Congratulations!

Many happy returns!

Solving high-energy QCD: 35 years later

or

How far is QCD from maximally supersymmetric Yang-Mills theory?

Gregory Korchemsky

IPhT, Saclay

Advances in Quantum Field Theory, October 12, 2021

35 years ago in Dubna...

A fresh PhD student at LTP, JINR

Joined QCD group

Great advisors: Anatoli Vasilievich Efremov   Anatoli Vladimirovich Radyushkin

Research project: infrared divergences of scattering amplitudes in perturbative QCD

35 years ago in Dubna...

No personel computers, no internet

No arXiv, but great library, limited access to preprints

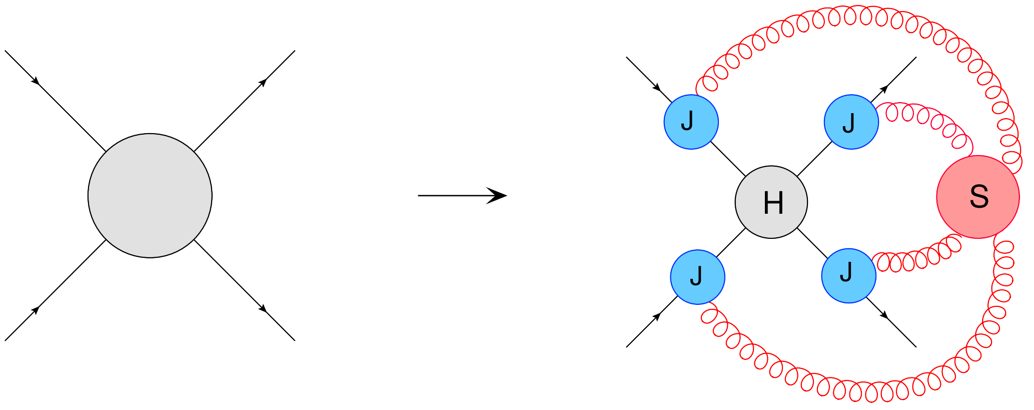
Need permission to make xerox copies

An invaluable source of information: desk in Anatoli Radyushkin's office

Very interesting paper

Scattering amplitudes in gauge theories

Scattering amplitudes at high energy

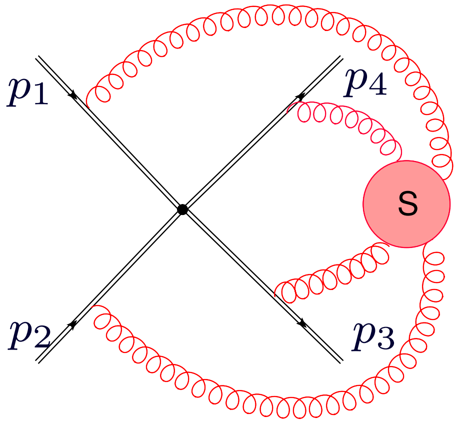


Factorize into a product of hard, jet and soft functions

$$A = H \otimes J \otimes \cdots \otimes J \otimes S$$

Infrared asymptotics is controlled by the soft function

Soft function = correlator of semi-infinite Wilson lines $\Phi_{p_i} = P\exp\left(i\int_0^\infty dt\, p_i \cdot A(p_i t)\right)$



$$= \qquad \langle \Phi_{-p_1} \Phi_{-p_2} \Phi_{p_3} \Phi_{p_4} \rangle$$

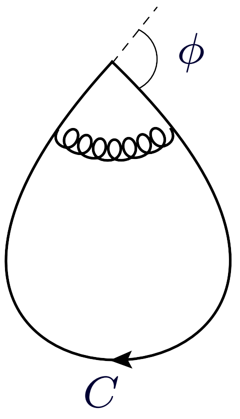
Can we compute the soft function exactly?

