Fixation of definite $\overline{\mathrm{MS}}$-scheme α_s^5 QCD contributions to Adler function and $R(s)$ in the β-expanded representation

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Outline

- Study of process of e^+e^- annihilation into hadrons continues to attract the close attention both theoreticians and experimenters: CERN, BINP RAS, KEK, BEPC
- Adler function and Bjorken polarized sum rule — quantities included in the generalized Crewther relation
- Two-fold representation an information source about definite $\{\beta\}$-expanded terms in Adler and Bjorken functions
- Results, comparison, consequences
- Conclusion

Adler function, R-ratio, Bjorken polarized sum rule

$$D^{(M)}(a_s(Q^2)) = d_R \left(\sum_f Q_f^2\right) D_{NS}^{(M)}(a_s(Q^2)) + d_R \left(\sum_f Q_f\right)^2 D_{SI}^{(M\geq 3)}(a_s(Q^2))$$

$$C^{(M)}(a_s(Q^2)) = C_{NS}^{(M)}(a_s(Q^2)) + d_R \sum_f Q_f C_{SI}^{(M\geq 4)}(a_s(Q^2))$$

$$D\left(Q^2\right) = Q^2 \int\limits_0^\infty \frac{R(s)}{(s+Q^2)^2}\, ds$$

$a_s = \alpha_s/\pi$

$M \geq 1$ is the order of PT

$$D_{NS}^{(M)}(a_s(Q^2)) = 1 + \sum_{m=1}^{M} d_m a_s^m(Q^2)$$

$$C_{NS}^{(M)}(a_s(Q^2)) = 1 + \sum_{m=1}^{M} c_m a_s^m(Q^2)$$

$$r_1 = d_1, \qquad r_2 = d_2,$$

$$r_3 = d_3 - \frac{\pi^2}{3} d_1 \beta_0^2, \quad r_4 = d_4 - \pi^2 \left(d_2 \beta_0^2 + \frac{5}{6} d_1 \beta_1 \beta_0\right),$$

$$r_5 = d_5 - \pi^2 \left(2 d_3 \beta_0^2 + \frac{7}{3} d_2 \beta_0 \beta_1 + \frac{1}{2} d_1 \beta_1^2 + d_1 \beta_0 \beta_2\right) + \frac{\pi^4}{5} d_1 \beta_0^4,$$

$\beta^{(N)}(a_s) = \mu^2 \frac{\partial a_s}{\partial \mu^2} = -\sum_{i=0}^{N-1} \beta_i a_s^{i+2}$, where $N \geq 1$ is the number of loops renormalizing a charge

$$M = N + 1$$

The generalized Crewther relation (GCR)

$$D^{(M)}_{NS}(a_s)C^{(M)}_{NS}(a_s) = 1+\Delta^{(N)}_{csb}(a_s)= 1+\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)K^{(N)}(a_s)= 1+\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)\sum_{i\geqslant 1} K_i a_s^i$$

The GCR was firstly detected in $\overline{\mathrm{MS}}$-scheme at $\mathcal{O}(a_s^3)$ level by *Broadhurst, Kataev in (93)*:

$$D^{(3)}_{NS}C^{(3)}_{NS} = 1+\left(\frac{\beta^{(2)}(a_s)}{a_s}\right)(K_1 a_s + K_2 a_s^2) + \mathcal{O}(a_s^4) = 1 - \beta_0 K_1 a_s^2 - (\beta_0 K_2 + \beta_1 K_1)a_s^3 + \mathcal{O}(a_s^4)$$

