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Sunrise integrals in terms of elliptic polylogarithms

OUTLINE

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Abstract

We consider a set of two-loop sunrise master integrals with two different internal masses at pseudo-threshold kinematics (i.e. $q^2 = m^2$ in Euclidean space) and we solve it including in terms of elliptic polylogarithms to all orders of the dimensional regulator.

0. Introduction

In the last decades much progress has been made in the understanding of the mathematical properties of Feynman integrals. Arguably many of the breakthroughs in this line of research originated from the identification of classes of special functions suited for the solution of Feynman integrals by means of various analytic methods. It is a well-known fact that while many Feynman integrals admit representations in terms of so-called multiple polylogarithms (MPLs) (Goncharov,1998); (Remiddi,Vermaseren,1999).

More recently, the scientific community has centered its attention to the study of Feynman integrals whose geometric properties are defined by elliptic curves. Following early investigations of (Sabry,1962), (Broadhurst,Fleischer,Tarasov,1993), many integrals involving elliptic curves have been computed in the literature (see the recent review in (Weinzierl,2020)).

In a parallel line of research, a class of functions, the so-called Elliptic Multiple Polylogarithms (eMPLs), describing all iterated integrals on the torus has been identified in the mathematics literature
(Brown,Andrey,2011); (Beilinson,Levin,1994); (Levin,Racined,2007).
While these functions formally solve the problem of generalising MPLs to more complicated geometries, their definition is not naturally suited for physical applications. Progress in this direction has been made in (Bloch,Vanhove,2013), (Weinzierl et al.,2013-2020); (Broedel,Duhr,Dulat,Tancredi,2017) [**below (Broedel,2017)**], where eMPLs are defined on the complex plane, and their structure naturally adapts to representations of Feynman integrals commonly used in the physics literature (e.g. Feynman parameters).

Special functions such as MPLs and eMPLs, frequently appear when computing Feynman integrals in dimensional regularisation. More specifically, Feynman integrals admit a Laurent expansion with respect to the dimensional regulator and the coefficients of this expansion can be often computed explicitly in terms of known special functions. In practice it is often possible to truncate the Laurent series, as the computation of physically relevant quantities requires only a few expansion orders. Nonetheless it is interesting to explore the analytic structure of these coefficients at higher orders or, more generally, to all orders of the dimensional regulator.

In this talk we consider a two-loop sunrise integral topology with two internal masses and pseudo-threshold kinematics (Kniehl,Kotikov,Onishchenko,Veretin,2005,2019); (Kalmykov,Kniehl,2008) [**below (Kniehl,2005,2019); (Kalmykov,2008)**].

More precisely, we consider two different internal masses, denoted by m and M, and external kinematics $q^2 = m^2$ (in Euclidean space). This integral family appears when considering non-relativistic limits of Quantum Chromodynamics (NRQCD) and Quantum Electrodynamics (NRQED).

This integral family admits a closed-form solution in terms of ${}_4F_3$-hypergeometric functions, as shown in (Kalmykov,2008) (the corresponding off-shell diagrams with equal masses are considerably more complicated and their explicit solution requires Appell's F_2 hypergeometric functions (Tarasov,2006)).
Moreover, rather similar results (but with $O(\varepsilon)$ accuracy) exist for three-point and four-point two-loop Feynman diagrams in NRQCD kinematics (see (Kniehl,2019)).

Here we consider the two-loop sunrise integral family discussed above and derive results in terms:

- of one- and two-fold integral representations.
- (In the first two ε-orders under consideration) multiple integrals containing the elliptic kernel and logarithms and dilogarithms in the integrands. In more general cases, such multiple integrals containing the elliptic kernel and Goncharov's MPLs in the integrand (see (Besuglov,Onishchenko,Veretin,2020);(Besuglov,2021)). [**below (Besuglov,2020,2021)**].
- of eMPLs (following (Broedel,2017)) valid to all orders of the dimensional regulator.

1. The sunrise integral

We study the sunrise integral topology defined as,

$$J_{i_1,i_2,i_3}(m^2, M^2) = \int \frac{d^D k_1 d^D k_2 \, (\mu^2)^{2\varepsilon}}{[k_2^2 + m^2]^{i_1}[k_1^2 + M^2]^{i_2}[(k_1 - k_2 - q)^2 + M^2]^{i_3}}\Bigg|_{q^2=m^2},$$

with $D = 4 - 2\epsilon$. This integral family has three master integrals, which can be chosen to be $J_{1,1,1}$, $J_{1,1,2}$, $J_{1,2,2}$ and which can be solved in closed form in terms of hypergeometric functions (Kalmykov,2008) as,

$$J_{1,2,2}(m^2,M^2) = -\frac{\hat{N}_1}{M^2}\frac{(1+\epsilon)}{\epsilon(1-\epsilon)} \times \left[\frac{1}{6}{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2},\frac{3+\varepsilon}{2},\frac{3}{2},1 \\ 2-\varepsilon,\frac{5}{4},\frac{7}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right.$$
$$-\left(\frac{M^2}{m^2}\right)^{1-\epsilon}\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},1+\varepsilon,1+\frac{\varepsilon}{2},1 \\ \frac{3-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)$$
$$\left.-\left(\frac{M^2}{m^2}\right)^{-\epsilon}\frac{1-\epsilon}{(2-\epsilon)(3+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{3}{2},1+\varepsilon,\frac{3+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{5+2\varepsilon}{4},\frac{7+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right],$$
$$J_{1,1,2}(m^2,M^2) = \frac{N_1}{2\epsilon(1-\epsilon)} \times \left[{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2},\frac{1+\varepsilon}{2},\frac{1}{2},1 \\ 2-\varepsilon,\frac{3}{4},\frac{5}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right.$$
$$-\left(\frac{M^2}{m^2}\right)^{1-\epsilon}\frac{1}{\epsilon}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},\varepsilon,\frac{\varepsilon}{2},1 \\ \frac{3-\varepsilon}{2},\frac{1+2\varepsilon}{4},\frac{3+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)$$
$$\left.-2\left(\frac{M^2}{m^2}\right)^{-\epsilon}\frac{1-\epsilon}{(2-\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},1+\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right],$$
$$J_{1,1,1}(m^2,M^2) = \frac{-M^2\hat{N}_1}{\epsilon^2(1-\epsilon)} \times \left[{}_4F_3\left(\begin{array}{c} \frac{\varepsilon}{2},\frac{1+\varepsilon}{2},\frac{1}{2},1 \\ 2-\varepsilon,\frac{3}{4},\frac{5}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right.$$
$$+\left(\frac{M^2}{m^2}\right)^{1-\epsilon}\frac{1}{(1-2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon-\frac{1}{2},\varepsilon,\frac{\varepsilon}{2},1 \\ \frac{3-\varepsilon}{2},\frac{1+2\varepsilon}{4},\frac{3+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)$$
$$\left.-\left(\frac{M^2}{m^2}\right)^{-\epsilon}\frac{1-\epsilon}{(2-\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4}\end{array}\middle| -\frac{m^4}{4M^4}\right)\right].$$

The normalization constant is,

$$\hat{N}_1 = \frac{\Gamma^2(1+\epsilon)(\mu^2)^{2\epsilon}}{(m^2M^2)^{\epsilon}}\frac{m^2}{M^2}\,.$$

2. Integral representations

Here we show that the above hypergeometric functions admit one-fold and two-fold integral representations:

$$J_{1,2,2} = \frac{\hat{N}_1}{M^2}[J^{(1)}_{1,2,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,2,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,2,2}(t)]\,,$$
$$J_{1,1,2} = \hat{N}_1[J^{(1)}_{1,1,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,1,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,1,2}(t)]\,,$$
$$J_{1,1,1} = M^2\hat{N}_1[J^{(1)}_{1,1,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,1,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,1,2}(t)]\,,$$

where $t = m^2/(2M^2)$ and

$$J^{(1)}_{1,2,2}(t) = -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{3+\varepsilon}{2}, \frac{3}{2}, 1 \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4}\end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{2+2\epsilon}\varepsilon t^2}\, I^{(1)}_1(t)\,,$$

$$J^{(1)}_{1,1,2}(t) = \frac{1}{2\varepsilon(1-\varepsilon)}\,{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4}\end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{1+2\epsilon}(1-2\varepsilon)\varepsilon t^2}\, I^{(1)}_2(t)\,,$$

$$J^{(1)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(1-\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4}\end{array}\middle| -t^2\right) = -\frac{1}{\varepsilon^2(1-\varepsilon)} - \frac{\hat{K}}{2^{2\epsilon-1}\varepsilon(1-2\varepsilon)^2 t^2}\, \tilde{I}^{(1)}_3(t)\,,$$

$$J^{(2)}_{1,2,2}(t) = \frac{1}{(1+2\varepsilon)(1-\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, 1+\varepsilon, 1+\frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{3+2\varepsilon}{4}, \frac{5+2\varepsilon}{4}\end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{1+4\epsilon}t^{1-\varepsilon}}\, I^{(2)}_1(t)\,,$$

