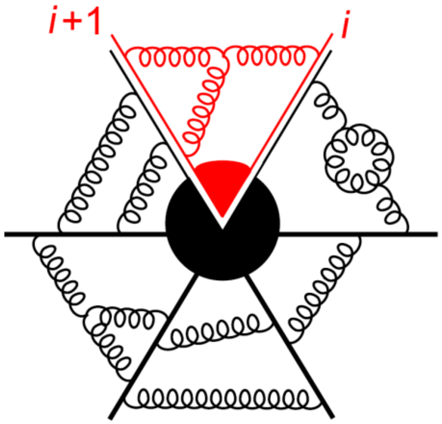
Editor: P.V. Landshoff

Zvi explained it to Volodya on a napkin

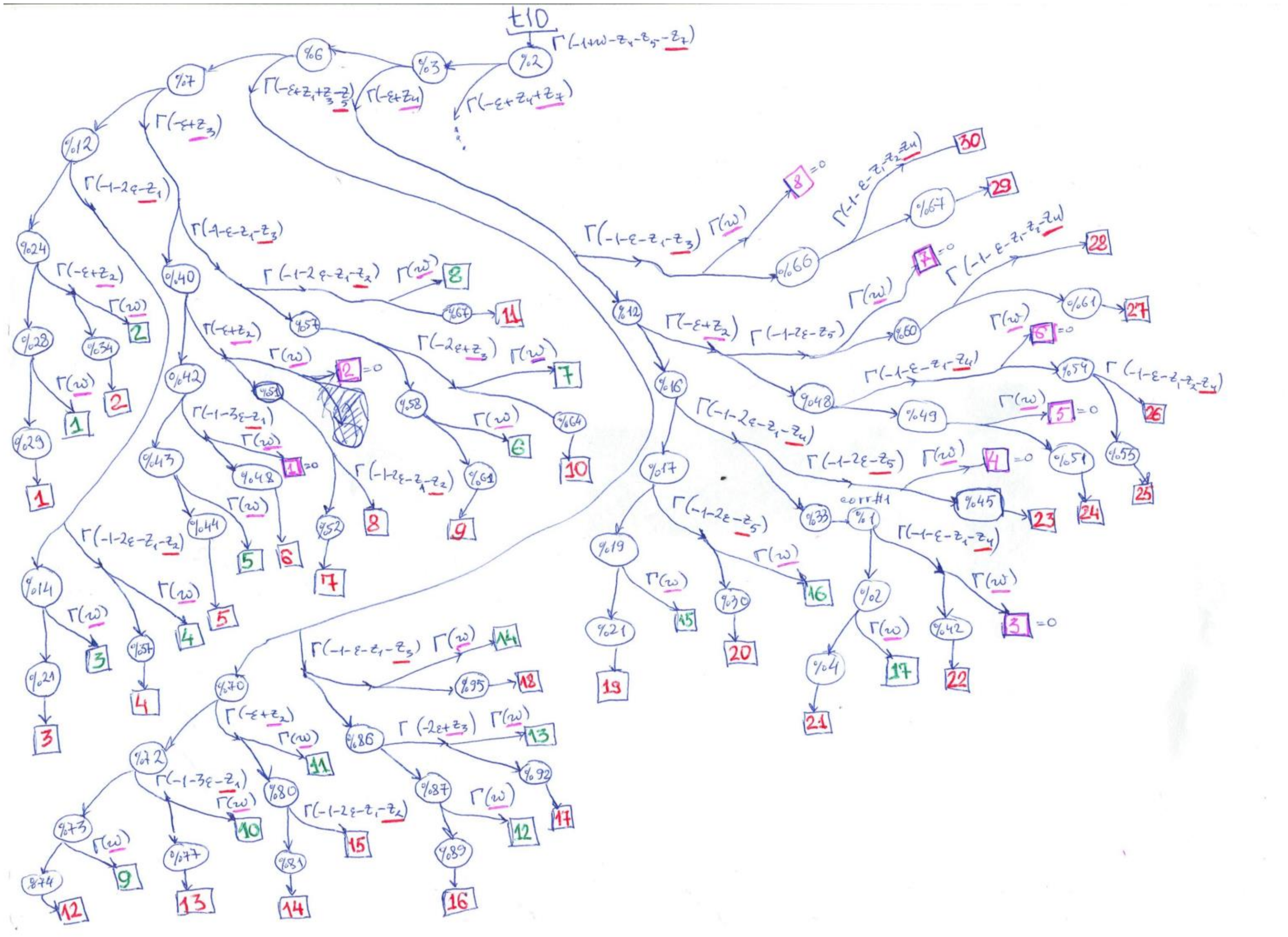
- At Loops and Legs in Zinnowitz, April 2004, after Volodya talked about 
- Christened the "tennis court integral" (although really only half a tennis court).

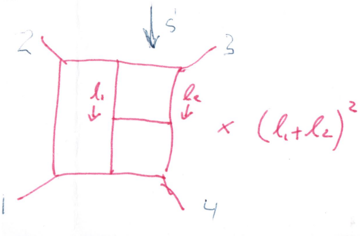


- Volodya got to work on it with Mellin-Barnes, and started producing more and more terms in the expansion in ϵ
- Zvi and I read some QCD papers which explained the **IR poles**, especially Magnea, Sterman, (1990) for the form factor, together with **planar** properties



Mellin-Barnes a la Smirnov, circa 2004-2009





ABDK → BDS Ansatz

Bern, LD, Smirnov, hep-th/0505205

- Combining Volodya's new tennis-court results with his old ladder results, we found we could write an "iterative relation" for 3-loop 4-point amplitude:

$$M_4^{(3)}(\epsilon) = \boxed{-\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon)} + C^{(3)} + \mathcal{O}(\epsilon)$$