Interesting papers on Anatoli's desk

Wilson loops in gauge theories

$$W_C = \frac{1}{N_R}\langle 0|\,\mathrm{tr}_R\, P\exp\left(ig\oint_C dx^\mu A^a_\mu(x)T^a\right)|0\rangle$$

- ✔ Nonlocal gauge invariant functional of the integration contour C
- ✔ Equations of motions in Yang-Mills theories $=$ Loop (Makeenko-Migdal) equations for W_C
- ✔ Cusped Wilson loops develop UV divergences making loop equations inconsistent with quantum corrections [Polyakov'80]

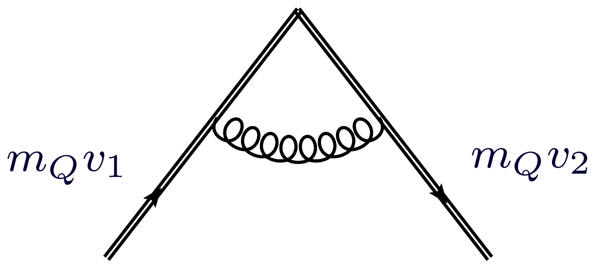
$W_C =$ 

$$\sim g^2 T^a T^a \int ds dt \frac{(\dot{x}(s)\dot{x}(t))}{[(x(s)-x(t))^2]^{1-\epsilon}} \sim \frac{1}{\epsilon} g^2 C_R(\phi\cot\phi - 1)$$

What this has to do with *infrared* divergences of scattering amplitudes?

Cusp anomalous dimension

Scattering of a heavy quark off an external potential $(m_Q \to \infty$ and $(v_1 v_2) = \cos\phi)$



$$\sim g^2_{\mathsf{YM}} C_R \int d^4k \frac{(v_1 v_2)}{k^2 (kv_1)(kv_2)} = -\underbrace{\frac{\alpha_s C_R}{\pi}(\phi\cot\phi - 1)}_{\text{cusp anom.dim.}} \ln\frac{\mu_{\rm UV}}{\mu_{\rm IR}}$$

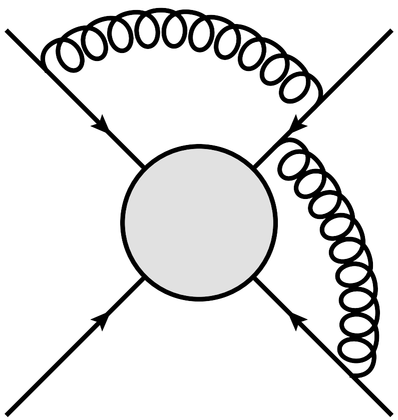
IR and UV divergences are in the one-to-one correspondence [GK,Radyushkin'86]

Renormalization group equation for IR divergences

$$\mu_{\mathsf{IR}} \frac{d}{d\mu_{\mathsf{IR}}} \log A = \mu_{\mathsf{IR}} \frac{d}{d\mu_{\mathsf{IR}}} \log S = \Gamma_{\rm cusp}(\phi, \alpha_s(\mu_{\mathsf{IR}}))$$

Infrared divergences of (planar) scattering amplitudes

$$\mu_{\mathsf{IR}} \frac{d}{d\mu_{\mathsf{IR}}} \log \Big[$$

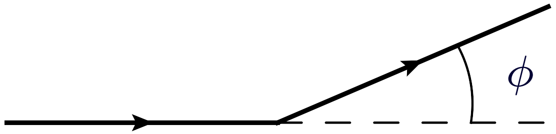


$$\Big] = \sum_i \Gamma_{\rm cusp}(\phi_i, \alpha_s(\mu_{\mathsf{IR}}))\,, \qquad \cos\phi_i = (v_i v_{i+1})$$

Interesting limits/important applications

- ✔ Small Euclidean cusp angle $\phi \to 0$

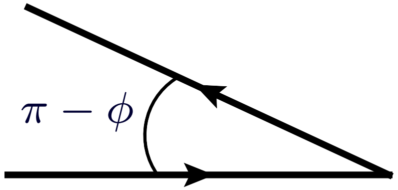
$$\Gamma_{\text{cusp}}(\phi) = -\phi^2 B(\alpha_s) + O(\phi^4),$$



$B(\alpha_s)$ bremsstrahlung function

- ✔ Backtracking Euclidean limit $\phi = \pi - \delta$ with $\delta \to 0$

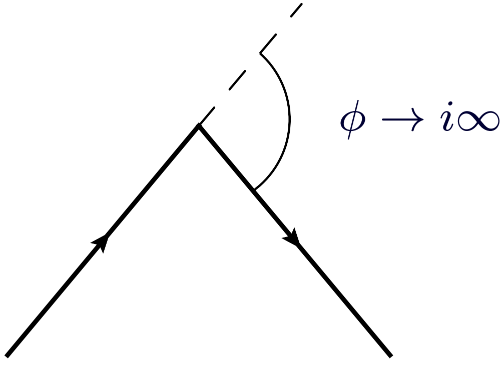
$$\Gamma_{\text{cusp}}(\phi) = -\frac{V(\alpha_s)}{\delta} + O(\delta^0),$$