$$J^{(2)}_{1,1,2}(t) = -\frac{1}{2\varepsilon^2(1-\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, \varepsilon, \frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{1+2\varepsilon}{4}, \frac{3+2\varepsilon}{4}\end{array}\middle| -t^2\right) = -\frac{1}{2\varepsilon^2(1-\varepsilon)} + \frac{1}{(1-2\varepsilon)}\,\frac{\hat{K}}{2^{4\epsilon}t^{1-\varepsilon}}\, \tilde{I}^{(2)}_2(t)\,,$$

$$J^{(2)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(1-\varepsilon)(1-2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon-\frac{1}{2},\varepsilon,\frac{\varepsilon}{2},1 \\ \frac{3-\varepsilon}{2},\frac{1+2\varepsilon}{4},\frac{3+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{1}{(1-\varepsilon)(1-2\varepsilon)\varepsilon^2} - \frac{2^{2-4\epsilon}\hat{K}}{(1-2\varepsilon)^2 t^{1-\varepsilon}}\,\tilde{I}^{(2)}_3(t)\,,$$

$$J^{(3)}_{1,2,2}(t) = \frac{1+\varepsilon}{\varepsilon(2-\varepsilon)(3+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{3}{2},1+\varepsilon,\frac{3+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{5+2\varepsilon}{4},\frac{7+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{\hat{K}}{2^{2+4\epsilon}\varepsilon t^2}\,I^{(3)}_1(t)\,,$$

$$J^{(3)}_{1,1,2}(t) = -\frac{1}{\varepsilon(2-\varepsilon)(1+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},1+\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{\hat{K}}{2^{1+4\epsilon}(1-2\varepsilon)\varepsilon t^2}\,I^{(3)}_2(t)\,,$$

$$J^{(3)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(2-\varepsilon)(1+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4} \end{array}\middle| -t^2\right) = \frac{1}{\varepsilon^2(2-\varepsilon)(1+2\varepsilon)} + \frac{2^{1-4\epsilon}\hat{K}}{(1-2\varepsilon)^2\varepsilon t^2}\,\tilde{I}^{(3)}_3(t)\,,$$

where $\hat{K}$ is defined as,

$$\hat{K} = \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+\varepsilon)},$$

The factors $I_j^{(i)}(t)$ and $\tilde{I}_3^{(i)}(t)$ $(j=1,2)$, $(i=1,3)$ are

$$I_j^{(i)}(t) = I_{j,1}^{(i)}(t) - \frac{2\epsilon}{1+i} I_{j,2}^{(1)}(t), \;\; \tilde{I}_3^{(i)}(t) = I_{3,1}^{(i)}(t) - \frac{2\epsilon}{1+i} I_{3,2}^{(i)}(t) \,,$$

with,

$$
\begin{aligned}
I_{1,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \left((p^2t^2+1)^{-\frac{1}{2}} - 1 \right), \\
I_{1,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-1}\, (1-p)^{-\epsilon-\frac{1}{2}}\, (p^2t^2+1)^{-\frac{1}{2}}\, J^{(1)}(q(p)), \\
I_{2,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}} \left((p^2t^2+1)^{\frac{1}{2}} - 1 \right), \\
I_{2,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-2}\, (1-p)^{-\epsilon-\frac{1}{2}}\, (p^2t^2+1)^{\frac{1}{2}}\, J^{(1)}(q(p)), \\
\tilde{I}_{3,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}} \left((p^2t^2+1)^{\frac{1}{2}} - 1 - \frac{(pt)^2}{2} \right), \\
\tilde{I}_{3,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-3}\, (1-p)^{-\epsilon+\frac{1}{2}}\, (p^2t^2+1)^{\frac{1}{2}}\, \tilde{J}^{(1)}(q(p)), \\
I_1^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \left(p^2t^2+1\right)^{-\epsilon-\frac{1}{2}}\, J^{(2)}(pt), \\
I_2^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}} \left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\, J^{(2)}(pt), \\
\tilde{I}_2^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}} \left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\, \tilde{J}^{(2)}(pt), \\
\tilde{I}_3^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}} \left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\, \tilde{J}^{(2)}(pt), \\
I_{1,1}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \left(\left(p^2t^2+1\right)^{-\frac{\epsilon}{2}-\frac{1}{2}} - 1 \right), \\
I_{1,2}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}} \left(p^2t^2+1\right)^{-\frac{\epsilon}{2}-\frac{1}{2}}\, J^{(3)}(q(p)), \\
I_{2,1}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}} \left((p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}} - 1 \right),
\end{aligned}
$$

$$I_{2,2}^{(3)}(t) = \int_0^1 dp\, p^{2\epsilon-2}\,(1-p)^{-\epsilon-\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}}\,J^{(3)}(q(p)),$$
$$\tilde{I}_{3,1}^{(3)}(t) = \int_0^1 dp\, p^{2\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}}\left((p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}} - 1 - (1-\varepsilon)\,\frac{(pt)^2}{2}\right),$$
$$\tilde{I}_{3,2}^{(3)}(t) = \int_0^1 dp\, p^{2\epsilon-3}\,(1-p)^{-\epsilon+\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}}\,\tilde{J}^{(3)}(q(p)),$$

where

$$J^{(1)}(q) = q^{\epsilon}\int_0^q dz\left((1-z)^{-\frac{1}{2}}-1\right)z^{-\epsilon-1},$$
$$J^{(2)}(pt) = \int_0^{pt} dz\, z^{-\epsilon}\left(z^2+1\right)^{\epsilon-\frac{1}{2}},$$
$$J^{(3)}(q) = q^{\frac{\epsilon}{2}}\int_0^q dz\left((1-z)^{-\frac{\epsilon}{2}-\frac{1}{2}}-1\right)z^{-\frac{\epsilon}{2}-1},$$
$$\tilde{J}^{(1)}(q) = J^{(1)}(q) - \frac{(pt)^2}{2(1-\varepsilon)},$$
$$\tilde{J}^{(2)}(pt) = J^{(2)}(pt) - \frac{1}{(p^2t^2+1)^{\frac{1}{2}-\epsilon}}\,\frac{(pt)^{1-\varepsilon}}{1-\varepsilon},$$
$$\tilde{J}^{(3)}(q) = J^{(3)}(q) - \frac{(1+\varepsilon)}{(2-\varepsilon)}(pt)^2$$

and

$$q(p) = \frac{p^2t^2}{p^2t^2+1}.$$

Integrals with tildes are used when the corresponding integrals have singularities for small p values and for small z values. They are constructed from the corresponding integrals by extracting the leading asymptotics of subintegral expressions for small p and for small z, respectively, and, therefore, they are finite.

3. Integral representations (continuation)

Using above avaluations, we have more convenient integral representations.

3.1. $J_{1,2,2}$

$$M^2\, J_{1,2,2} = \hat{N}_1\, \frac{\hat{K}_1}{4t^2}\, \hat{J}_{1,2,2},$$

$$\hat{J}_{1,2,2} = [\frac{1}{\varepsilon} I_1^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I_1^{(3)}(t)]$$

$$= \left[\frac{1}{\varepsilon} I_{1,1}^{(1)}(t) - I_{1,2}^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \left(\frac{t}{2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}\, I_{1,1}^{(3)}(t) - \frac{1}{2}\, I_{1,2}^{(3)}(t)\right)\right],$$

where

$$\hat{K}_1 = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \epsilon\right)\Gamma(\epsilon + 1)} = \frac{\hat{K}}{2^{2\varepsilon}}$$

and $\hat{K}$ is defined above.

3.2. $J_{1,1,1}$ and $J_{1,1,2}$

We split the expressions for $J_{1,1,2}$ and $J_{1,1,1}$ to singular and regular parts:

$$J_{1,1,j} = J^{\text{sing}}_{1,1,j} + J^{reg}_{1,1,j} \quad (j = 1, 2),$$

$$J^{\text{sing}}_{1,1,2} = -\frac{\hat{N}_2}{2(1-\varepsilon)\varepsilon^2}, \quad J^{reg}_{1,1,2} = \frac{\hat{K}_1 \hat{N}_1}{2(1-2\varepsilon)t^2} \hat{J}_{1,1,2},$$

$$\hat{J}^{\text{reg}}_{1,1,2} = [\frac{1}{\varepsilon} I^{(1)}_2(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} \tilde{I}^{(2)}_2(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_2(t)]\,,$$

$$J^{\text{sing}}_{1,1,1} = \left\{\frac{m^2}{\varepsilon^2}\left[\frac{1}{(2-\varepsilon)(1+2\varepsilon)} - \frac{(2t)^{-\varepsilon}}{(1-\varepsilon)}\right] - \frac{M^2}{(1-\varepsilon)(1-2\varepsilon)\varepsilon^2}\right\} \hat{N}_2,$$

$$J^{\text{reg}}_{1,1,1} = -M^2 \frac{2\hat{K}_1 \hat{N}_1}{(1-2\varepsilon)^2 t^2} \hat{J}_{1,1,1},$$

$$\hat{J}_{1,1,1} = [\frac{1}{\varepsilon} \tilde{I}^{(1)}_3(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} \tilde{I}^{(2)}_3(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} \tilde{I}^{(3)}_3(t)]\,,$$

where the new normalization constant is

$$\hat{N}_2 = (2t)^{\varepsilon-1}\,\hat{N}_1 = \frac{\Gamma^2(1+\varepsilon)(\mu^2)^{2\varepsilon}}{(M^2)^{2\varepsilon}}\,.$$