IR poles

$$f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)$$

$\propto \Gamma_{\rm cusp}^{(3)}$ $\qquad$ $\propto \mathcal{G}_0^{(3)}$

cusp, collinear anomalous dimensions

constant, independent of *s/t*
foreshadowing
dual conformal symmetry

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2$$

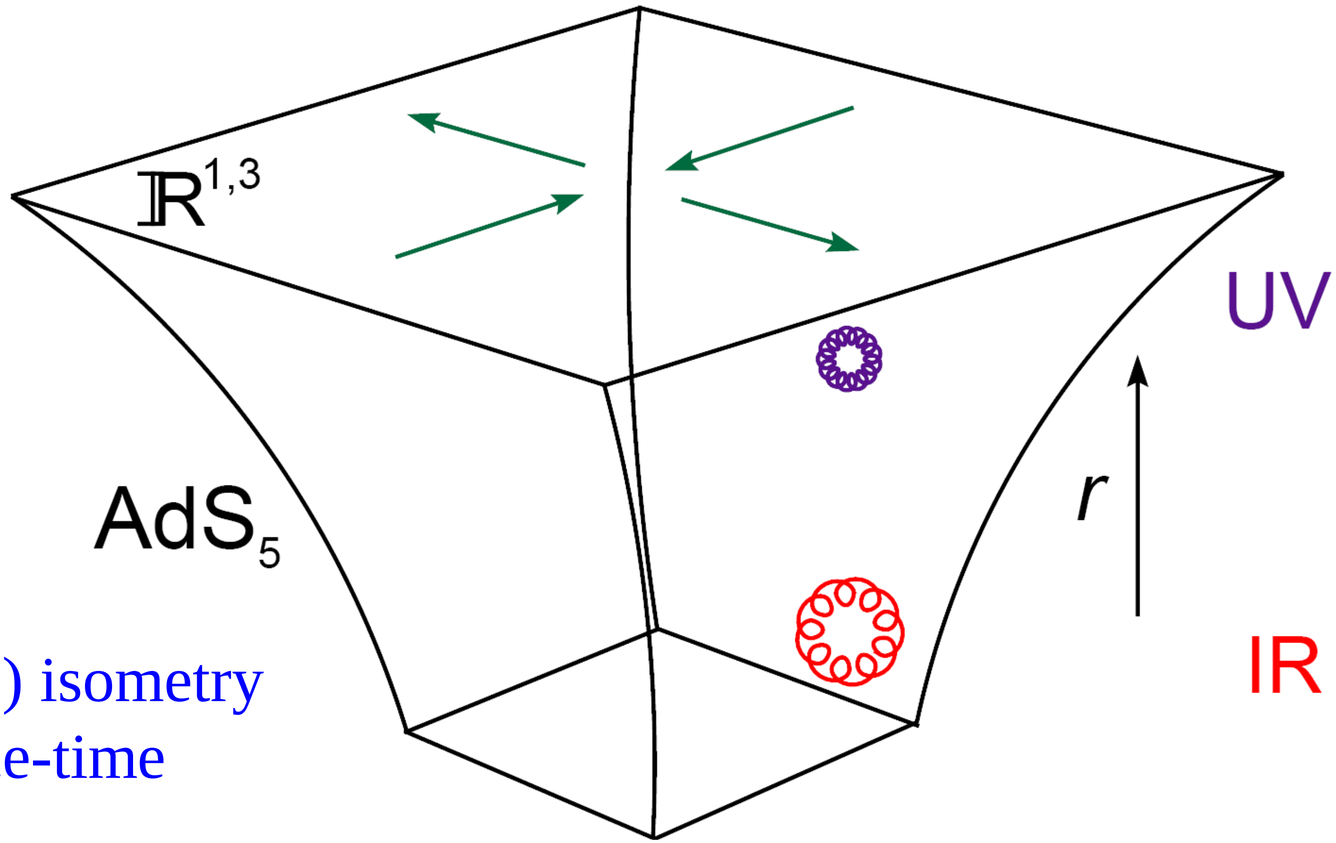
- With this result we could conjecture essentially the same relation to all loop orders and all numbers of legs, the BDS ansatz:

$$A_n^{\rm BDS} \sim \exp\left[\frac{\Gamma_{\rm cusp}}{4}\frac{A_n^{\rm 1-loop}}{A_n^{\rm tree}}\right] \times R_n\left(\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}\right)$$

- Turned out to be right for $n = 4{,}5$ (due to dual conformal invariance) but wrong for $n > 5$

Dual conformal invariance is geometric: from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303



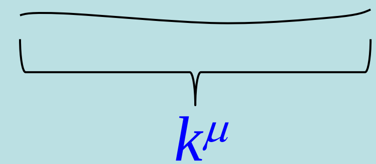
SO(4,2) isometry
of space-time

T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

→ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



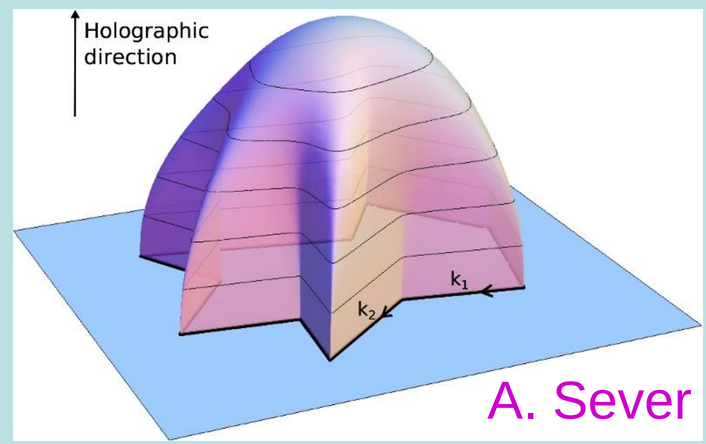
- Strong coupling limit of planar gauge theory

is semi-classical limit of string theory:

world-sheet stretches tight around

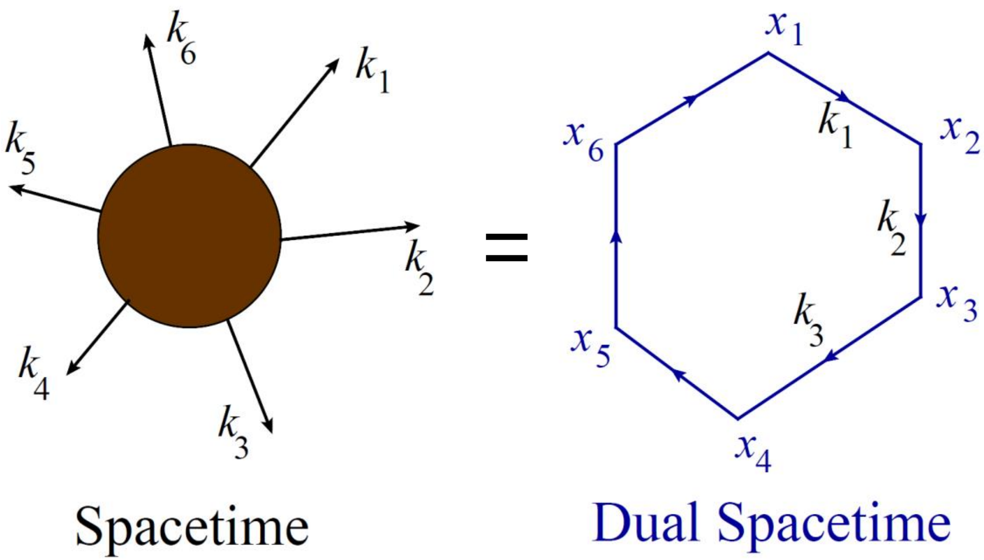
minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

Amplitudes = Wilson loops



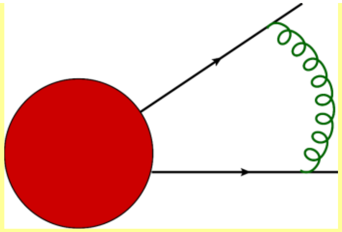
- Polygon vertices x_i are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

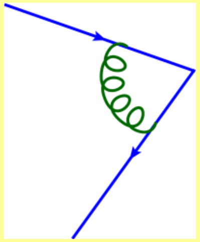
Duality verified to hold at weak coupling too!

IR Renormalization Schemes

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$



known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by either BDS ansatz or BDS-like ansatz   Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized amplitude finite, dual conformal invariant.
- BDS-like → also maintains important relation due to causality (Steinmann) Caron-Huot, LD, McLeod, von Hippel, 1609.00669

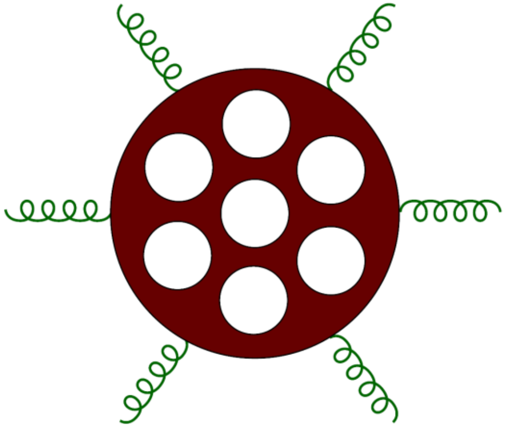
BDS begat the Hexagon function bootstrap

3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890 and 1906.07116

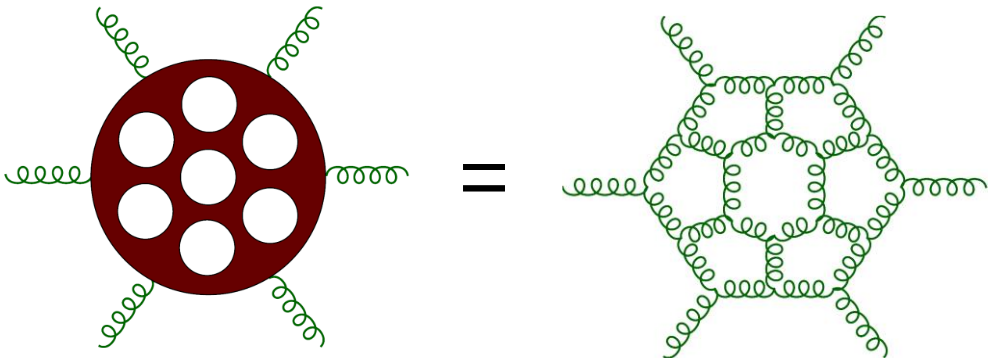
- Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
- Write answer as a linear combination of transcendental functions living in same function space for all loops
- Fix all parameters using “boundary data”



- First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

How far can we go?