$V(\alpha_s)$ quark-antiquark potential (up to conformal symmetry breaking corrections)

- ✔ Large Minkowskian cusp angle $x = \mathrm{e}^{i\phi} \to 0$

$$\Gamma_{\text{cusp}}(\phi) = K(\alpha_s) \ln(1/x) + O(x^0),$$



$K(\alpha_s)$ *light-like cusp anomalous dimension*

From QCD to $\mathcal{N}=4$ SYM

Two classes of Yang-Mills theories:

(i) QCD – gauge field coupled to n_f fermions in the fundamental representation of the $SU(N)$

(ii) Supersymmetric extensions – gauge field coupled to interacting n_s scalars and n_f fermions all in the adjoint representation of the $SU(N)$:

$$\mathcal{N}=1: \quad \big(n_f=1, n_s=0\big)$$
$$\mathcal{N}=2: \quad \big(n_f=2, n_s=2\big)$$
$$\mathcal{N}=4: \quad \big(n_f=4, n_s=6\big)$$

$\mathcal{N}=4$ SYM is special:

- ✔ the most (super) symmetric field theory that does not involve gravity
- ✔ exact conformal symmetry, AdS/CFT correspondence, integrability
- ✔ light-like cusp anom.dim. is known for any coupling in planar $\mathcal{N}=4$ SYM

Main question for this talk:

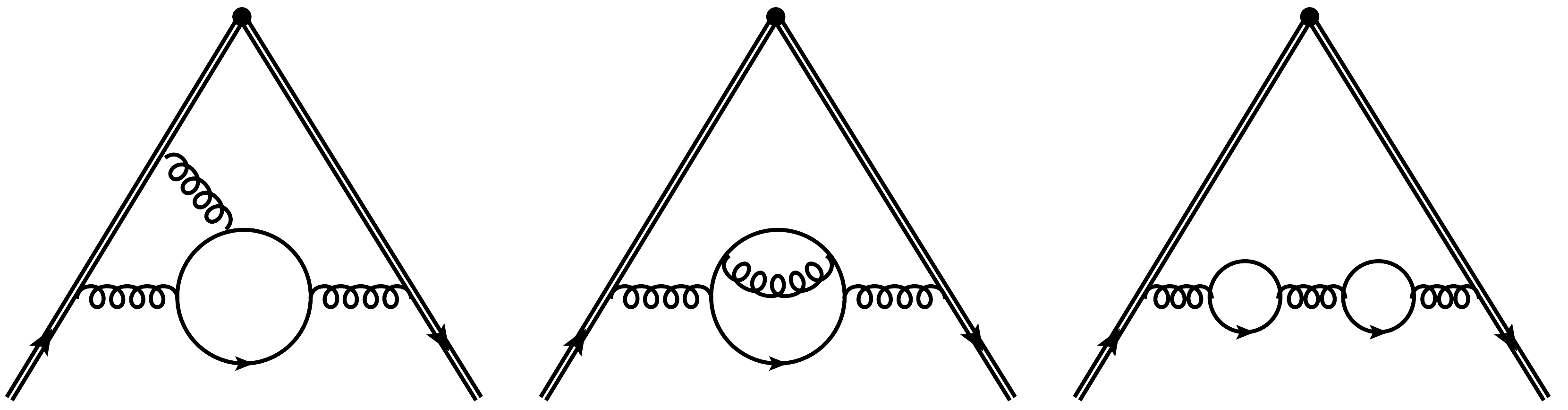
Are there any common properties of the cusp anomalous dimension in QCD and SYM theories?

General properties

General form of the cusp anomalous dimension in QCD

$$\Gamma_{\text{cusp}}(\phi,\alpha_s) = C_R\left[\frac{\alpha_s}{\pi}\gamma + \left(\frac{\alpha_s}{\pi}\right)^2 \left(C_A\gamma_A + T_F n_f \gamma_f\right)\right.$$
$$\left. + \left(\frac{\alpha_s}{\pi}\right)^3 \left(C_A^2\gamma_{AA} + C_A T_F n_f \gamma_{Af} + C_F T_F n_f \gamma_{Ff} + \left(T_F n_f\right)^2 \gamma_{ff}\right)\right] + \mathcal{O}(\alpha_s^4)$$

Sample diagrams contributing to $C_A T_F n_f$, $C_F T_F n_f$ and $(T_F n_f)^2$ terms, respectively



Time scale:

one loop	1980	[Polyakov]
two loops	1987	[GK,Radyushkin]
three loops	2015	[Grozin,Henn,GK,Marquard]
four loops	???	