3.3. Example: the first hypergeometric function in $J_{1,2,2}$.

The first ${}_4F_3$-hypergeometric function of $J_{1,2,2}$, i.e.

$$J^{(1)}_{1,2,2}(t) = -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,{}_4F_3\left(\left.\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{3+\varepsilon}{2}, \frac{3}{2}, 1 \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{array}\right| - t^2\right) \equiv -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,F^{(1)}_1(t)\,,$$

admits the following series representation,

$$F^{(1)}_1(t) \equiv {}_4F_3\left(\left.\begin{array}{c} 1, \frac{3}{2}, 1+\frac{\varepsilon}{2}, \frac{3}{2}+\frac{\varepsilon}{2} \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{array}\right| - t^2\right),$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})\Gamma(m+1+\frac{\varepsilon}{2})\Gamma(m+\frac{3}{2}+\frac{\varepsilon}{2})}{\Gamma(m+2-\varepsilon)\Gamma(m+\frac{5}{4})\Gamma(m+\frac{7}{4})}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{4})\Gamma(\frac{7}{4})}{\Gamma(\frac{3}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(\frac{3}{2}+\frac{\varepsilon}{2})}\,(-t^2)^m\,,$$

where $t = m^2/(2M^2)$. The product $\Gamma(\alpha)\Gamma(1/2+\alpha)$ can be written as,

$$\Gamma(\alpha)\Gamma(1/2+\alpha) \;=\; 2^{1-2\alpha}\,\sqrt{\pi}\,\Gamma(2\alpha)\,,$$

which results in the following simplified expression for $F_1^{(1)}(t)$,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})\Gamma(2m+2+\varepsilon)}{\Gamma(m+2-\varepsilon)\Gamma(2m+\frac{5}{2})}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\varepsilon)}\,(-t^2)^m\,.$$

It is convenient to use the following integral representations for the ratio of gamma functions,

$$\frac{\Gamma(2m+2+\varepsilon)}{\Gamma(2m+\frac{5}{2})} = \int_0^1 dp\,\frac{p^{2m+1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\,.$$

We find,

$$F_1^{(1)}(t) = \int_0^1 dp\,\frac{p^{1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\varepsilon)}\,(-(tp)^2)^m\,.$$

In order to proceed with our analysis it is convenient to consider first the series on the right hand side in the limit $\varepsilon = 0$,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{(m+1)!}\left(-(tp)^2\right)^m = \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!}\left(-(tp)^2\right)^{m-1}$$

$$= -\frac{\Gamma(\frac{1}{2})}{(tp)^2}\left[\frac{1}{\left(1+t^2p^2\right)^{1/2}} - 1\right]$$

In the general case we have,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)}\left(-(tp)^2\right)^m = \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1-\varepsilon)}\left(-(tp)^2\right)^{m-1}$$

$$= -\frac{\Gamma(\frac{1}{2})}{\Gamma(1-\varepsilon)(tp)^2}\left[{}_2F_1\left(1, \frac{1}{2}; 1-\varepsilon; -p^2t^2\right) - 1\right]$$

Using standard properties of the ${}_2F_1$-function,

$$ {}_2F_1(a,b;c;z) = (1-z)^b \, {}_2F_1\left(c-a, b; c; \frac{z-1}{z}\right) $$

we obtain,

$$ {}_2F_1\left(1, \frac{1}{2}; 1-\varepsilon; -p^2t^2\right) = \frac{1}{\left(1+t^2p^2\right)^{1/2}} \, {}_2F_1\left(\frac{1}{2}, -\varepsilon; 1-\varepsilon; q(p)\right) \, , $$

where $q = p^2t^2/(1+p^2t^2)$ and

$$ {}_2F_1\left(\frac{1}{2}, -\varepsilon; 1-\varepsilon; q\right) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})} \frac{-\varepsilon}{m-\varepsilon} q^m = 1 - \varepsilon \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})} \frac{q^m}{m-\varepsilon} \, . $$

Using the integral representation for the factor $1/(m-\varepsilon) = \int_0^1 dz \, z^{m-1-\varepsilon}$,

we have,

$$ {}_2F_1\left(\frac{1}{2}, -\varepsilon; 1-\varepsilon; q\right) = 1 - \varepsilon \int_0^1 \frac{dz}{z^{1+\varepsilon}} \left[\frac{1}{\sqrt{1-zq}} - 1\right] $$

$$ = 1 - \varepsilon \int_0^q \frac{dz_1 q^{\varepsilon}}{z_1^{1+\varepsilon}} \left[\frac{1}{\sqrt{1-z_1}} - 1\right] \equiv 1 - \varepsilon J^{(1)}(q) \, . $$

Combining these results, we have

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)} \left(-(tp)^2\right)^m = -\frac{\Gamma(\frac{1}{2})}{\Gamma(1-\varepsilon)(tp)^2}\, f_1^{(1)}(pt)\,,$$

where

$$f_1^{(1)}(pt) = \frac{1}{\left(1+t^2p^2\right)^{1/2}} \left\{1-\varepsilon J^{(1)}(q)\right\} - 1\,.$$

The final result for $F_1^{(1)}(t)$ reads,

$$F_1^{(1)}(t) = -\frac{3(1-\varepsilon)}{2(1+\varepsilon)t^2}\,\hat{K}_1\, I_1^{(1)}(t)\,, \quad I_1^{(1)}(t) = \int_0^1 dp\, p^{\varepsilon-1}(1-p)^{-1/2-\varepsilon}\, f_1^{(1)}(pt)\,,$$

where the normalization $\hat{K}_1$ was determined above. The elliptic structure is carried by the product $(1-p)^{-1/2-\varepsilon}(1+t^2p^2)^{-1/2}$.

4. Leading terms of the ε-expansion and one-fold integrals

Here we derive a one-fold integral representation for the first two orders of the ϵ-expansion of $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$. The first two ϵ orders considered here can be expressed as one-fold integrals over logarithms and dilogarithms (in General Goncharov MPLs as it was shown in (Besuglov,2020,2021)) with algebraic prefactors. A similar analysis shows that to arbitrary order of the dimensional regulator the result is in terms of one-fold integrals over higher weight MPLs.

4.1. Inner integrals

We start by considering integral $J^{(2)}(p)$, at order ε^0,

$$J^{(2)}(pt, \varepsilon = 0) = \int_0^{tp} \frac{ds}{\sqrt{1+s^2}}\,,$$

which can be evaluated directly by means of the variable change,

$$s_2 = \frac{\sqrt{1+s^2} - s}{\sqrt{1+s^2} + s}\,,$$

leading to,

$$J^{(2)}(pt, \varepsilon = 0) = \frac{1}{2} \int_{R_2}^{1} \frac{ds_2}{s_2} = -\frac{1}{2} \log R_2 \equiv J_0^{(2)}(p)\,,$$

with

$$R_2 = \frac{\sqrt{1+t^2p^2} - tp}{\sqrt{1+t^2p^2} + tp} = \frac{1-\sqrt{q}}{1+\sqrt{q}}\,.$$

By means of the same variable change, we evaluate the next ε order,

$$J^{(2)}(pt) = \frac{1}{2^{1+\varepsilon}} \int_{R_2}^{1} \frac{ds_2}{s_2^{1+\varepsilon/2}} \frac{(1+s_2)^{2\varepsilon}}{(1-s_2)^{2\varepsilon}} = J_0^{(2)}(pt) + \varepsilon J_1^{(2)}(pt) + O(\varepsilon^2)\,,$$

where $J_0^{(2)}(pt)$ is given above and,

$$J_1^{(2)}(p) = \frac{1}{8}\log^2 R_2 + \zeta_2 + \mathrm{Li}_2(-R_2) - \frac{1}{2}\mathrm{Li}_2(R_2)\,.$$

We now consider integrals $J^{(1)}(p)$ and $J^{(3)}(p)$ at order ε^0,

$$J^{(1)}(p,\varepsilon=0) = J^{(3)}(p,\varepsilon=0) = \int_0^q \frac{dz}{z}\left(\frac{1}{\sqrt{1-z}} - 1\right)\,.$$

By introducing a regulator δ we have,

$$\int_\delta^y \frac{dz}{z} = \log q - \log\delta\,,$$

while the remaining term can be evaluated by the variable change,

$$z = 1 - s^2, \quad s = \frac{(1-s_1)}{(1+s_1)}\,,$$

and,

$$\int_{\delta}^{q} \frac{dz}{z\sqrt{1-z}} = \int_{\delta/4}^{R_1} \frac{ds_1}{s_1} = \log R_1 - \log\frac{\delta}{4}\,, \qquad R_1 = \frac{1-\sqrt{1-q}}{1+\sqrt{1-q}}\,.$$

The full result can be written as,

$$J^{(i)}(p,\varepsilon=0) = \log(4R_1) - \log q = \log\frac{4R_1}{q} \equiv J_0^{(i)}(p), \qquad (i=1,3)\,.$$