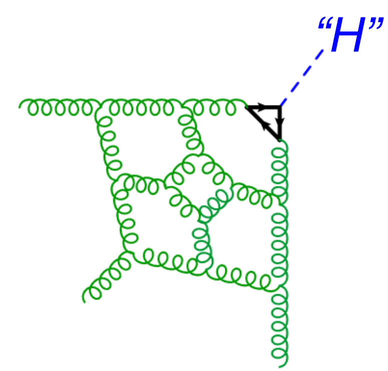
- So far, through 7 loops at 6 points



+ ~10^9 more diagrams

- Also 4 loops at 7 points …, LD, Liu, 2007.12966
- And 8 loops for a 3-gluon form factor

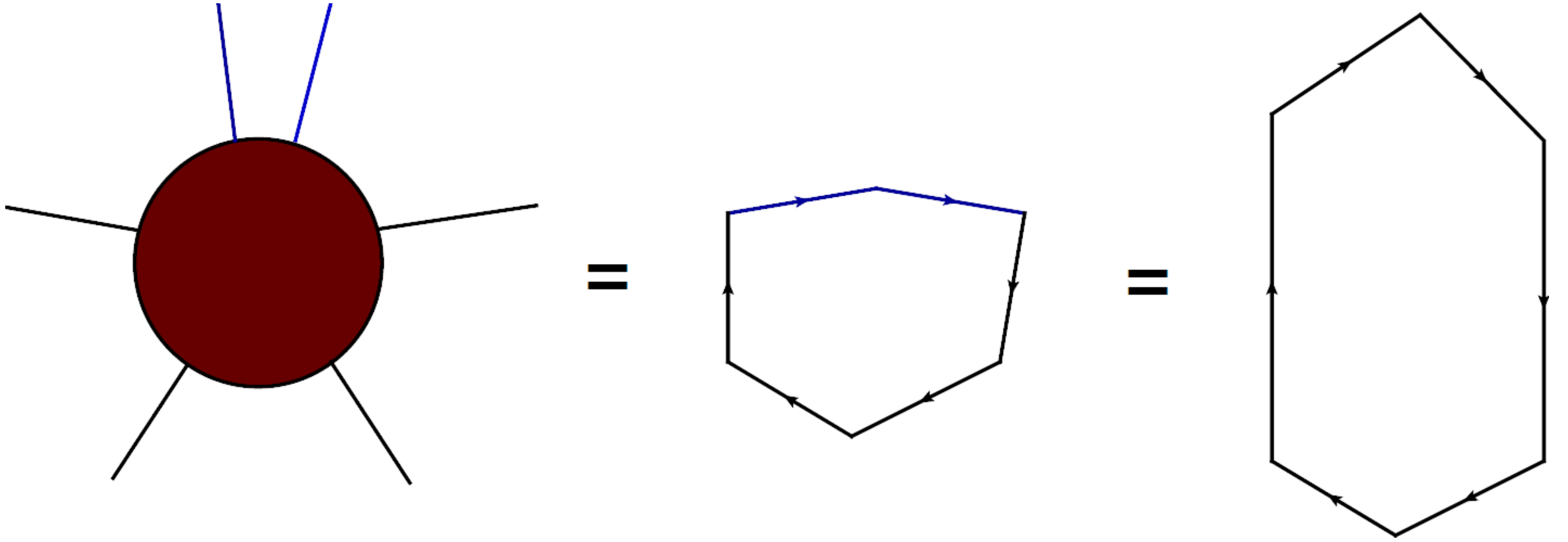
LD, A. McLeod, M. Wilhelm, 2012.12286 $L = 3{,}4{,}5$ loops
+ in progress also with Ö. Gürdoğan $L = 6{,}7{,}8$ loops



Rich theoretical “data” mine

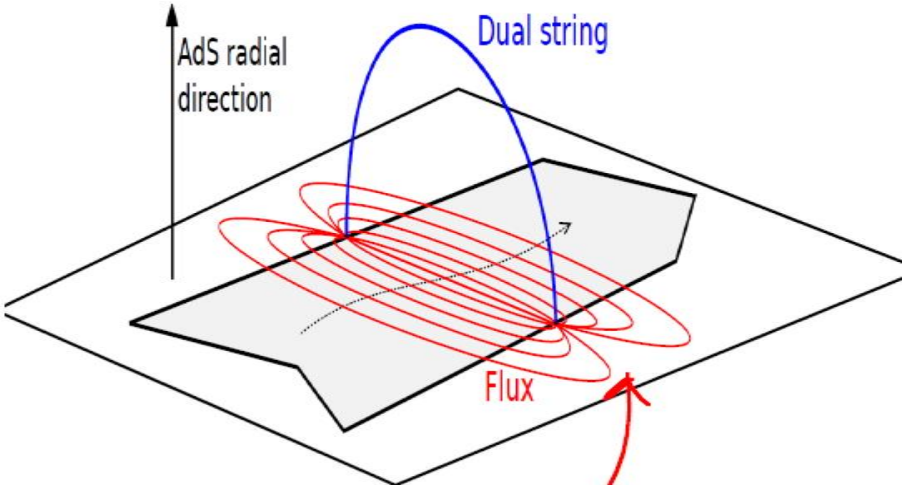
- Rare to have perturbative results to 7 or 8 loops
- Usually high loop order → single numbers such as β functions or anomalous dimensions
- Here we have analytic functions of 3 variables (6 variables in 7-point case, 2 variables for form-factor)
- Many limits to study (and exploit)