Two-loop result

$$\Gamma^{\overline{\rm MS}} = \frac{\alpha_s}{\pi} C_R \tilde{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_R \left[\frac{1}{2} C_A \left(\tilde{A}_2 + \tilde{A}_3\right) + \left(\frac{67}{36} C_A - \frac{5}{9} T_F n_f\right) \tilde{A}_1\right]$$

Coefficient functions

$$\tilde{A}_i(x) = A_i(x) - A_i(1)$$

$$A_1(x) = \xi \, \frac{1}{2} H_1(y) \,, \qquad A_2(x) = \left[\frac{\pi^2}{3} + \frac{1}{2} H_{1,1}(y)\right] + \xi \left[-H_{0,1}(y) - \frac{1}{2} H_{1,1}(y)\right] \,,$$

$$A_3(x) = \xi \left[-\frac{\pi^2}{6} H_1(y) - \frac{1}{4} H_{1,1,1}(y)\right] + \xi^2 \left[\frac{1}{2} H_{1,0,1}(y) + \frac{1}{4} H_{1,1,1}(y)\right]$$

Scaling variables

$$x = \mathrm{e}^{i\phi} \,, \qquad \xi = \frac{1+x^2}{1-x^2} = i \cot\phi \,, \qquad y = 1 - x^2$$

The harmonic polylogarithms (HPL)

$$H_1(x) = -\log(1-x) \,, \qquad H_0(x) = \log(x) \,, \qquad H_{-1}(x) = \log(1+x)$$

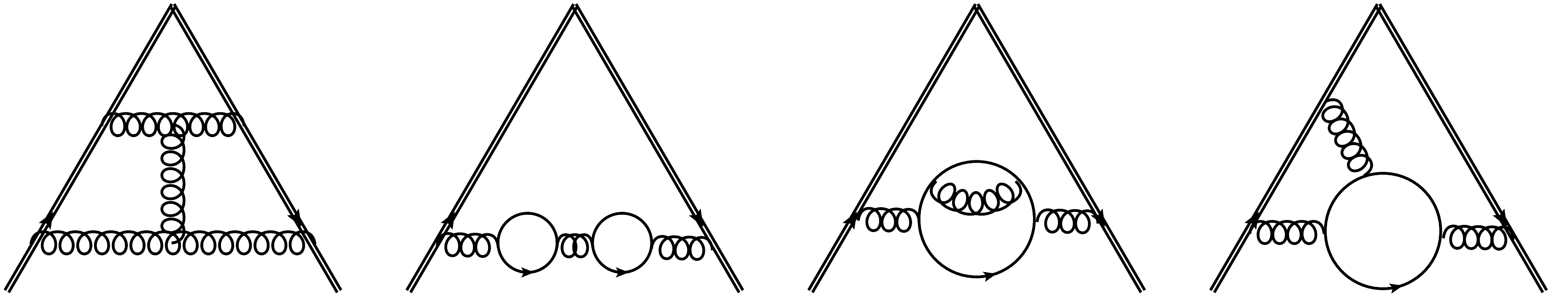
$$H_{a_1,a_2,\ldots,a_n}(y) = \int_0^y \frac{dt}{t+a_1} H_{a_2,\ldots,a_n}(t) dt \,,$$

Assign weight n

Three-loop result

$$\Gamma^{\overline{\text{MS}}} = \cdots + \left(\frac{\alpha_s}{\pi}\right)^3 C_R \left[C_A^2\, \gamma_{AA} + (T_F n_f)^2 \gamma_{ff} + C_F T_F n_f \gamma_{Ff} + C_A T_F n_f \gamma_{Af} \right]$$

Sample diagrams



Coefficient functions

$$\gamma_{AA} = \frac{1}{4}\left(\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3\right) + \frac{67}{36}\tilde{A}_3 + \frac{29}{18}\tilde{A}_2 + \left(\frac{245}{96} + \frac{11}{24}\zeta_3\right)\tilde{A}_1\,,$$

$$\gamma_{ff} = -\frac{1}{27}\tilde{A}_1\,, \qquad \gamma_{Ff} = \left(\zeta_3 - \frac{55}{48}\right)\tilde{A}_1\,,$$

$$\gamma_{Af} = -\frac{5}{9}\left(\tilde{A}_2 + \tilde{A}_3\right) - \frac{1}{6}\left(7\zeta_3 + \frac{209}{36}\right)\tilde{A}_1$$

