Non-compact duality, super-Weyl invariance and effective actions

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Based on:

- SMK, JHEP **07**, 222 (2020) [arXiv:2006.00966 [hep-th]].

What this talk is about

- This talk is about the implications of (i) (super) Weyl invariance and (ii) $\mathrm{SL}(2,\mathbb{R})$ duality invariance on the structure of effective actions in supersymmetric theories with local couplings.
- Related earlier works:
  H. Osborn, *Local couplings and Sl(2,R) invariance for gauge theories at one loop*, Phys. Lett. B **561**, 174 (2003);
  I. L. Buchbinder, N. G. Pletnev and A. A. Tseytlin, *Induced N=4 conformal supergravity*, Phys. Lett. B **717**, 274 (2012).

Outline

1. U(1) duality in (nonlinear) electrodynamics
2. Local couplings in electrodynamics and $SL(2, \mathbb{R})$ duality
3. Local couplings in supersymmetric electrodynamics
4. Open problems

Electromagnetic duality: Maxwell's theory

- Maxwell's electrodynamics is the simplest and oldest example of a duality-invariant theory in four spacetime dimensions.

$$L_{\rm Maxwell}(F) = -\frac{1}{4}F^{ab}F_{ab} = \frac{1}{2}\big(\vec{E}^2 - \vec{B}^2\big) \ , \qquad F_{ab} = \partial_a A_b - \partial_b A_a$$

- The Bianchi identity and the equation of motion read

$$\partial^b \widetilde{F}_{ab} = 0 \ , \qquad \partial^b F_{ab} = 0$$

where $\widetilde{F}_{ab} := \frac{1}{2}\,\varepsilon_{abcd}\,F^{cd}$ is the Hodge dual of F.

- Since these differential equations have the same functional form, one may consider so-called duality rotations

$$F + {\rm i}\widetilde{F} \to {\rm e}^{{\rm i}\varphi}\big(F + {\rm i}\widetilde{F}\big) \quad \Longleftrightarrow \quad \vec{E} + {\rm i}\vec{B} \to {\rm e}^{{\rm i}\varphi}\big(\vec{E} + {\rm i}\vec{B}\big) \ , \quad \varphi \in \mathbb{R}$$

- Lagrangian $L_{\rm Maxwell}(F)$ changes, but the energy-momentum tensor

$$T^{ab} = \frac{1}{2}\big(F + {\rm i}\widetilde{F}\big)^{ac}\big(F - {\rm i}\widetilde{F}\big)^{bd}\,\eta_{cd} = F^{ac}F^{bd}\eta_{cd} - \frac{1}{4}\eta^{ab}F^{cd}F_{cd}$$

is invariant under U(1) duality transformations.

Electromagnetic duality: Nonlinear electrodynamics

- U(1) duality invariance of Born-Infeld theory — Schrödinger (1935)
- Patterns of duality invariance in extended supergravity

  Ferrara, Scherk & Zumino (1977)
  Cremmer & Julia (1979)
- General theory of duality invariance in four dimensions

  Gaillard & Zumino (1981)
  Gibbons & Rasheed (1995)
  Gaillard & Zumino (1997)
- General theory of duality invariance in higher dimensions

  Gibbons & Rasheed (1995)
  Araki &Tanii (1999)
  Aschieri, Brace, Morariu & Zumino (2000)
- General theory of duality invariance for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric nonlinear electrodynamics

  SMK & Theisen (2000)

  Partial SUSY breaking often implies U(1) duality invariance
- Duality invariance as manifest U(1) invariance of self-interaction

  Ivanov & Zupnik (2001,2002)

U(1) duality in nonlinear electrodynamics

- Nonlinear electrodynamics

$$L(F_{ab}) = -\frac{1}{4} F^{ab} F_{ab} + \mathcal{O}(F^4)$$

- Using the definition

$$\widetilde{G}_{ab}(F) := \frac{1}{2}\,\varepsilon_{abcd}\,G^{cd}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}\;, \qquad G(F) = \widetilde{F} + \mathcal{O}(F^3),$$

  the Bianchi identity (BI) and the equation of motion (EoM) read

$$\partial^b \widetilde{F}_{ab} = 0\;, \qquad \partial^b \widetilde{G}_{ab} = 0\;.$$

- The same functional form of BI and EOM gives us a rationale to introduce a duality transformation

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$$

  For $G'(F')$ one should require

$$\widetilde{G}'_{ab}(F') \;=\; 2\,\frac{\partial L'(F')}{\partial F'^{ab}}$$

  Transformed Lagrangian, $L'(F)$, always exists.

U(1) duality in nonlinear electrodynamics

The above considerations become nontrivial if the model is required to be duality invariant (self-dual)

$$L'(F) = L(F) \ .$$

The requirement of self-duality implies the following:

- Only U(1) duality transformations can consistently be defined in the nonlinear case.

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}$$

  Maxwell's theory also allows scale duality transformations which, however, are forbidden if the energy-momentum tensor is required to be duality invariant.