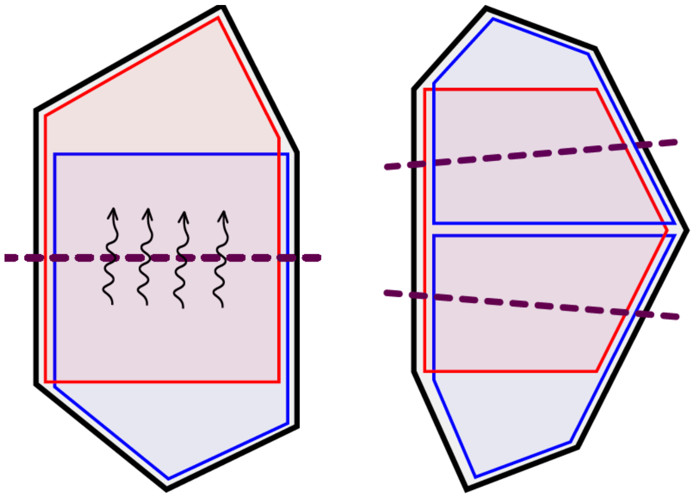
(Near) collinear (OPE) limit



Flux tubes at finite coupling

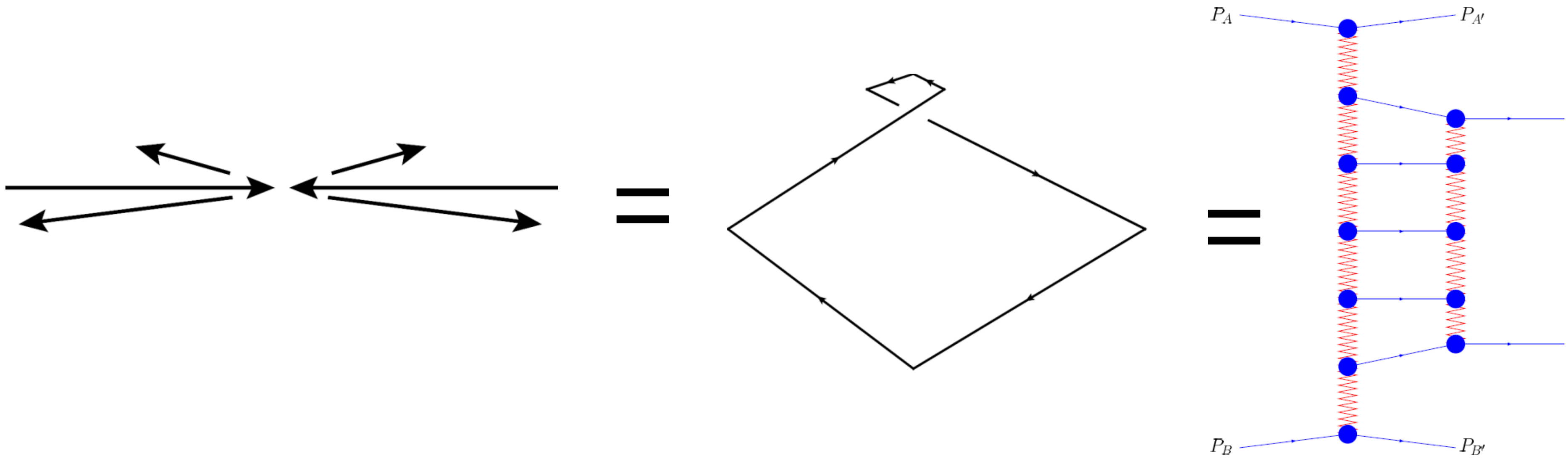
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

Multi-regge limit



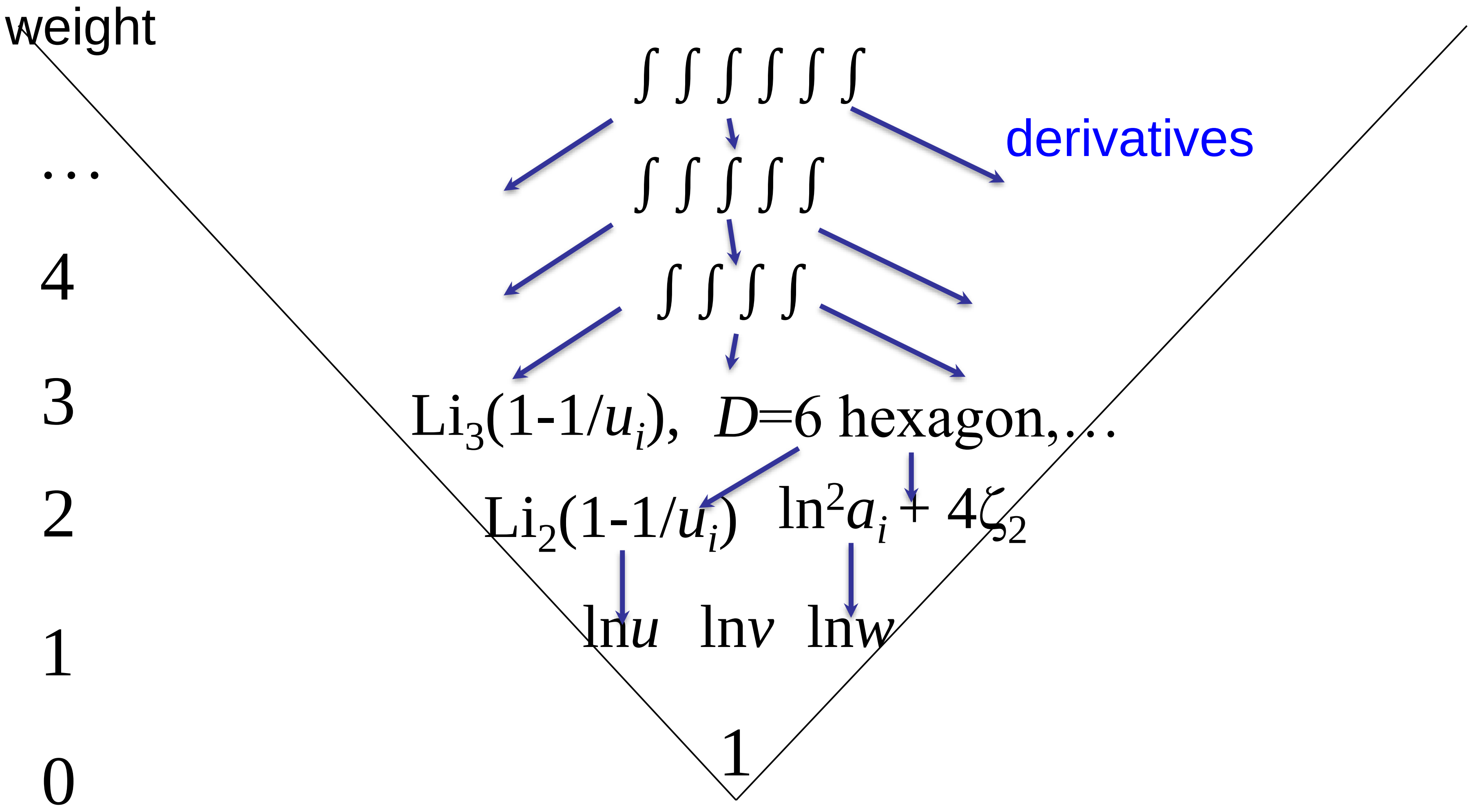
- Amplitude factorizes in Fourier-Mellin space

Bartels, Lipatov, Sabio Vera, 0802.2065, Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE limit);
Broedel, Sprenger, 1512.04963; Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1606.08807;…

Key "initial" condition

- Two-loop 6-gluon result first computed numerically from both amplitude and Wilson loop pictures
  Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1466;
  Drummond, Henn, Korchemsky, Sokatchev, 0803.1466
- Wilson loop side evaluated analytically by Volodya and friends → 17 pages of [Goncharov] polylogarithms
  Del Duca, Duhr, V. Smirnov, 0911.5332, 1003.1702
- Simplified to a few lines in term of classical polylogs $\mathrm{Li}_n(x)$, demonstrating power of symbol
  Goncharov, Spradlin, Vergu, Volovich, 1006.5703
- Told us what types of functions were likely to occur at higher loops (symbol alphabet)

Heuristic view of polylogarithmic space



Constraining Parameters in Linear Combination

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. All functions	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,3692?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear limit	(0,0)	(0,0)	$(0^*, 0^*)$	$(0^*, 2^*)$	$(1^{*3}, 5^{*3})$	$(6^{*2}, 17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^{*2}, 2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^*, 0^*)$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2 F^2 \ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all $T^2 F^2$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

Also have $L = 7$

Properties of Amplitudes

- Having determined the 6-point amplitudes to 7 loops, can study their physical, numerical and (number-theoretic) properties.
- Analytic behavior in various factorization limits.
- What kinds of transcendental numbers appear?
- Numerics feasible on "simple lines" like $(u,u,1)$, $(u,1,1)$, (u,u,u).
- Planar N=4 SYM should have finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic).
- For BES solution to cusp anomalous dimension, using coupling $g^2 = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$

→ Ratio of successive coefficients: $$\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}} \to -16$$

At $(u,v,w) = (1,1,1)$, amplitude → MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