At the next order we have,

$$J^{(i)}(p) = J_0^{(i)}(p) + \varepsilon J_1^{(i)}(p) + O(\varepsilon^2), \quad (i = 1, 3)\,,$$

where $J_0^{(1)}(p) = J_0^{(3)}(p)$ are given

$$J_1^{(1)}(p) = \overline{J}_1^{(1)}(p) - 2\mathrm{Li}_2(-R_1),$$
$$J_1^{(3)}(p) = \frac{1}{2}\overline{J}_1^{(1)}(p) + 2\mathrm{Li}_2(R_1) - 4\mathrm{Li}_2(-R_1)\,,$$

with,

$$\overline{J}_1^{(1)}(p) = \log q \log(4R_1) - \frac{1}{2}\log^2 q - \log 4 \log R_1\,.$$

4.2. Results for the sunrise integrals

In order to obtain the ε-expansions of $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$ up to and including $O(\varepsilon)$, we use the expressions for integrals $I^{(i)}(t)$ $(i=1,3)$ and their integral representations. We have, for example, for $J_{1,2,2}$

$$I^{(1)}_{1,1}(t) = \int_0^1 \frac{dp}{p\sqrt{1-p}} \left(\frac{1}{\sqrt{1+p^2t^2}} - 1 \right) \left[1 + \varepsilon\, l_1 + \frac{\varepsilon^2}{2}\, l_1^2 \right] + O(\varepsilon^2)\,,$$

$$I^{(1)}_{1,2}(t) = \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(1)} + \varepsilon\,(l_1 J_0^{(1)} + J_1^{(1)}) \right] + O(\varepsilon^2),$$

$$\left(\frac{t^2}{2}\right)^{\varepsilon} I^{(2)}_{1}(t) = \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(2)} + \varepsilon\,(l_2 J_0^{(2)} + J_1^{(2)}) \right] + O(\varepsilon^2)\,,$$

$$\left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_{1,1}(t) = \int_0^1 \frac{dp}{p\sqrt{1-p}} \left(\frac{1}{\sqrt{1+p^2t^2}} \left[1 + \varepsilon\, l_{32} + \frac{\varepsilon^2}{2}\, l_{32}^2 \right] - \left[1 + \varepsilon\, l_{31} + \frac{\varepsilon^2}{2}\, l_{31}^2 \right] \right) + O(\varepsilon^2)\,,$$

$$\left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_{1,2}(t) = \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(3)} + \varepsilon\,(l_{32} J_0^{(3)} + J_1^{(3)}) \right] + O(\varepsilon^2)\,,$$

where,

$$l_1 = \log\left(\frac{p}{1-p}\right), \quad l_2 = \log\left(\frac{p^3t^2}{2(1-p)(1+p^2t^2)}\right) = \log\left(\frac{pq}{2(1-p)}\right),$$

$$l_{31} = \log\left(\frac{p^2t}{2(1-p)}\right), \quad l_{32} = \log\left(\frac{p^2t}{2(1-p)\sqrt{1+p^2t^2}}\right) = \log\left(\frac{p\sqrt{q}}{2(1-p)}\right),$$

with $J_0^{(i)}$ and $J_1^{(i)}$ $(i=1,2,3)$ given above.

Combining all terms, we obtain the following finite expression,

$$\hat{J}_{1,2,2} = \int_0^1 \frac{dp}{p\sqrt{1-p}} \left[b_0 + \frac{1}{\sqrt{1+p^2t^2}} B_0 + \varepsilon \left(b_1 + \frac{1}{\sqrt{1+p^2t^2}} B_1 \right) \right] + O(\varepsilon^2)\,,$$

$$\hat{J}^{\text{reg}}_{1,1,2} = \int_0^1 \frac{dp}{p^2\sqrt{1-p}} \left[b_0 - (pt) + \frac{1}{\sqrt{1+p^2t^2}} B_0 + \varepsilon \Big(b_1 - (pt)(1+l_{31}) + \frac{1}{\sqrt{1+p^2t^2}} B_1 \Big) \right] + O(\varepsilon^2)\,,$$

$$\hat{J}^{\text{reg}}_{1,1,1} = \int_0^1 \frac{dp\sqrt{1-p}}{p^3} \Bigg[b_0 - (pt) + \frac{(pt)^2}{4}(2b_0-1) + \frac{1}{\sqrt{1+p^2t^2}} B_0$$
$$+\varepsilon \Big(b_1 - (pt)(1+l_{31}) + \frac{(pt)^2}{4} \left[b_0^2 - 3b_0 + \frac{1}{2} + l_1(2b_0-1) \right] + \frac{1}{\sqrt{1+p^2t^2}} B_1 \Big) \Bigg] + O(\varepsilon^2)\,,$$

where,

$$b_0 = \log\left(\frac{pt}{2}\right), \quad B_0 = \log(R_1R_2), \quad b_1 = \log\left(\frac{pt}{2}\right)\log\left(\frac{p^3t}{2(1-p)^2}\right),$$
$$B_1 = J_1^{(2)} + \text{Li}_2(R_1) + \frac{1}{4}\log^2(R_1) - \frac{1}{2}\log(R_1R_2)\log\left(\frac{pq}{4(1-p)}\right)$$
$$+\frac{1}{4}\log\left(\frac{q}{4}\right)\log(R_1) - \frac{1}{8}\log^2\left(\frac{q}{4}\right) - \log^2 2\,,$$

where $J_1^{(2)}$ is given above as

$$J_1^{(2)}(p) = \frac{1}{8}\log^2 R_2 + \zeta_2 + \text{Li}_2(-R_2) - \frac{1}{2}\text{Li}_2(R_2)\,.$$

Up to $O(\varepsilon^0)$, the results are in full agreement with (Kniehl,2019). Here, however, the results are given up to $O(\varepsilon^1)$.

5. All orders result in terms of elliptic polylogarithms

In this section we derive eMPL representations for the sunrise integrals $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$ valid to all orders of the dimensional regulator. Specifically, we start with a short review of eMPLs (following to (Brodel,2017)) , discussing their definition and the basic analytic properties.

We do not discuss the general structure of the integral representations (it was presented in (CMK,2020)) but we show a simple example by calculation of the integral $I_2^{(1)}$.

We express all considered integrals in terms of eMPLs to all orders of the dimensional regulator, and present our final results.

5.1. Elliptic polylogarithms

We are interested in the computation of iterated integrals of the form,

$$\int_0^x dx_1 R_1(x_1, y(x_1)) \int_0^{x_1} dx_2 R_2(x_2, y(x_2)) \ldots \int_0^{x_{n-1}} dx_n R_n(x_n, y(x_n)),$$

where R_i are rational functions of their arguments and $y(x)$ is an elliptic curve,

$$y(x) = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)},$$

All iterated integrals can be expressed in terms of eMPLs. In the complex plane, eMPLs are defined as

$$E_4\begin{pmatrix} n_1, ..., n_k \\ c_1, ..., c_k \end{pmatrix}; x = \int_0^x dt\, \varphi_{n_1}(c_1, t)\, E_4\begin{pmatrix} n_2, ..., n_k \\ c_2, ..., c_k \end{pmatrix}; t$$

with $n_i \in Z$ and $c_i \in C$.

Elliptic polylogarithms are a generalisation of ordinary multiple polylogarithms (MPLs), defined recursively as,

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t),$$

with $G(; x) \equiv 1$ and,

$$G(\vec{0}, x) \equiv \frac{\log(x)^n}{n!} \, .$$

By definition we see that MPLs are a subset of eMPLs,

$$E_4 \left(\begin{matrix} 1, \ldots, 1 \\ c_1, \ldots, c_n \end{matrix} ; x \right) = G(c_1, c_2, \ldots, c_n; x) \, ,$$

where $c_i \neq \infty$.

The recursion starts at $E_4(\ ; x) = 1$. By construction, the kernels $\varphi_n(c, x)$ have at most simple poles, and they are (see (Broedel,2017) for a detailed discussion)

$$
\begin{aligned}
&\varphi_0(0, x) = \frac{c_4}{y(x)}\,, \\
&\varphi_1(c, x) = \frac{1}{x-c}\,, \qquad \varphi_{-1}(c, x) = \frac{y(c)}{(x-c)y(x)} - (\delta_{c0} + \delta_{c1})\frac{1}{x-c}\,, \\
&\varphi_{-1}(\infty, x) = \frac{x}{y(x)}\,, \qquad \varphi_1(\infty, x) = \frac{c_4}{y(x)}\, Z_4(x)\,, \\
&\varphi_n(\infty, x) = \frac{c_4}{y(x)}\, Z_4^{(n)}(x)\,, \qquad \varphi_{-n}(\infty, x) = \frac{x}{y(x)}\, Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,, \\
&\varphi_n(c, x) = \frac{1}{x-c}\, Z_4^{(n-1)}(x) - \delta_{n2}\, \Phi_4(x)\,, \\
&\varphi_{-n}(c, x) = \frac{y(c)}{(x-c)y(x)}\, Z_4^{(n-1)}(x)\,, \quad (n > 1),
\end{aligned}
$$

where $y(c)$ and c_4 are independent of x with,

$$
c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}} \quad \text{with} \quad a_{ij} = a_i - a_j\,.
$$

Moreover we define,

$$E_4\left(\begin{matrix}\vec{1}\\ \vec{0}\end{matrix};x\right) \equiv \frac{\log(x)^n}{n!},$$

where $\vec{1}$ and $\vec{0}$ are vectors with entries equal to 1 and 0 respectively, and $n = \text{length}(\vec{1}) = \text{length}(\vec{0})$. The function $Z_4(x)$ is defined by first introducing an auxiliary function $\Phi_4(x)$,

$$\Phi_4(x) \equiv \widetilde{\Phi}_4(x) + 4c_4\frac{\eta_1}{\omega_1}\frac{1}{y} = \frac{1}{c_4\, y}\left(x^2 - \frac{s_1}{2}x + \frac{s_2}{6}\right) + 4c_4\frac{\eta_1}{\omega_1}\frac{1}{y}\,,$$

where $\eta_1, \omega_1, s_1, s_2, c_4$ are independent of x and they are defined in (Broedel,2017), and,

$$\widetilde{\Phi}_4(x) = \frac{1}{c_4 y}\left(x^2 - \frac{s_1 x}{2} + \frac{s_2}{6}\right)\,,$$

whose primitive is,

$$Z_4(x) = \int_{a_1}^{x} dt\, \Phi_4(t)\,.$$

In the next sections we will see that, in our integral representations, the function $\Phi_4(x)$ appears only in the last (outer) integration, and only the case $Z_4^{(1)}(x) = Z_4(x)$ need to be considered.