It was confirmed later on at $\mathcal{O}(a_s^4)$ level by *Baikov, Chetyrkin, Kühn in (10)*:

$$D^{(4)}_{NS}C^{(4)}_{NS} = 1+\left(\frac{\beta^{(3)}(a_s)}{a_s}\right)(K_1 a_s + K_2 a_s^2 + K_3 a_s^3) + \mathcal{O}(a_s^5)$$

$$K_1 = \left(-\frac{21}{8} + 3\zeta_3\right)C_F,$$

$$K_2 = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{629}{32} + \frac{221}{12}\zeta_3\right)C_F C_A + \left(\frac{163}{24} - \frac{19}{3}\zeta_3\right)C_F T_F \underline{n_f},$$

$$K_3 = k_1 C_F^3 + k_2 C_F^2 C_A + k_3 C_F C_A^2 + k_4 C_F^2 T_F \underline{n_f} + k_5 C_F T_F^2 \underline{n_f^2} + k_6 \times C_F C_A T_F \underline{n_f^2}$$

Proof indications of the validity of the GCR were given in *(97) by Crewther*

and in *(03) by Braun, Korchemsky, Müller* (consideration in x-space).

Then the GCR will be fulfilled in the gauge-dependent MOM-like schemes

in Landau gauge $\xi = 0$ in all orders of PT as well *(Garkusha, Kataev, Molokoedov (18))*.

Another representation: two-fold form

The conformal symmetry breaking term $\Delta_{csb}^{(N)}(a_s) = \left(\frac{\beta^{(N)}(a_s)}{a_s}\right) K^{(N)}(a_s)$

can be represented in the following form *(Kataev, Mikhailov (10, 12))*:

$$\Delta_{csb}^{(N)}(a_s) = \sum_{n=1}^{N} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s) = \sum_{n=1}^{N} \left(\frac{\beta(a_s)}{a_s}\right)^n \sum_{r\geq 1} P_n^{[r]} a_s^r$$

where coefficients $P_n^{[r]}$ are uniquely defined and do not depend on $T_F n_f$-structures and contain C_F and C_A quadratic Casimir operator of $SU(N_c)$ group (exception - light-by-light scattering effects).

For e.g.

$$\Delta_{csb}^{(2)}(a_s) = \left(\frac{\beta^{(2)}(a_s)}{a_s}\right) P_1^{(2)}(a_s) + \left(\frac{\beta^{(1)}(a_s)}{a_s}\right)^2 P_2^{(1)}(a_s)$$

$$= (-\beta_0 a_s - \beta_1 a_s^2)(P_1^{[1]} a_s + P_1^{[2]} a_s^2) + \beta_0^2 a_s^2 P_2^{[1]} a_s$$

$$= -\beta_0 P_1^{[1]} a_s^2 + (-\beta_0 P_1^{[2]} - \beta_1 P_1^{[1]} + \beta_0^2 P_2^{[1]}) a_s^3$$

$$P_1^{[1]} = \left(-\frac{21}{8} + 3\zeta_3\right) C_F,$$

$$P_1^{[2]} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right) C_F^2 + \left(-\frac{47}{48} + \zeta_3\right) C_F C_A,$$

$$P_2^{[1]} = \left(\frac{163}{8} - 19\zeta_3\right) C_F, \quad \ldots$$

Two-fold expansion for Adler function

The double sum expression for $\Delta_{csb}(a_s)$ motivated *(Cvetič, Kataev (16))* to propose similar representation for Adler (and Bjorken) function:

$$D^{(M)}_{NS}(a_s) = 1 + D^{(M)}_0(a_s) + \sum_{n\geq 1}\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)^n D^{(N)}_n(a_s)$$
$$= 1 + D^{(M)}_0(a_s) + \sum_{n\geq 1}\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)^n \sum_{r\geq 1} D^{[r]}_n a_s^r$$

At the four-loop level :

$$D_n(a_s) = \sum_{r=1}^{4-n} a_s^r \sum_{k=1}^{r} D^{[r]}_n[k, r-k] C_F^k C_A^{r-k}$$
$$+a_s^4\delta_{n0}\underbrace{\left(D^{(4)}_0[F,A]\frac{d_F^{abcd}d_A^{abcd}}{d_R} + D^{(4)}_0[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right)}_{light-by-light\ scattering\ effects}$$

In $SU(N_c)$ QCD in $\overline{\text{MS}}$-scheme coefficients $D^{[r]}_n[k, r-k]$ are unambiguously determined from the corresponding system of linear equations at the $\mathcal{O}(a_s^4)$ level at least.