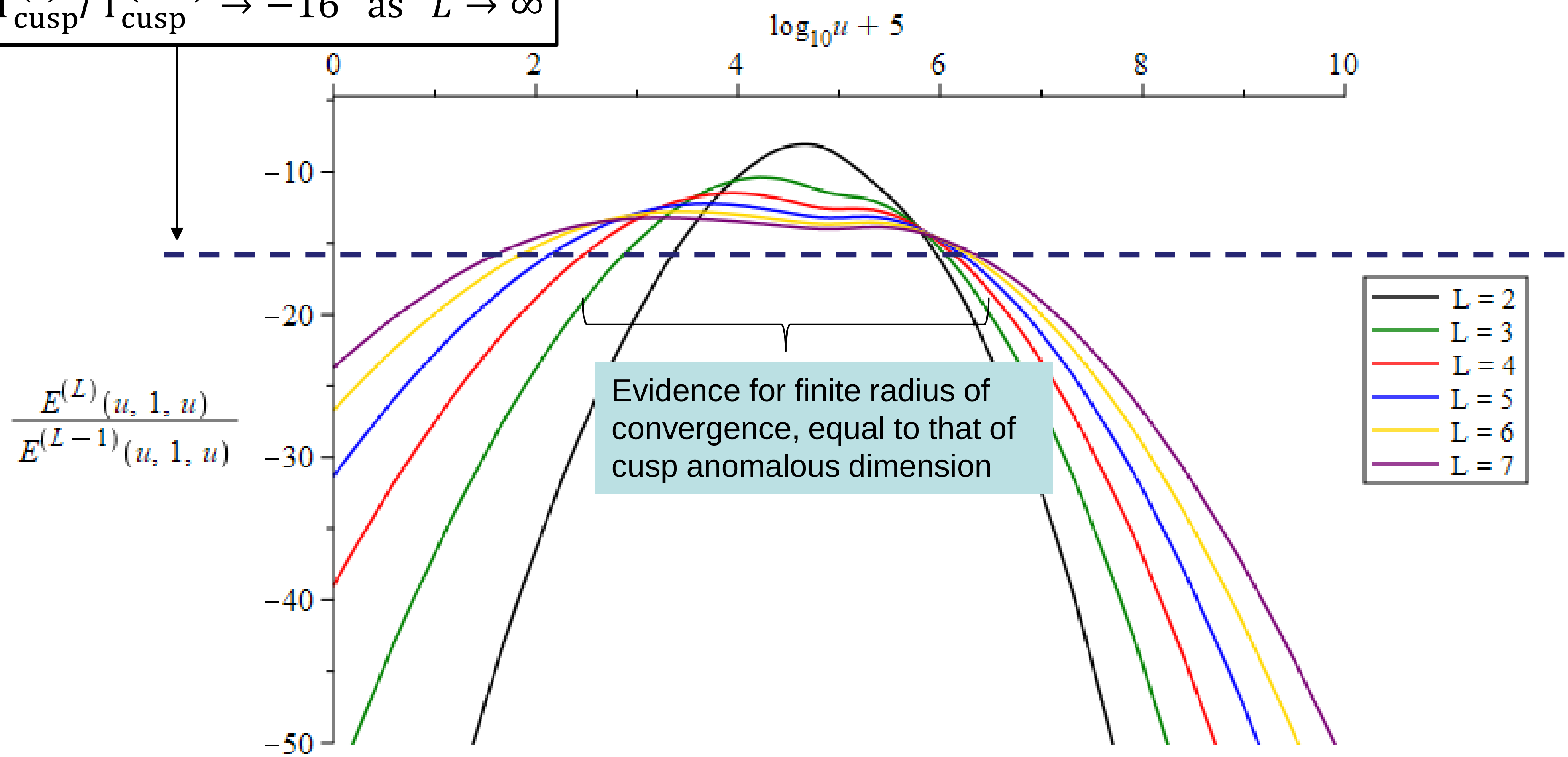
$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

Successive loop ratios for NMHV Amplitude on $(u,u,1)$

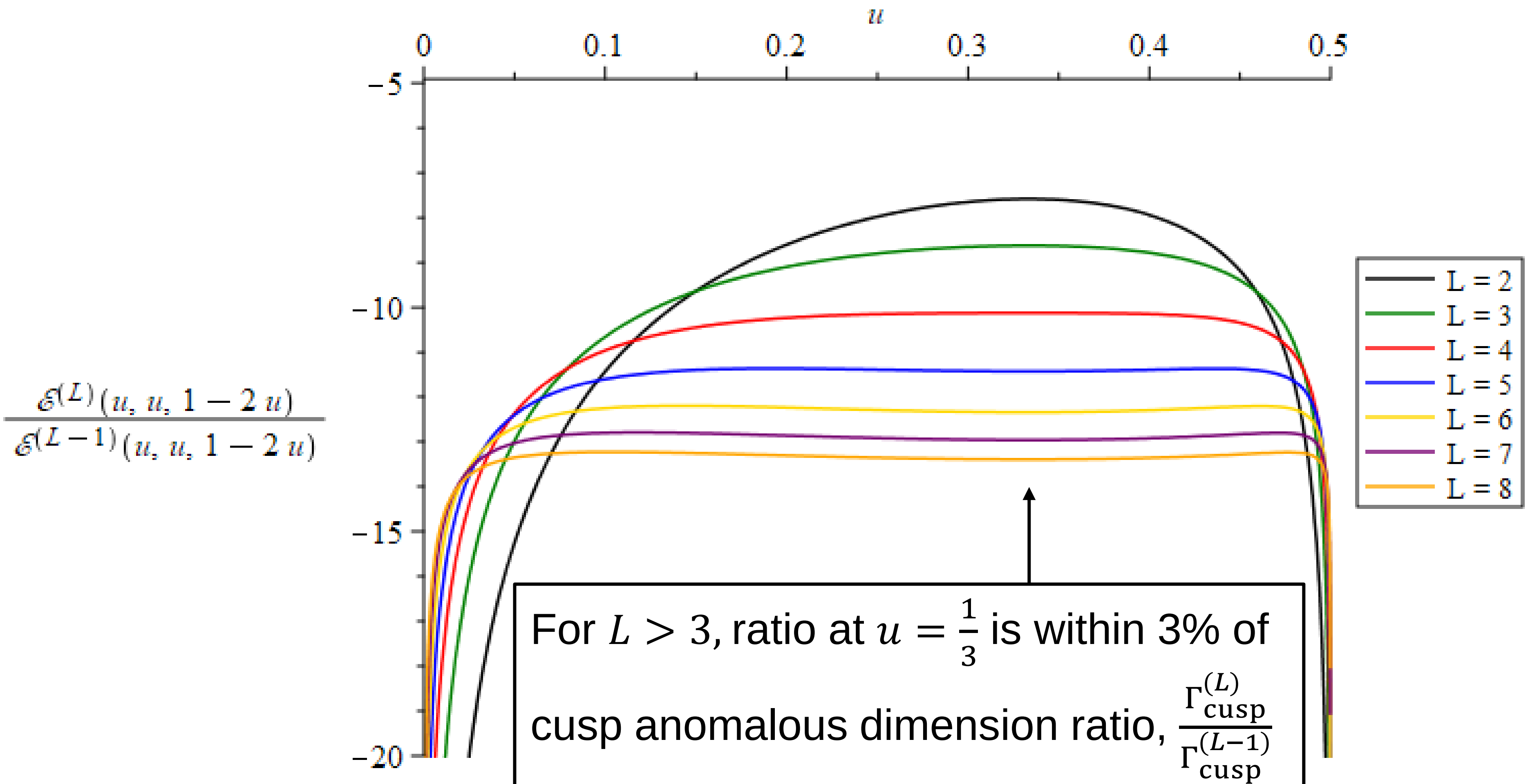
$\Gamma_{\text{cusp}}^{(L)} / \Gamma_{\text{cusp}}^{(L-1)} \to -16$ as $L \to \infty$



Evidence for finite radius of convergence, equal to that of cusp anomalous dimension