As for all iterated integrals, eMPLs satisfy a shuffle algebra, with the shuffle product defined as,

$$E_4\left(\begin{matrix} a_1, \ldots, a_n \\ a'_1, \ldots, a'_n \end{matrix}; x\right) E_4\left(\begin{matrix} b_1, \ldots, b_m \\ b'_1, \ldots, b'_m \end{matrix}; x\right) = \sum_{\vec{c}=\vec{a}\sqcup\!\sqcup\vec{b}} E_4\left(\begin{matrix} c_1, \ldots, c_{n+m} \\ c'_1, \ldots, c'_{n+m} \end{matrix}; x\right) .$$

The vector $\vec{c}$ is the vector obtained by performing all the shuffles of $\vec{a}$ and $\vec{b}$, preserving the ordering of the elements of $\vec{a}$ and $\vec{b}$ respectively.

5.2. Regularisation

As we will see in the next sections we are interested in computing definite integrals of the form,

$$\int_0^1 f(x)dx = F(1) - F(0), \quad \frac{\partial F(x)}{\partial x} = f(x)\,.$$

In some cases individual functions inside the primitive develop divergences when evaluated at the integration bounds, and in order to compute the definite integral one needs to perform two limits,

$$\int_0^1 f(x)dx = \lim_{x\to 1} F(x) - \lim_{x\to 0} F(x) \equiv \mathrm{Reg}_{0,1}F(x).$$

5.3. Elliptic polylogarithms and all orders result

Contrary to general cases considered in [CMK] we consider here an example which clear up the obtained results.

5.3.1. Example

We show how the solution strategy of the previous section works in practice by considering integral $I_1^{(2)}(t)$. The dependence on the elliptic curve is made explicit by applying the variable change,

$$I_1^{(2)}(t) = \int_0^1 dx \, \frac{2}{t\left(1-x^2\right)y(x)} \left(\frac{\left(1-x^2\right)^3}{t^2x^2y(x)^2}\right)^{\varepsilon} J^{(2)}(x)$$

where

$$p(x) = 1 - x^2, \quad 0 < x < 1; \quad y(x) = \sqrt{\frac{1}{t^2} + (1-x^2)^2} \equiv \frac{1}{t}\sqrt{1 + t^2p^2}\,.$$

The inner integral can be expressed as

$$J^{(2)}(x) = -\int_1^x dz_1 \frac{2z_1}{y(z_1)} \left(\frac{ty^2(z_1)}{1-z_1^2} \right)^{\varepsilon} .$$

All the ε-powers can be expanded in ε

$$\left(\frac{ty^2(x)}{1-x^2} \right)^{\varepsilon} = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \log^i \left(\frac{ty^2(x)}{1-x^2} \right) ,$$

The resulting logarithm can be expressed in terms of eMPLs,

$$\log \left(\frac{ty^2(x)}{1-x^2} \right) = \log(ty^2(0))) + \int_0^x dz \, \frac{d}{dz} \, \log \left(\frac{ty^2(z)}{1-z^2} \right)$$

$$= \log\left(t^2+1\right) - \log(t) + \int_0^x dz \frac{2z \left(t^2 \left(z^2-1\right)^2 - 1\right)}{t^2 \left(z^2-1\right) y(z)^2} .$$

The integrand above can be written in terms of the integration kernels as,

$$\frac{2z\left(t^2\left(z^2-1\right)^2-1\right)}{t^2\left(z^2-1\right)y(z)^2}=\sum_{i=1}^{4}\varphi_1\left(a_i,z\right)-\varphi_1(-1,z)-\varphi_1(1,z)\,,$$

where we denoted with a_i the four roots of the elliptic curve,

$$a_1=-\frac{\sqrt{t-i}}{\sqrt{t}}\,,\ a_2=\frac{\sqrt{t-i}}{\sqrt{t}}\,,\ a_3=-\frac{\sqrt{t+i}}{\sqrt{t}}\,,\ a_4=\frac{\sqrt{t+i}}{\sqrt{t}}\,.$$

Upon integration we find,

$$L_4\equiv\log\left(\frac{ty^2(x)}{1-x^2}\right)=\sum_{i=1}^{4}E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right)-E_4\left(\begin{matrix}1\\-1\end{matrix};x\right)-E_4\left(\begin{matrix}1\\1\end{matrix};x\right)$$
$$+\log\left(t^2+1\right)-\log(t)\,.$$

The prefactor is

$$-\frac{2z_1}{y(z_1)} = -2\varphi_{-1}(\infty, z_1) \equiv k_5(z_1)$$

and, thus, the inner integral has the following form

$$- \int_1^x dz_1 \frac{2z_1}{y(z_1)} \left(\frac{ty^2(z_1)}{1-z_1^2}\right)^{\varepsilon}$$
$$= \int_1^x dz_1\, k_5(z_1) \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} L_4^j(z_1) \equiv \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \left[K_5 * L_4^j\right]_1^x ,$$

where the primitive K_5 is

$$K_5 = -2E_4\left(\begin{matrix} -1 \\ \infty \end{matrix}; x\right) .$$

The evalution of the rest is very siimilar and we have

$$I_1^{(2)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * L_5^i \left[K_5 * L_4^j\right]_1^x ,$$

where,

$$\begin{aligned} L_5 = & -\sum_{i=1}^{4} E_4\left(\begin{matrix} 1 \\ a_i \end{matrix}; x\right) + 3E_4\left(\begin{matrix} 1 \\ -1 \end{matrix}; x\right) - 2E_4\left(\begin{matrix} 1 \\ 0 \end{matrix}; x\right) + E_4\left(\begin{matrix} 1 \\ 1 \end{matrix}; x\right) \\ & - \log\left(t^2+1\right), \\ K_4 = & \sum_{i=1}^{4} E_4\left(\begin{matrix} -1 \\ -1 \end{matrix}; x\right) - E_4\left(\begin{matrix} -1 \\ 1 \end{matrix}; x\right) - E_4\left(\begin{matrix} 1 \\ 1 \end{matrix}; x\right). \end{aligned}$$

5.4. Sunsets

By applying the procedure described above we obtain one of the main results of this paper, i.e. an explicit expression for the above integrals in terms of eMPLs valid to all orders of the dimensional regulator.

5.4.1. $J_{1,2,2}$

We obtain the following expression for $J_{1,2,2}$,

$$M^2 \, J_{1,2,2} = \hat{N}_1 \, \frac{\hat{K}_1}{4t^2} \, \hat{J}_{1,2,2},$$

$$\hat{J}_{1,2,2} = [\frac{1}{\varepsilon} I_1^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I_1^{(3)}(t)]$$

$$= [\frac{1}{\varepsilon} I_{1,1}^{(1)}(t) - I_{1,2}^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \left(\frac{t}{2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon} \, I_{1,1}^{(3)}(t) - \frac{1}{2} \, I_{1,2}^{(3)}(t)\right)] \, .$$

where,

$$
\begin{aligned}
I^{(1)}_{1,1}(t) &= \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_1 * L_1^i, \quad ,\\
I^{(1)}_{1,2}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_2 * L_3^i \left[K_3 * L_2^j\right]_1^x ,\\
I^{(2)}_{1}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * L_5^i \left[K_5 * L_4^j\right]_1^x ,\\
I^{(3)}_{1,1}(t) &= \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_6 * L_6^i + \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_4 * L_7^i ,\\
I^{(3)}_{1,2}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * L_8^i \left[K_9 * L_4^j\right]_1^x\\
&\qquad + \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * L_8^i \left[K_8 * L_9^j\right]_1^x ,
\end{aligned}
$$

where K_i and L_i are depth one eMPLs.