$\tilde{A}_n$, $\tilde{B}_n$ are linear combinations of HPL's of weight n

Checks of result

✔ Light-like limit $\phi \to i\infty$

$$\Gamma_{\rm cusp}(\phi,\alpha_s) = K(\alpha_s)|\phi| + \mathcal{O}(\phi^0)$$

Light-like cusp anomalous dimension

$$K^{\overline{\rm MS}}_{\rm QCD}(\alpha_s) = C_R\Bigg\{\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left[C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-\frac{5}{9}T_F n_f\right] + \left(\frac{\alpha_s}{\pi}\right)^3\left[C_A^2\left(\frac{245}{96}-\frac{67\pi^2}{216}+\frac{11\pi^4}{720}+\frac{11}{24}\zeta_3\right)-\frac{1}{27}(T_F n_f)^2 + C_A T_F n_f\left(-\frac{209}{216}+\frac{5\pi^2}{54}-\frac{7}{6}\zeta_3\right)+C_F T_F n_f\left(\zeta_3-\frac{55}{48}\right)\right]\Bigg\}.$$

Agrees with the known result [Moch,Vermaseren,Vogt'04]

$$K^{\overline{\rm DR}}_{\mathcal{N}=4}(\alpha_s) = C_R\left[\frac{\alpha_s}{\pi} - \frac{\pi^2}{12}\left(\frac{\alpha_s}{\pi}\right)^2 C_A + \frac{11\pi^4}{720}\left(\frac{\alpha_s}{\pi}\right)^3 C_A^2\right] + \mathcal{O}(\alpha_s^4)\,,$$

✔ Verifies principle of maximal transcendentality

$$K_{\rm QCD}(\alpha_s)\Big|_{\rm maximal\ weight} = K_{\mathcal{N}=4}(\alpha_s)$$

Universal scaling function

- ✔ Introduce a new effective coupling constant

$$a = \frac{\pi}{C_R} K(\alpha_s) = \alpha_s \left[1 + \frac{\alpha_s}{\pi} K^{(2)} + \left(\frac{\alpha_s}{\pi} \right)^2 K^{(3)} + O(\alpha_s^3) \right]$$

  $K^{(2)}$ and $K^{(3)}$ depend on n_f and n_s

- ✔ Define a new function

$$\Omega(\phi, a) := \Gamma_{\text{cusp}}(\phi, \alpha_s)$$

  General form

$$\Omega(\phi, a) = \sum_{k \geq 0} \left(\frac{a}{\pi} \right)^k \Omega^{(k)}(\phi) \, , \qquad \Gamma_{\text{cusp}}(\phi, \alpha_s) = \sum_{k \geq 0} \left(\frac{\alpha_s}{\pi} \right)^k \Gamma^{(k)}(\phi)$$

  Iterative solution

$$\Omega^{(1)} = \Gamma^{(1)}$$

$$\Omega^{(2)} = \Gamma^{(2)} - K^{(2)} \Gamma^{(1)}$$

$$\Omega^{(3)} = \Gamma^{(3)} - K^{(3)} \Gamma^{(1)} - 2K^{(2)} \Gamma^{(2)} + 2(K^{(2)})^2 \Gamma^{(1)}$$

  $K^{(\ell)}$ and $\Gamma^{(\ell)}$ (with $\ell \geq 2$) depend on n_f and n_s

Universal scaling function (2)

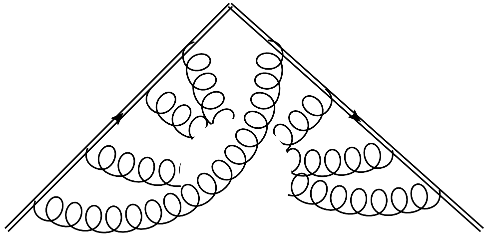
- ✔ $\Omega_{\rm QCD}$ is independent on the number of fermions