3-gluon form factor, Euclidean region, to 8 loops

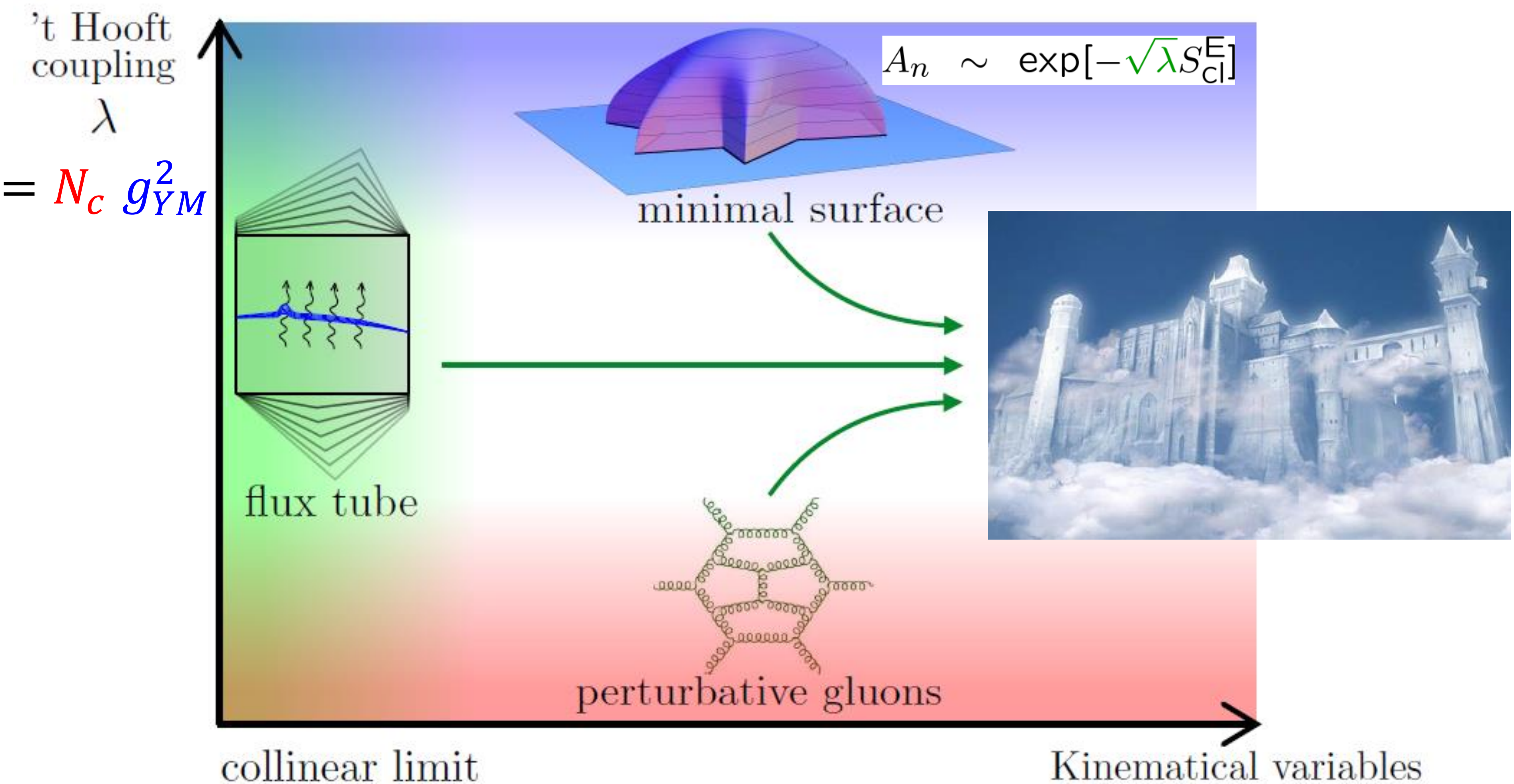
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, to appear



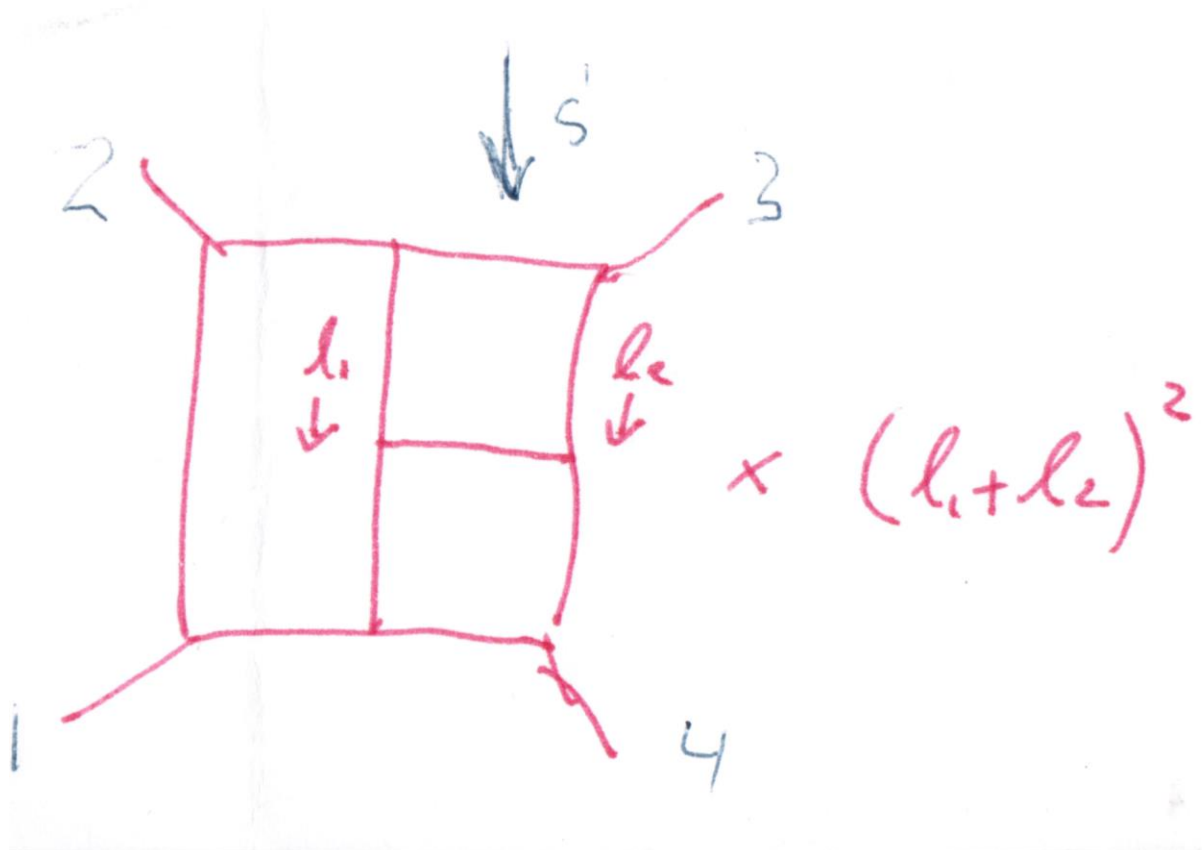
For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\text{cusp}}^{(L)}}{\Gamma_{\text{cusp}}^{(L-1)}}$