Since the integrals $I_1^{(2)}(t)$ and $I_1^{(3)}(t)$ contribute to $\hat{J}_{1,2,2}$ with the corresponding factors, it is convenient to present also

$$\begin{aligned}
\left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * \hat{L}_5^i \left[K_5 * L_4^j\right]_1^x \,, \\
\left(\frac{t}{2}\right)^{\varepsilon} I_{1,1}^{(3)}(t) &= \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^{i}}{i!} K_6 * \hat{L}_6^i + \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^{i}}{i!} K_4 * \hat{L}_7^i \,, \\
\left(\frac{t}{2}\right)^{\varepsilon} I_{1,2}^{(3)}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_9 * L_4^j\right]_1^x \\
&\quad + \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_8 * L_9^j\right]_1^x \,,
\end{aligned}$$

where,

$$\hat{L}_5 = L_5 + 2\log t - \log 2, \quad \hat{L}_k = L_k + \log t - \log 2 \quad (k = 6, 7, 8)\,.$$

5.4.2. Results for L_i and K_i

Here we provide the definitions for the eMPLs expressions

$$L_1 = E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\0\end{matrix};x\right) + E_4\left(\begin{matrix}1\\1\end{matrix};x\right),$$

$$L_2 = \sum_{i=1}^{4} E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\1\end{matrix};x\right) + \log\left(t^2+1\right) - 2\log(t),$$

$$L_3 = -\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\0\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\1\end{matrix};x\right)$$
$$- \log\left(t^2+1\right) + 2\log(t),$$

$$L_4 = \sum_{i=1}^{4} E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right) - E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\1\end{matrix};x\right) + \log\left(t^2+1\right) - \log(t),$$

$$L_5 = -\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\0\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\1\end{matrix};x\right)$$
$$- \log\left(t^2+1\right),$$

$$L_6 = 2E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\0\end{matrix};x\right) + 2E_4\left(\begin{matrix}1\\1\end{matrix};x\right),$$

$$L_7 = -\frac{1}{2}\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\a_i\end{matrix};x\right) + 2E_4\left(\begin{matrix}1\\-1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\0\end{matrix};x\right) + 2E_4\left(\begin{matrix}1\\1\end{matrix};x\right)$$

$$-\frac{1}{2}\log\left(t^2+1\right),$$

$$L_8 = -\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ 0\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right)$$

$$-\log\left(t^2+1\right) + \log(t),$$

$$L_9 = \frac{1}{2}\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right) + \frac{1}{2}\log\left(t^2+1\right) - \log(t).$$

while the primitives of the relevant integration kernels are defined as,

$$K_1 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right),$$

$$K_2 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right),$$

$$K_3 = \sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) + 2E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) + 2E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right),$$

$$K_4 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right),$$

$$K_5 = -2E_4\left(\begin{matrix}-1\\ \infty\end{matrix};x\right),$$

$$K_6 = E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix} - E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix},$$

$$K_7 = E_4\begin{pmatrix} -1 & \\ & ;x \\ -1 & \end{pmatrix} - E_4\begin{pmatrix} -1 & \\ & ;x \\ 1 & \end{pmatrix} - E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix},$$

$$K_8 = -\sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix},$$

$$K_9 = 2E_4\begin{pmatrix} -1 & \\ & ;x \\ -1 & \end{pmatrix} + 2E_4\begin{pmatrix} -1 & \\ & ;x \\ 1 & \end{pmatrix} + 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix}.$$

We see that the integral $J_{1,2,2}$ in the form of the elliptic integrals is finite since $K_1 = K_4 + K_6$ and,

$$I^{(1)}_{1,1}(t) - \left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_{1,1}(t) = \mathrm{Reg}_{0,1} \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} K_1 * (L_1^i - \hat{L}_6^i)$$
$$+\mathrm{Reg}_{0,1} \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} K_2 * (L_1^i - \hat{L}_7^i) \sim O(\varepsilon).$$

5.5. $J_{1,1,1}$ and $J_{1,1,2}$

We obtain the following expression for $J_{1,1,2}$ and $J_{1,1,1}$,

$$J_{1,1,j} = J_{1,1,j}^{\text{sing}} + J_{1,1,j}^{reg} \quad (j = 1, 2),$$

$$J_{1,1,2}^{\text{sing}} = -\frac{\hat{N}_2}{2(1-\varepsilon)\varepsilon^2}, \quad J_{1,1,2}^{reg} = \frac{\hat{K}_1\,\hat{N}_1}{2(1-2\varepsilon)t^2}\,\hat{J}_{1,1,2},$$

$$\hat{J}_{1,1,2}^{\text{reg}} = [\frac{1}{\varepsilon}I_2^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon}\tilde{I}_2^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon}I_2^{(3)}(t)]\,,$$

$$J_{1,1,1}^{\text{sing}} = \left\{\frac{m^2}{\varepsilon^2}\left[\frac{1}{(2-\varepsilon)(1+2\varepsilon)} - \frac{(2t)^{-\varepsilon}}{(1-\varepsilon)}\right] - \frac{M^2}{(1-\varepsilon)(1-2\varepsilon)\varepsilon^2}\right\}\hat{N}_2,$$

$$J_{1,1,1}^{\text{reg}} = -M^2\,\frac{2\hat{K}_1\,\hat{N}_1}{(1-2\varepsilon)^2t^2}\,\hat{J}_{1,1,1},$$

$$\hat{J}_{1,1,1} = [\frac{1}{\varepsilon}\tilde{I}_3^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon}\tilde{I}_3^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon}\tilde{I}_3^{(3)}(t)]\,,$$

where

$$\hat{N}_2 = (2t)^{\varepsilon-1}\,\hat{N}_1 = \frac{\Gamma^2(1+\varepsilon)(\mu^2)^{2\varepsilon}}{(M^2)^{2\varepsilon}}\,.$$

Results for $I_2^{(i)}(t)$, $\tilde{I}_2^{(2)}(t)$, $\tilde{I}_3^{(2)}(t)$ and $\tilde{I}_3^{(i)}(t)$ $(i=1,3)$ can be expressed in terms of eMPLs by using the strategy of the previous sections and by applying integration by parts identities (see (Broedel,2017)). However, upon variable change, one encounters kernels of the form $\frac{x^a}{y(x)}$ with $a>2$, which render the integration by parts identities quite cumbersome. In [CMK] it was shown that one can reduce these integrals to the computation of integrals $I_{21}^{(j)}(t)$ and $I_{31}^{(j)}(t)$ $(j=1,2,3)$ which require considering only kernels with at most simple poles, and $\frac{x^2}{y(x)}$, which has a double pole at infinity.

5.5.1. Improved representations

Here we restrict our analysis by consideration of the integral $I_2^{(1)}(t)$ only (other results can be found in [CMK]). It shows the inportance interation by part procedure to get results easier.

Consider the integrals $I_1^{(1)}(t)$ and $I_2^{(1)}(t)$, which are rather similar:

$$I_1^{(1)}(t) = \int_0^1 \frac{dp}{p^{1-\varepsilon}(1-p)^{1/2+\varepsilon}} f_1^{(1)}(pt), \quad f_1^{(1)}(pt) = \frac{1}{(1+t^2p^2)^{1/2}} \left\{1 - \varepsilon J^{(1)}(q)\right\},$$

$$I_2^{(1)}(t) = \int_0^1 \frac{dp}{p^{2-\varepsilon}(1-p)^{1/2+\varepsilon}} f_2^{(1)}(pt), \quad f_2^{(1)}(pt) = \left(1+t^2p^2\right)^{1/2} \left\{1 - \varepsilon J^{(1)}(q)\right\}.$$

Using the following property,

$$\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p},$$

we can split the result for $I_2^{(1)}(t)$ as the combination of two parts:

$$I_2^{(1)}(t) = \int_0^1 \frac{dp}{p^{1-\varepsilon}(1-p)^{1/2+\varepsilon}} f_2^{(1)}(pt)$$
$$+ \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{2-\varepsilon}} f_2^{(1)}(pt) = \hat{I}_{21}^{(1)}(t) + \hat{I}_{22}^{(1)}(t).$$

We note that,

$$f_2^{(1)}(pt) = f_1^{(1)}(pt) + \frac{t^2p^2}{\left(1+t^2p^2\right)^{1/2}} \left\{1 - \varepsilon J^{(1)}(q)\right\} .$$

Then,

$$\hat{I}_{21}^{(1)}(t) = I_1^{(1)}(t) + t^2 \int_0^1 \frac{dp\, p^{1+\varepsilon}(1-p)^{1/2-\varepsilon}}{\left(1+t^2p^2\right)^{1/2}} \left\{1 - \varepsilon J^{(1)}(q)\right\} = I_1^{(1)}(t) + t^2\, I_{21}^{(1)}(t) ,$$

$$\hat{I}_{22}^{(1)}(t) = \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{2-\varepsilon}} f_1^{(1)}(p) + t^2 \int_0^1 \frac{dp\, p^{\varepsilon}(1-p)^{1/2-\varepsilon}}{\left(1+t^2p^2\right)^{1/2}} \left\{1 - \varepsilon J^{(1)}(q)\right\}$$

$$= \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{2-\varepsilon}} f_1^{(1)}(p) + t^2\, I_{22}^{(1)}(t) .$$