- ✔ $\Omega_{\rm SUSY}$ is independent on the number of fermions and scalars

- ✔ *The function $\Omega(\phi, a)$ is the same in any gauge theory to three loops !*

$$\Omega(\phi,a) = C_R\left[\frac{a}{\pi}\tilde{A}_1 + \left(\frac{a}{\pi}\right)^2 \frac{N}{2}\left(\frac{\pi^2}{6}\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3\right)\right.$$
$$\left. + \left(\frac{a}{\pi}\right)^3 \frac{N^2}{4}\left(-\tilde{A}_2 + \tilde{A}_4 + \tilde{A}_5 + \tilde{B}_3 + \tilde{B}_5 - \frac{\pi^4}{180}\tilde{A}_1 + \frac{\pi^2}{3}(\tilde{A}_2 + \tilde{A}_3)\right)\right]$$

- ✔ New effect at four loops – the appearance of quartic Casimirs of the $SU(N_c)$ [Frenkel,Taylor'84]



$$\sim \quad g^8 \frac{d_R^{abcd} d_A^{abcd}}{N_R}\,, \qquad d_R^{abcd} = \frac{1}{4!}\sum_\sigma {\rm tr}[T_R^{\sigma(a)} T_R^{\sigma(b)} T_R^{\sigma(c)} T_R^{\sigma(d)}]$$

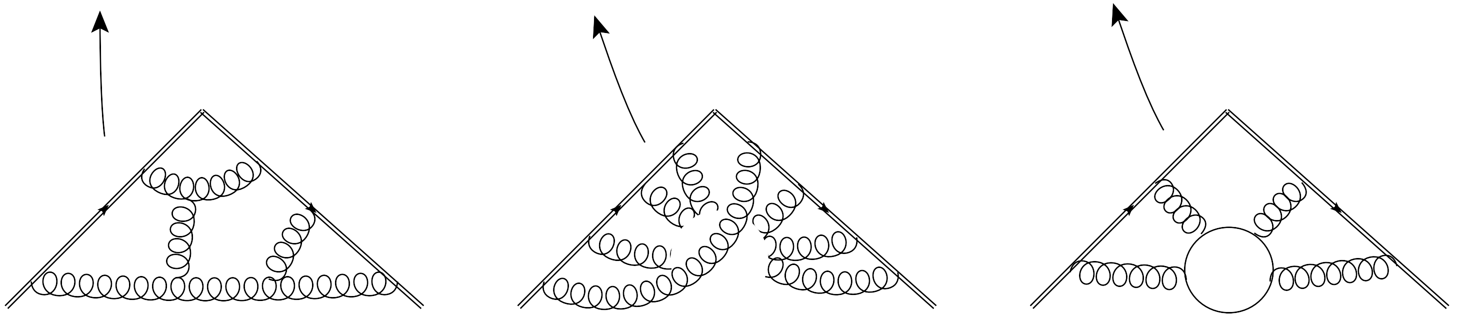
Generates nonplanar corrections, breaks Casimir scaling

Light-like cusp at four loops

Generic Yang-Mills theory with matter

$$K(\alpha_s) = \ldots + \left(\frac{\alpha_s}{\pi}\right)^4 \left[C_R N^3 K^{(4)} + \frac{d_R^{abcd} d_A^{abcd}}{N_R} K_g^{(4)} + \underbrace{\frac{d_R^{abcd} d_f^{abcd}}{N_R} n_f K_f^{(4)} + \frac{d_R^{abcd} d_s^{abcd}}{N_R} n_s K_s^{(4)}}_{\text{[Henn et al 19],[Lee, Smirnov}^2\text{,Steinhauser'19]}} \right]$$

Sample diagrams:



The only missing term in QCD [Henn,GK,Mistlberger'19]

$$K_g^{(4)} = \frac{\zeta_3}{6} - \frac{3\zeta_3^2}{2} + \frac{55\zeta_2}{12} - \frac{\pi^2}{12} - \frac{31\pi^6}{7560}$$

Checked independently [von Manteuffel,Panzer,Schabinger'20]

Full result in $\mathcal{N} = 4$ SYM

$$K(\alpha_s) = \ldots + \left(\frac{\alpha_s}{\pi}\right)^4 \left[-\frac{73\pi^6}{20160} - \frac{\zeta_3^2}{8} - \frac{1}{N^2}\left(\frac{31\pi^6}{5040} + \frac{9\zeta_3^2}{4}\right)\right]$$

Challenge for AdS/CFT integrability approach

Summary and open questions

- ✔ The cusp anomalous dimension is a ubiquitous quantity in gauge theories
- ✔ $\mathcal{N} = 4$ SYM is much closer to QCD than one might expect!
- ✔ What is the reason for universality of the cusp anomalous dimension?
- ✔ Are there other examples of universal quantities?
- ✔ Yet another hint for existence of new structures in QCD