“Holy Grail” for Planar N=4 SYM Scattering

Images: A. Sever, N. Arkani-Hamed

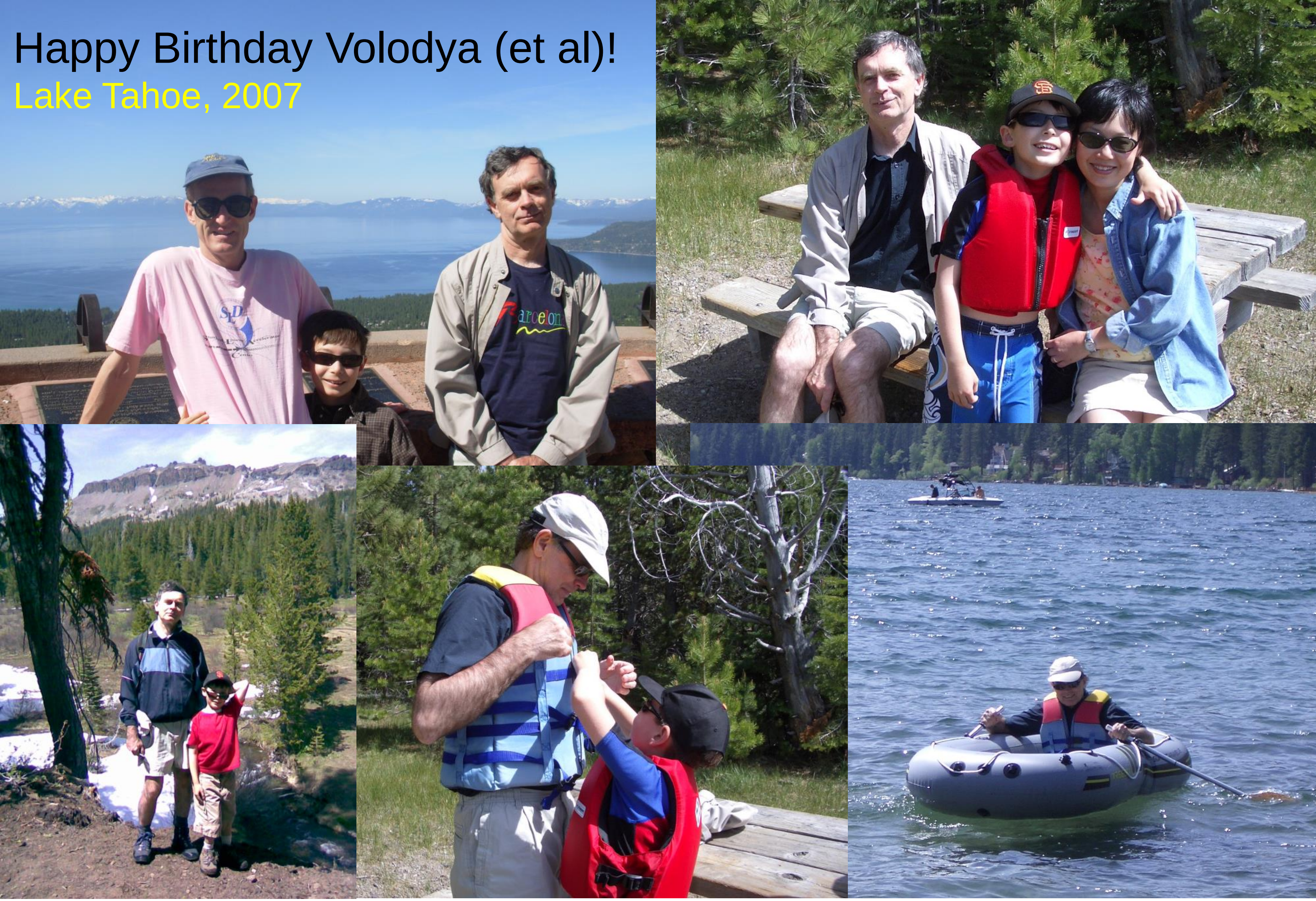


It all started with a napkin!



Happy Birthday Volodya (et al)!

Lake Tahoe, 2007



6 loops → Sheldon Cooper ("Big Bang Theory")?

Extras

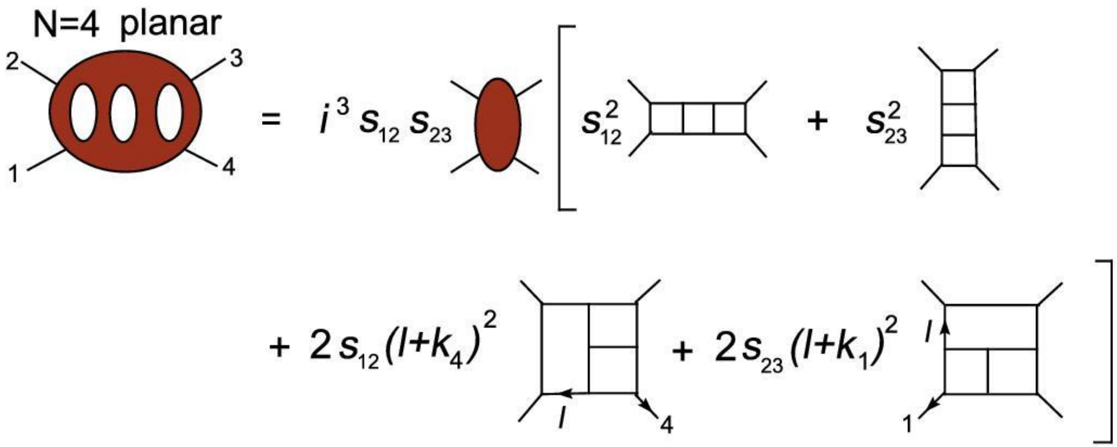
Test at Three Loops Soon?

From talk at STRINGS 2004

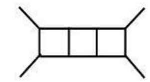
- 3-loop planar N=4 amplitude also dates from 1997

(3-particle cuts checked later)

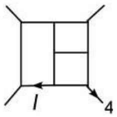
Bern, Rozowsky, Yan (1997)



- Integrals now being computed:



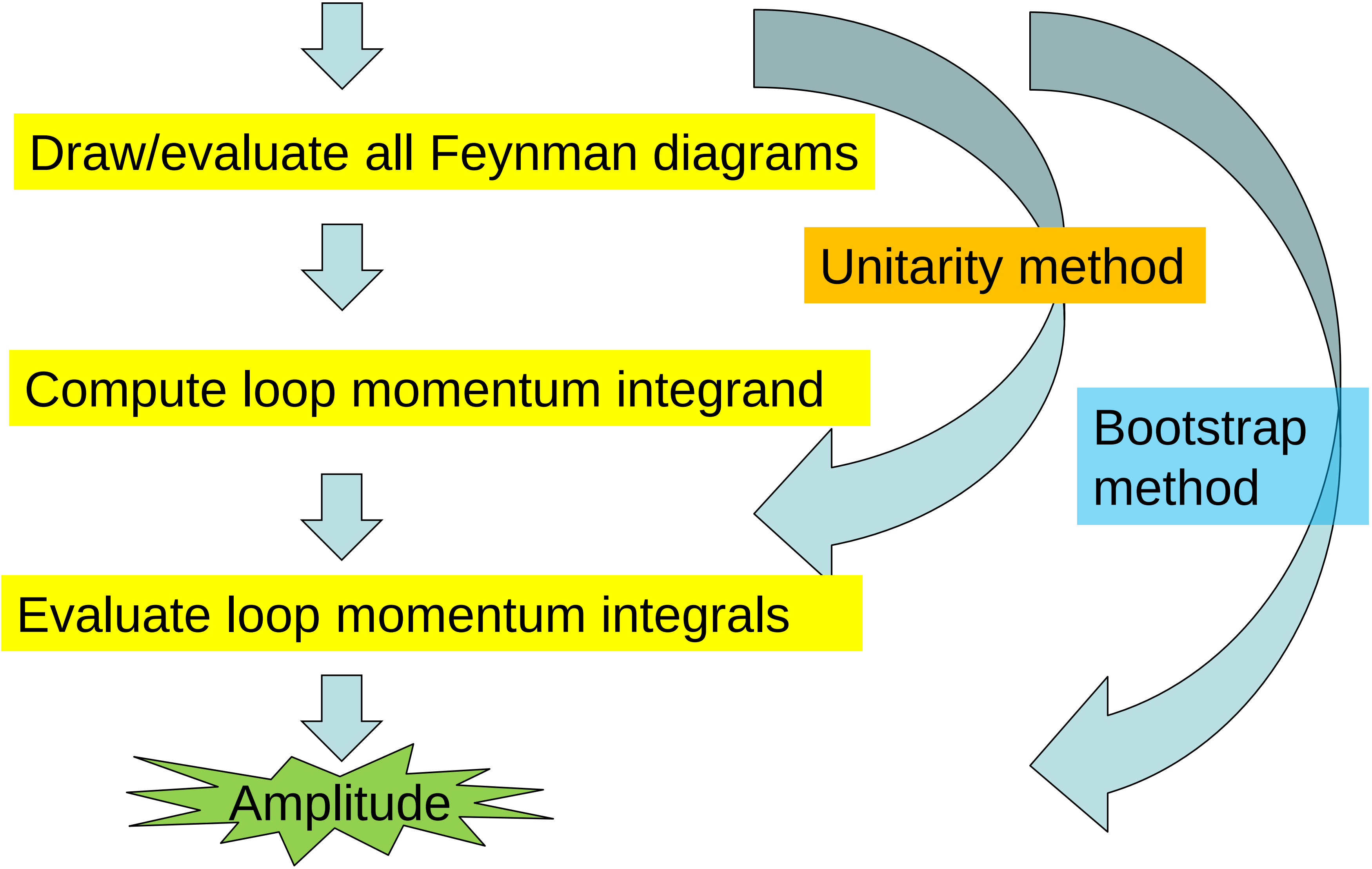
Smirnov, hep-ph/0305142



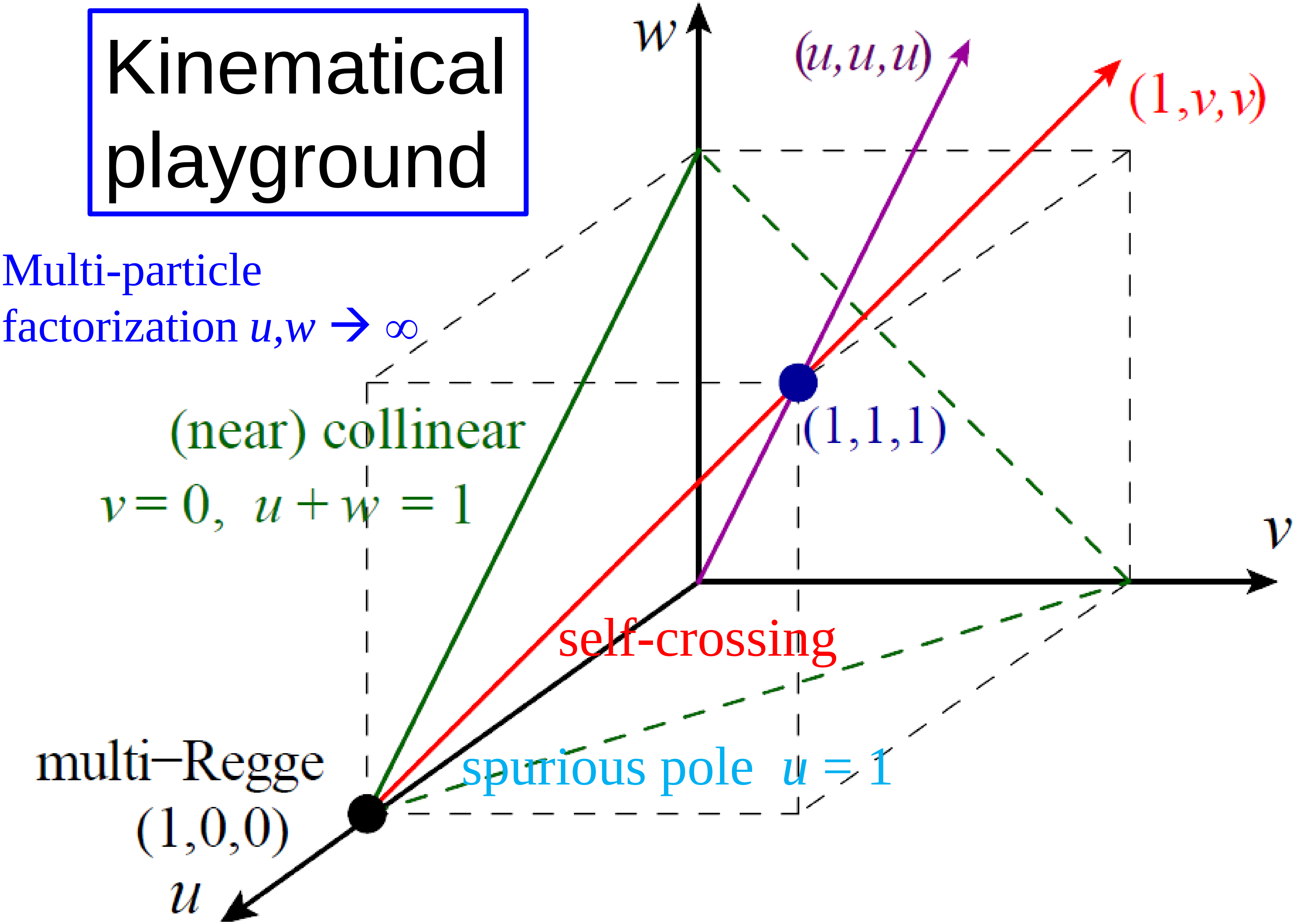
Smirnov, in progress

- $1/\varepsilon^6$ to $1/\varepsilon^2$ poles predictable — Sterman, Tejeda-Yeomans, hep-ph/0210130
- Works to order computed so far -- $1/\varepsilon^3$

Usual approaches vs. bootstrap

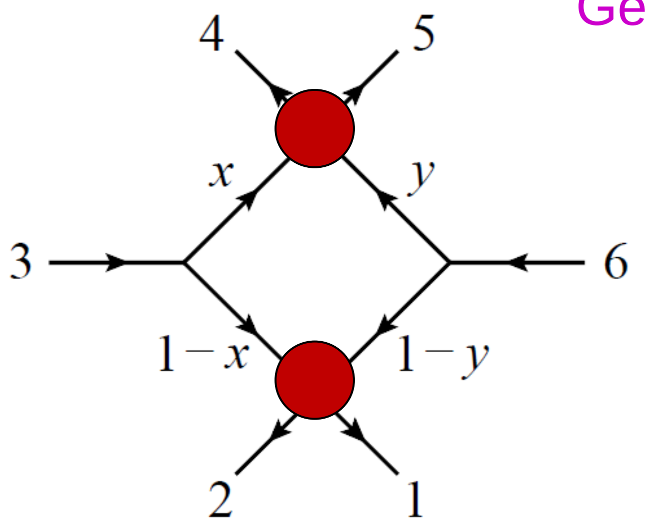


Kinematical playground

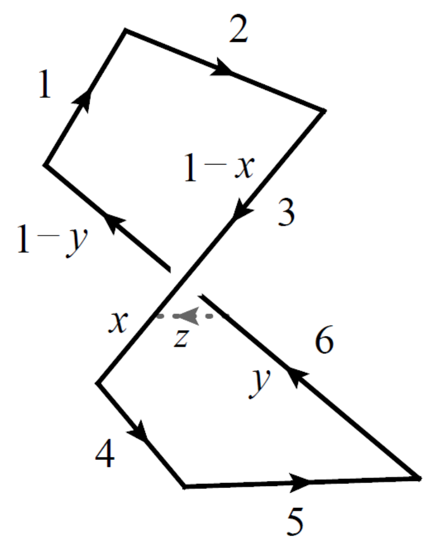


Double-parton-scattering-like limit

Georgiou, 0904.4675; LD, Esterlis, 1602.02107



=



- Self-crossing limit of Wilson loop, $\delta \sim |z|^2 \rightarrow 0$
- Overlaps MRK limit
- A virtual Sudakov region, $A \sim \exp[-\ln^2\delta]$
- Singularities ~ Wilson line RGE

Korchemsky and Korchemskaya hep-ph/9409446