We now integrate $\hat{I}_{22}^{(1)}(t)$ by parts as,

$$\int_0^1 \frac{dp}{p^{\alpha+1}}\, F(p) = -\frac{1}{\alpha}\left[\frac{F(p)}{p^\alpha}\,\Big|_0^1 - \int_0^1 \frac{dp}{p^\alpha}\,\frac{dF(p)}{dp}\right],$$

where $F(p)$ is some function and $\alpha = 1-\varepsilon$. We have,

$$\hat{I}_{22}^{(1)}(t) = -\frac{1}{1-\varepsilon}\,\Big[\frac{(1-p)^{1/2-\varepsilon}}{p^{1-\varepsilon}}\, f_2^{(1)}(p)\,\Big|_0^1 + \left(\frac{1}{2}-\varepsilon\right)\, \hat{I}_{21}^{(1)}(t)$$
$$-t^2\, I_{21}^{(1)}(t) + \varepsilon \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{1-\varepsilon}\left(1+t^2p^2\right)^{-1/2}}\,\frac{dJ^{(1)}(q)}{dp}\Big],$$

where,

$$\frac{(1-p)^{1/2-\varepsilon}}{p^{1-\varepsilon}}\, f_2^{(1)}(p)\,\Big|_0^1 = 0\,,$$

and,

$$\frac{dJ^{(1)}(q)}{dp} = \frac{2}{p(1+p^2t^2)}[\varepsilon J^{(1)}(q) + \sqrt{1+p^2t^2} - 1] = -\frac{2}{p(1+p^2t^2)^{1/2}}\, f_1^{(1)}(p)\,.$$

So, we have

$$\varepsilon \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{1-\varepsilon}\left(1+t^2p^2\right)^{-1/2}} \frac{dJ^{(1)}(q)}{dp} = -2\varepsilon \int_0^1 \frac{dp\,(1-p)^{1/2-\varepsilon}}{p^{2-\varepsilon}} f_1^{(1)}(p)$$

$$= -2\varepsilon \left[\hat{I}_{22}^{(1)}(t) - t^2\, I_{22}^{(1)}(t)\right],$$

and, the r.h.s. takes the following form,

$$-\frac{1}{1-\varepsilon}\,[\left(\frac{1}{2}-\varepsilon\right)\,\hat{I}_{21}^{(1)}(t) - (1-2\varepsilon)\,t^2 I_{22}^{(1)}(t) - 2\varepsilon\,\hat{I}_{22}^{(1)}(t)],$$

which leads to,

$$\hat{I}_{22}^{(1)}(t) = \frac{1-2\varepsilon}{2(1-3\varepsilon)}\,[-\hat{I}_{21}^{(1)}(t) + 2t^2 I_{22}^{(1)}(t)].$$

Therefore, integral $I_2^{(1)}(t)$ can be represented as,

$$I_2^{(1)}(t) = \frac{1}{2(1-3\varepsilon)} \left[(1-4\varepsilon)\left(I_1^{(1)}(t) + t^2 I_{21}^{(1)}(t)\right) + 2(1-2\varepsilon)t^2\, I_{22}^{(1)}(t)\right].$$

After the replacement $p = 1 - x^2$, we have,

$$I_{21}^{(1)}(t) \sim \int_0^1 \frac{dp\, p^{\varepsilon}(1-p)^{1/2-\varepsilon}}{\left(1+t^2p^2\right)^{1/2}} = 2 \int_0^1 \frac{dx\, x^2}{ty}\left[\frac{1-x^2}{x^2}\right]^{\varepsilon},$$

$$I_{22}^{(1)}(t) \sim \int_0^1 \frac{dp\, p^{1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\left(1+t^2p^2\right)^{1/2}} = 2 \int_0^1 \frac{dx\,(1-x^2)}{ty}\left[\frac{1-x^2}{x^2}\right]^{\varepsilon}.$$

5.5.2. The results

The results take the form, $(i = 1, 3)$

$$I_2^{(i)}(t) = \frac{1}{2(1-3\varepsilon)}\left[(1-4\varepsilon)\, I_1^{(i)}(t) - t^2 I_{21}^{(i)}(t) + 2(1-2\varepsilon)t^2 I_{31}^{(i)}(t)\right],$$

$$\tilde{I}_3^{(i)}(t) = \frac{(1-2\varepsilon)}{2(2-3\varepsilon)}\left[2t^2 \left(I_1^{(i)}(t) - I_{31}^{(i)}(t)\right) - I_2^{(i)}(t) + C_3^{(i)}\, t^2\right],$$

where,

$$C_3^{(1)} = \frac{5-6\varepsilon}{1-\varepsilon}\,\frac{1}{\hat{K}_1}, \quad C_3^{(3)} = \frac{2(5-4\varepsilon)}{(1+2\varepsilon)(2-\varepsilon)}\,\frac{1}{\hat{K}_2},$$

and,

$$I_{21}^{(1)}(t) = \int_0^1 \frac{dp\, p^{1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\left(1+t^2p^2\right)^{(1/2)}} \left\{1 - \varepsilon J^{(1)}(q)\right\} ,$$

$$I_{31}^{(1)}(t) = \int_0^1 \frac{dp\, p^{\varepsilon}(1-p)^{-1/2-\varepsilon}}{\left(1+t^2p^2\right)^{1/2}} \left\{1 - \varepsilon J^{(1)}(y)\right\} ,$$

and,

$$I_{21}^{(3)}(t) = \int_0^1 \frac{dp\, p^{1+2\varepsilon}(1-p)^{-1/2-\varepsilon}}{\left(1+t^2p^2\right)^{(1+\varepsilon)/2}} \left\{1 - \frac{\varepsilon}{2} J^{(3)}(q)\right\} ,$$

$$I_{31}^{(3)}(t) = \int_0^1 \frac{dp\, p^{2\varepsilon}(1-p)^{-1/2-\varepsilon}}{\left(1+t^2p^2\right)^{(1+\varepsilon)/2}} \left\{1 - \frac{\varepsilon}{2} J^{(3)}(y)\right\} .$$

The results for $I_2^{(2)}(t)$ and $I_3^{(2)}(t)$ are,

$$\tilde{I}_2^{(2)}(t) = \frac{1}{2(1-3\varepsilon)} \left[(1-4\varepsilon)\, I_1^{(2)}(t) - t^2\, I_{21}^{(2)}(t) - \frac{2(1-4\varepsilon) t^{1-\varepsilon}}{(1-\varepsilon)(1+2\varepsilon)} \frac{1}{\hat{K}_2} \right.$$
$$\left. + 2(1-2\varepsilon)\, t^2 I_{31}^{(2)}(t) \right],$$
$$\tilde{I}_3^{(2)}(t) = \frac{1-2\varepsilon}{2(2-3\varepsilon)} \left[2t^2 \left(I_1^{(2)}(t) - I_{31}^{(2)}(t) \right) - \tilde{I}_2^{(2)}(t) \right],$$

where,

$$I_{21}^{(2)}(t) = \int_0^1 \frac{dp\, p^{1+3\varepsilon}}{(1-p)^{1/2+\varepsilon} \left(1+t^2p^2\right)^{1/2+\varepsilon}} J^{(2)}(tp),$$
$$I_{31}^{(2)}(t) = \int_0^1 \frac{dp\, p^{3\varepsilon}}{(1-p)^{1/2+\varepsilon} \left(1+t^2p^2\right)^{1/2+\varepsilon}} J^{(2)}(p).$$

For integrals $I_j^{(i)}(t)$ with $(j = 21, 31)$ and $(i = 1, 2, 3)$ we have,

$$I_j^{(1)}(t) = \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_j * L_1^i - \varepsilon \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_j * L_3^i \left[K_3 * L_2^j\right]_1^x ,$$

$$\left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_j * \hat{L}_5^i \left[K_5 * L_4^j\right]_1^x ,$$

$$\left(\frac{t}{2}\right)^{\varepsilon} I_j^{(3)}(t) = \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_j * \hat{L}_6^i$$

$$-\frac{\varepsilon}{2} (\mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_9 * L_4^j\right]_1^x$$

$$+\mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_8 * L_9^j\right]_1^x) ,$$

where K_i and L_i are depth one eMPLs:

$$K_{31} = \frac{1}{c_4} E_4\left(\begin{matrix} 0 \\ 0 \end{matrix}; x\right),$$

$$K_{21} = \frac{1}{c_4} E_4\left(\begin{matrix} 0 \\ 0 \end{matrix}; x\right) - K_{21}^{(a)} - K_{21}^{(b)},$$

$$K_{21}^{(a)} = \frac{s_1}{2} E_4\left(\begin{matrix} -1 \\ \infty \end{matrix}; x\right) - \left(\frac{s_2}{6c_4} + \frac{4c_4\eta_1}{\omega_1}\right) E_4\left(\begin{matrix} 0 \\ 0 \end{matrix}; x\right), \quad K_{21}^{(b)} = c_4\, \hat{Z}_4(x),$$

where,

$$\hat{Z}_4(x) * L_a^i(x) \equiv \int dx\, \Phi_4(x)\, L_a^i(x) = Z_4(x) \star L_a^i(x)\,,$$

where the definition of the product of the $\star$-operator for a eMPL is provided in the next subsection.

The finiteness of $\hat{J}^{\text{reg}}_{1,1,2}$ and $\hat{J}^{\text{reg}}_{1,1,1}$ can be shown in sam way as it was done for $\hat{J}_{1,2,2}$.