Supposing that two-fold expansion for Adler function is valid at the five-loop level, one can obtain:

$$D^{(5)}_{NS}(a_s) = 1 + D^{(5)}_0(a_s) + \left(\frac{\beta^{(4)}(a_s)}{a_s}\right) D^{(4)}_1(a_s) + \left(\frac{\beta^{(3)}(a_s)}{a_s}\right)^2 D^{(3)}_2(a_s) + \left(\frac{\beta^{(2)}(a_s)}{a_s}\right)^3 D^{(2)}_3(a_s) + \left(\frac{\beta^{(1)}(a_s)}{a_s}\right)^4 D^{(1)}_4(a_s)$$

This representation is in full agreement with $\{\beta\}$-expansion, proposed by *(Mikhailov (07))* :

$$d_1 = d_1[0], \qquad d_2 = \beta_0 d_2[1] + d_2[0],$$

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0],$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1\beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0]$$

At a_s^5 expansion will have the form :

$$d_5 = \beta_0^4 d_5[4] + \beta_1\beta_0^2 d_5[2,1] + \beta_0^3 d_5[3] + \beta_2\beta_0 d_5[1,0,1] + \beta_1^2 d_5[0,2] + \beta_1\beta_0 d_5[1,1] + \beta_0^2 d_5[2] + \beta_3 d_5[0,0,0,1] + \beta_2 d_5[0,0,1] + \beta_1 d_5[0,1] + \beta_0 d_5[1] + d_5[0]$$

Comparing these representations, we lead to the following equalities:

$$D^{(1)}_1 = -d_2[1] = -d_3[0,1] = -d_4[0,0,1] = -d_5[0,0,0,1],$$

$$D^{(2)}_1 = -d_3[1] = -d_4[0,1] = -d_5[0,0,1],$$

$$D^{(3)}_1 = -d_4[1] = -d_5[0,1],$$

$$D^{(1)}_2 = d_3[2] = d_4[1,1]/2 = d_5[0,2] = d_5[1,0,1]/2,$$

$$D^{(2)}_2 = d_4[2] = d_5[1,1]/2, \qquad D^{(1)}_3 = -d_4[3] = -d_5[2,1]/3.$$

Known $\{\beta\}$-coefficients: three-loop level *(Cvetič, Kataev (16))*

Coefficients	Color structures	
$d_1[0]$	C_F	$\frac{3}{4}$
$d_2[0]$	C_F^2	$-\frac{3}{32}$
	$C_F C_A$	$\frac{1}{16}$
$d_2[1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_3[0]$	C_F^3	$-\frac{69}{128}$
	$C_F^2 C_A$	$-\frac{101}{256} + \frac{33}{16}\zeta_3$
	$C_F C_A^2$	$-\frac{53}{192} - \frac{33}{16}\zeta_3$
$d_3[1]$	C_F^2	$-\frac{111}{64} - 12\zeta_3 + 15\zeta_5$
	$C_F C_A$	$\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5$
$d_3[0,1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_3[2]$	C_F	$\frac{151}{6} - 19\zeta_3$