5.5.3. Definition of the $\star$-operator

In order to obtain an eMPLs expression with the $\star$-operator we need to perform integrals of the form:

$$Z_4(x) \star E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right) \equiv \int dx\, \Phi_4(x)\, E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right) .$$

When $E_4 = 1$, the $\star$-operator is trivial. For more complicated cases, we can use the integration by parts. We have,

$$\int dx\, \Phi_4(x)\, E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right) = Z_4(x)\, E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right)$$

$$- \int dx\, Z_4(x) \frac{\partial}{\partial x}\, E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right)$$

$$= Z_4(x)\, E_4\left(\begin{matrix} a_1\, a_2 \ldots \\ b_1\, b_2 \ldots \end{matrix}; x\right) - \int dx\, Z_4(x)\, \phi_{a_1}(b_1, x)\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right) .$$

By noticing that $Z_4^{(1)}(x) = Z_4(x)$ and that $Z_4^{(n)}(x)$ appears only in the last integration, we have the following possible cases.

1. Let $a_1 = b_1 = 0$, then,

$$Z_4(x)\,\phi_0(0,x) = \frac{c_4 Z_4(x)}{y(x)} = \phi_1(\infty, x),$$

and, thus, the last term in r.h.s. evaluates to,

$$\int dx\,\phi_1(\infty,x)\,E_4\left(\begin{matrix} a_2 \dots \\ b_2 \dots \end{matrix}\,;x\right) = E_4\left(\begin{matrix} 1\,a_2 \dots \\ \infty\,b_2 \dots \end{matrix}\,;x\right).$$

2. Let $a_1 = 1$ and $b_1 = c$, where c is some constant. Then we have,

$$Z_4(x)\,\phi_1(c,x) = \frac{Z_4(x)}{x-c} = \phi_2(c,x) + \Phi_4(x)\,.$$

Thus, for the last term in r.h.s. we have the following results,

$$\int dx\,Z_4(x)\,\phi_1(c,x)\,E_4\left(\begin{matrix} a_2 \dots \\ b_2 \dots \end{matrix}\,;x\right) = E_4\left(\begin{matrix} 2\,a_2 \dots \\ c\,b_2 \dots \end{matrix}\,;x\right) + \int dx\,\Phi_4(x)\,E_4\left(\begin{matrix} a_2 \dots \\ b_2 \dots \end{matrix}\,;x\right).$$

3. Let $a_1 = -1$ and $b_1 = c$, where c is also some constant. Then we have,

$$Z_4(x)\,\phi_{-1}(c,x) = \frac{Z_4(x)y(c)}{(x-c)y(x)} - \left(\delta_c^0 + \delta_c^1\right)\frac{Z_4(x)}{x-c}\,,$$

where δ_c^0 and δ_c^1 are Cronecker symbols. We have,

$$Z_4(x)\,\phi_{-1}(c,x) = \phi_{-2}(c,x) - \left(\delta_c^0 + \delta_c^1\right)\left[\phi_2(c,x) + \Phi_4(x)\right]\,.$$

Thus, for the last term in r.h.s. we have the following results,

$$\begin{aligned}\int dx\, Z_4(x)\,\phi_{-1}(c,x)\,E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix};x\right) &= E_4\left(\begin{matrix} -2\,a_2 \ldots \\ c\,b_2 \ldots \end{matrix};x\right) \\ - \left(\delta_c^0 + \delta_c^1\right)\left[E_4\left(\begin{matrix} 2\,a_2 \ldots \\ c\,b_2 \ldots \end{matrix};x\right) + \int dx\, \Phi_4(x) E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix};x\right)\right]\,.\end{aligned}$$

Therefore, in cases **2** and **3** we expressed the initial integral as a combination of eMPLs and integrals with lower depth.

Therefore these integrals can be expressed fully in terms of eMPLs by iterating the application of the IBP identities and by using the definition of $Z_4(x)$.

4. Let $a_1 = -1$ and $b_1 = \infty$, then we have,

$$Z_4(x)\,\phi_{-1}(\infty, x) = \frac{x Z_4(x)}{y(x)} = \phi_{-2}(\infty, x) + \frac{1}{c_4}\,.$$

Thus, for the last term in r.h.s. we have the following results,

$$\int dx\, Z_4(x)\,\phi_{-1}(\infty, x)\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right) = E_4\left(\begin{matrix} -2\, a_2 \ldots \\ \infty\, b_2 \ldots \end{matrix}; x\right) + \frac{1}{c_4}\int dx\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right)$$

The last term in the r.h.s. can be transformed by integration by part as,

$$\int dx\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right) = (x+d)\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right) - \int dx\, (x+d)\,\phi_{a_2}(b_2, x)\, E_4\left(\begin{matrix} a_2 \ldots \\ b_2 \ldots \end{matrix}; x\right),$$

where d is arbitrary constant.

Finally, the following cases need to be considered.

4.1. Let $a_2 = b_2 = 0$, then in the case $d = 0$ we have,

$$\frac{c_4 x}{y(x)} = \phi_{-1}(\infty, x),$$

and the last term in r.h.s. evaluates to,

$$\int dx\, E_4\left(\begin{matrix} 0, \ , a_3 \ldots \\ 0, \ b_3 \ldots \end{matrix} ; x\right) = E_4\left(\begin{matrix} -1, \ 0, \ a_3 \ldots \\ \infty, \ 0, \ b_2 \ldots \end{matrix} ; x\right).$$

4.2. Let $a_2 = 1$ and $b_2 = c_1$, where c_1 is some constant. Then, we have, when $d = -c_1$,

$$\frac{x+d}{x-c_1} = 1\,.$$

Then, the last term in r.h.s. has the following form,

$$\int dx\, E_4\left(\begin{matrix} 1, \ a_3 \ldots \\ c_1, \ b_3 \ldots \end{matrix} ; x\right) = (x - c_1)\, E_4\left(\begin{matrix} 1\, a_3 \ldots \\ c_1, \ b_2 \ldots \end{matrix} ; x\right) - \int dx\, E_4\left(\begin{matrix} a_3 \ldots \\ b_3 \ldots \end{matrix} ; x\right).$$

As in cases **2** and **3** we expressed reduced integrals in the r.h.s. as a combination of eMPLs and lower depth integrals. The iterative application of the IBP procedure terminates with,

$$\int dx = x + d_1\,,$$

where d is arbitrary constant.

4.3. Let $a_2 = -1$ and $b_2 = c_2$, where c_1 is some constant. Then, we have, when $d = -c_2$,

$$\frac{x+d}{(x-c_2)y(x)} = \frac{1}{y(x)} = \frac{1}{c_4}\,\phi_0(0,x),$$

and, thus, the last term in r.h.s. has the closed form,

$$\int dx\, E_4\left(\begin{matrix} -1,\ a_3\ldots \\ c_2,\ b_3\ldots \end{matrix};x\right)(x-c_2)\,E_4\left(\begin{matrix} -1,\ a_2\ldots \\ c_2,\ b_2\ldots \end{matrix};x\right) - \frac{1}{c_4}\,E_4\left(\begin{matrix} 0,\ a_3\ldots \\ 0,\ b_3\ldots \end{matrix};x\right).$$

4.4. Let $a_2 = -1$ and $b_2 = \infty$, then in the case $d = 0$, we have,

$$\frac{(x+d)x}{y(x)} = \frac{x^2}{y(x)} = c_4\Phi_4(x) + \frac{s_1}{2}\,\frac{x}{y(x)} - \frac{s_2}{6}\,\frac{1}{y(x)} - 4c_4^2\frac{\eta_1}{\omega_1}\,\frac{1}{y(x)},$$

and, thus, the last term in r.h.s. has the following form,

$$\int dx\, E_4\begin{pmatrix} -1, \; a_3 \dots \\ \infty, \; b_3 \dots \end{pmatrix} ; x = \left(x + \frac{s_1}{2}\right)\; E_4\begin{pmatrix} -1, \; a_3 \dots \\ \infty, \; b_3 \dots \end{pmatrix} ; x$$
$$- \left(\frac{s_2}{6} + 4c_4^2 \frac{\eta_1}{\omega_1}\right) E_4\begin{pmatrix} 0, \; a_3 \dots \\ 0, \; b_3 \dots \end{pmatrix} ; x$$
$$+ c_4 \int dx\, \Phi(x)\, E_4\begin{pmatrix} a_3 \dots \\ b_3 \dots \end{pmatrix} ; x \; .$$

Therefore, we reduced any integral to the computation of integrals of the same form with lower depth. Iterating the procedure several times, we can compute any integral in terms of eMPLs.

6. Conclusions

In this paper we studied a family of sunrise integrals with two different internal masses and pseudo-threshold kinematics in dimensional regularisation. These integrals admit a closed-form solution in terms of hypergeometric functions (Kalmykov,2008)
and we use this representation as the starting point of our analysis.

- In particular, we show that all corresponding hypergeometric functions can be represented in terms of one- and two-fold integral representations.
- In each ε-order, these representations can be represented as multiple integrals containing the elliptic kernel and Goncharov's MPLs in their integrands (see (Besuglov; 2020,2021)). In the first two ε-orders under consideration, there are only logarithms and dilogarithms.
- Moreover, integral representations make it possible to represent them as a combination of eMPLs, which were obtained using the procedure (Brodel,2017).