Known $\{\beta\}$-coefficients: four-loop level

Coefficients	Color structures	
$d_4[0]$	C_F^4	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$
	$C_F^3 C_A$	$-\frac{3509}{1536} - \frac{73}{128}\zeta_3 - \frac{165}{32}\zeta_5$
	$C_F^2 C_A^2$	$\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5$
	$C_F C_A^3$	$-\frac{30863}{36864} - \frac{147}{128}\zeta_3 + \frac{165}{64}\zeta_5$
	$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$
	$\frac{d_F^{abcd} d_F^{abcd}}{d_R} n_f$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$
$d_4[1]$	C_F^3	$-\frac{785}{128} - \frac{9}{16}\zeta_3 + \frac{165}{2}\zeta_5 - \frac{315}{4}\zeta_7$
	$C_F^2 C_A$	$-\frac{3737}{144} + \frac{3433}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{615}{16}\zeta_5 + \frac{315}{8}\zeta_7$
	$C_F C_A^2$	$-\frac{2695}{384} - \frac{1987}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{175}{32}\zeta_5 - \frac{105}{16}\zeta_7$
$d_4[0,1]$	C_F^2	$-\frac{111}{64} - 12\zeta_3 + 15\zeta_5$
	$C_F C_A$	$\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5$
$d_4[2]$	C_F^2	$-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5$
	$C_F C_A$	$\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5$
$d_4[0,0,1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_4[1,1]$	C_F	$\frac{151}{3} - 38\zeta_3$
$d_4[3]$	C_F	$\frac{6131}{36} - \frac{203}{2}\zeta_3 - 45\zeta_5$

Coefficients	Color structures	
$d_5[0,1]$	C_F^3	$-\frac{785}{128} - \frac{9}{16}\zeta_3 + \frac{165}{2}\zeta_5 - \frac{315}{4}\zeta_7$
	$C_F^2 C_A$	$-\frac{3737}{144} + \frac{3433}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{615}{16}\zeta_5 + \frac{315}{8}\zeta_7$
	$C_F C_A^2$	$-\frac{2695}{384} - \frac{1987}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{175}{32}\zeta_5 - \frac{105}{16}\zeta_7$
$d_5[0,0,1]$	C_F^2	$-\frac{111}{64} - 12\zeta_3 + 15\zeta_5$
	$C_F C_A$	$\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5$
$d_5[1,1]$	C_F^2	$-\frac{4159}{192} - \frac{2997}{8}\zeta_3 + 54\zeta_3^2 + 375\zeta_5$
	$C_F C_A$	$\frac{14615}{128} + \frac{39}{8}\zeta_3 - 9\zeta_3^2 - \frac{185}{2}\zeta_5$
$d_5[0,2]$	C_F	$\frac{151}{6} - 19\zeta_3$
$d_5[1,0,1]$	C_F	$\frac{151}{3} - 38\zeta_3$
$d_5[2,1]$	C_F	$\frac{6131}{12} - \frac{609}{2}\zeta_3 - 135\zeta_5$
$d_5[4]$	C_F	$\frac{91865}{72} - \frac{4955}{9}\zeta_3 - 570\zeta_5$

$SU(3)$ case:

$$
\begin{aligned}
D_{NS}(a_s) &= 1 + a_s + \left(0.6918\beta_0 + \underline{\underline{0.0833}}\right)a_s^2 \\
&+ \left(3.1035\beta_0^2 + 0.6918\beta_1 + 4.9402\beta_0 - \underline{\underline{23.2227}}\right)a_s^3 \\
&+ \left(2.1800\beta_0^3 + 6.2069\beta_0\beta_1 + 17.6990\beta_0^2 + 0.6918\beta_2 + 4.9402\beta_1\right. \\
&- \left.101.928\beta_0 + \underline{\underline{81.1571}} + 0.0802 n_f\right)a_s^4 \\
&+ \left(30.7398\beta_0^4 + 6.5401\beta_0^2\beta_1 + 6.2069\beta_0\beta_2 + 3.1035\beta_1^2 + 35.3981\beta_0\beta_1\right. \\
&+ 0.6918\beta_3 + 4.9402\beta_2 - 101.928\beta_1 \\
&+ \left.\underset{\sim}{d_5[3]}\beta_0^3 + \underset{\sim}{d_5[2]}\beta_0^2 + \underset{\sim}{d_5[1]}\beta_0 + \underset{\sim}{d_5[0]}\right)a_s^5 + \mathcal{O}(a_s^6)
\end{aligned}
$$

4 wavy terms of 12 remain unknown in d_5 (but with fixed ζ_4-contributions in $d_5[0], d_5[1], d_5[2]$ *(Goriachuk, Kataev (20), arXiv:2011.14746)*)

$d_5[4]$ is the known renormalon contribution *(Broadhurst, Kataev (93))*

The solid underlined conformal-invariant terms are in good agreement with ones, obtained by PMC/BLM scale setting (absorption of n_f-dependence in the scale) by *(Brodsky, Wu (12))*

The first two of them are also confirmed by the results of generalization of BLM procedure and application of the effective charge approach *(Grunberg, Kataev (92))*

R-ratio

$$r_3[2] = d_3[2] - \left(\pi^2/3\right) d_1[0],$$

$$r_4[2] = d_4[2] - \pi^2 d_2[0],$$

$$r_4[1,1] = d_4[1,1] - \left(5\pi^2/6\right) d_1[0],$$

$$r_4[3] = d_4[3] - \pi^2 d_2[1],$$

$$r_5[2] = d_5[2] - 2\pi^2 d_3[0],$$

$$r_5[1,1] = d_5[1,1] - \left(7\pi^2/3\right) d_2[0],$$

$$r_5[0,2] = d_5[0,2] - \left(\pi^2/2\right) d_1[0],$$

$$r_5[1,0,1] = d_5[1,0,1] - \pi^2 d_1[0],$$

$$r_5[3] = d_5[3] - 2\pi^2 d_3[1],$$

$$r_5[2,1] = d_5[2,1] - 2\pi^2 d_3[0,1] - \left(7\pi^2/3\right) d_2[1],$$

$$r_5[4] = d_5[4] - 2\pi^2 d_3[2] + \left(\pi^4/5\right) d_1[0].$$

The rest terms are the same as its $\{\beta\}$-expanded analogs for Adler function, for e.g.

$$r_1[0] = d_1[0], \quad r_2[1] = d_2[1], \quad r_3[0] = d_3[0], \quad r_4[1] = d_4[1], \quad r_5[0,0,1] = d_5[0,0,1], \ldots$$

R-ratio

$$
\begin{aligned}
R_{NS}(a_s) &= 1 + a_s + \Big(0.6918\beta_0 + \underline{\underline{0.0833}}\Big)a_s^2 \\
&+ \Big(-0.1864\beta_0^2 + 0.6918\beta_1 + 4.9402\beta_0 - \underline{\underline{23.2227}}\Big)a_s^3 \\
&+ \Big(-4.6475\beta_0^3 - 2.0178\beta_0\beta_1 + 16.8766\beta_0^2 + 0.6918\beta_2 + 4.9402\beta_1 \\
&- 101.928\beta_0 + \underline{\underline{81.1571}} + 0.0802 n_f\Big)a_s^4 \\
&+ \Big(-11.0380\beta_0^4 - 23.0458\beta_0^2\beta_1 - 3.6627\beta_0\beta_2 - 1.8314\beta_1^2 + 33.4790\beta_0\beta_1 \\
&+ 0.6918\beta_3 + 4.9402\beta_2 - 101.928\beta_1 \\
&+ \underset{\sim\sim}{r_5[3]}\beta_0^3 + \underset{\sim\sim}{r_5[2]}\beta_0^2 + \underset{\sim\sim}{r_5[1]}\beta_0 + \underset{\sim\sim}{r_5[0]}\Big)a_s^5 + \mathcal{O}(a_s^6)
\end{aligned}
$$

Conclusion

- Two-fold representation for Adler function is considered. It does not contradict but confirms results of $\{\beta\}$-expansion procedure. It is also in correspondence with results of some previous works on the PMC/BLM topic.

- This two-fold representation unambiguously determines the coefficients $D_n^{[r]} \to d_n[r]$

- 8 terms of 12 are defined in $\{\beta\}$-expansion of the five-loop correction to Adler function

- $\{\beta\}$-expanded terms being found can serve as an estimate of the $\mathcal{O}(a_s^5)$ contribution to $D_{NS}(a_s)$

Thank you for your attention!

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