- The Lagrangian is a solution of the self-duality equation

$$G^{ab}\,\widetilde{G}_{ab} + F^{ab}\,\widetilde{F}_{ab} = 0 \ , \qquad \widetilde{G}_{ab}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}$$

Gibbons & Rasheed (1995) Gaillard & Zumino (1997)

Properties of U(1) duality-invariant models

- Duality invariance of the energy-momentum tensor.
- $\mathsf{SL}(2,\mathbb{R})$ duality invariance in the presence of dilaton and axion.
- Self-duality under Legendre transformation.

Legendre transformation for nonlinear electrodynamics $L(F)$.

1. Associate with $L(F)$ an equivalent auxiliary model defined by

$$L(F, F_{\rm D}) = L(F) - \frac{1}{2}\,F \cdot \widetilde{F}_{\rm D}~, \qquad\qquad {F_{\rm D}}^{ab} = \partial^a {A_{\rm D}}^b - \partial^b {A_{\rm D}}^a~,$$

   in which F_{ab} is an unconstrained two-form (auxiliary field).

2. Eliminate F_{ab} using its equation of motions $G(F) = F_{\rm D}$ to yield

$$L_{\rm D}(F_{\rm D}) := \Big(L(F) - \frac{1}{2}F \cdot \widetilde{F}_{\rm D}\Big)\Big|_{F=F(F_{\rm D})}~.$$

3. If $L(F)$ solves the self-duality equation $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, then

$$L_{\rm D}(F) = L(F)~.$$

Self-dual electrodynamics

General structure of dualiy-invariant electrodynamics

- Given a model for nonlinear electrodynamics, its Lagrangian $L(F_{ab})$ can be realised as a real function of one complex variable,

$$L(F_{ab}) = L(\omega, \bar{\omega}) \ , \qquad \omega = \alpha + \mathrm{i}\,\beta = F^{\alpha\beta} F_{\alpha\beta} \ ,$$

where $\alpha = \frac{1}{4} F^{ab} F_{ab}$ and $\beta = \frac{1}{4} F^{ab} \widetilde{F}_{ab}$ are the EM invariants.

$$L(\omega, \bar{\omega}) = -\frac{1}{2} (\omega + \bar{\omega}) + \omega \bar{\omega}\, \Lambda(\omega, \bar{\omega}) \ .$$

- Self-duality equation (SDE), $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, turns into

$$\mathrm{Im} \left\{ \frac{\partial(\omega \Lambda)}{\partial \omega} - \bar{\omega} \left(\frac{\partial(\omega \Lambda)}{\partial \omega} \right)^2 \right\} = 0 \ .$$

- Simplest solutions:
  1. Maxwell's theory $\quad L(\omega, \bar{\omega}) = -\frac{1}{2} (\omega + \bar{\omega})$
  2. Born-Infeld theory

$$L_{\mathrm{BI}}(\omega, \bar{\omega}) = \frac{1}{g^2} \left\{ 1 - \sqrt{1 + g^2(\omega + \bar{\omega}) + \frac{1}{4} g^4 (\omega - \bar{\omega})^2} \right\}$$

Conformal duality-invariant electrodynamics

- ModMax theory

$$L_{\mathrm{conf}}(\omega, \bar{\omega}) = -\frac{1}{2} \cosh\gamma \left(\omega + \bar{\omega}\right) + \sinh\gamma \sqrt{\omega\bar{\omega}}\ ,$$

with γ a positive parameter.

Bandos, Lechner, Sorokin & Townsend arXiv:2007.09092

Kosyakov arXiv:2007.13878

Coupling to the dilaton and axion

- Given a U(1) duality-invariant model, $L(F_{ab}) = L(\omega, \bar{\omega})$, its compact duality group U(1) is enhanced to the non-compact $\mathsf{SL}(2, \mathbb{R})$ group by coupling F_{ab} to dilaton φ and axion $\mathfrak{a}$ by replacing

$$L(\omega, \bar{\omega}) \quad \rightarrow \quad L(\omega, \bar{\omega}, \tau, \bar{\tau}) = L(\tau_2 \omega, \tau_2 \bar{\omega}) + \frac{\mathrm{i}}{2} \tau_1 (\bar{\omega} - \omega) \ ,$$
$$\tau = \tau_1 + \mathrm{i}\, \tau_2 = \mathfrak{a} + \mathrm{i}\, \mathrm{e}^{-\varphi} \ .$$

- The duality group acts by transformations

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \quad \tau' = \frac{a\tau + b}{c\tau + d} \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbb{R})$$

- Maxwell's theory coupled to the dilaton and axion

$$L(F, \tau, \bar{\tau}) = -\frac{1}{4} \mathrm{e}^{-\varphi} F^{ab} F_{ab} + \frac{1}{4} \mathfrak{a} \tilde{F}^{ab} F_{ab}$$

is Weyl invariant (conformal in flat space), with τ being Weyl inert.
τ & $\bar{\tau}$ local couplings.
Non-minimal operator $\Longrightarrow$ generalised heat kernel techniques.

Quantum theory

- Let $\Gamma(\tau,\bar\tau)$ be the effective action obtained by integrating out the gauge field in the path integral.
- Both Weyl and rigid $\mathsf{SL}(2,\mathbb{R})$ duality transformations are anomalous at the quantum level. However, the logarithmically divergent part of $\Gamma(\tau,\bar\tau)$ is invariant under these transformations.
- General structure of the logarithmic divergence of $\Gamma(\tau,\bar\tau)$

$$
\begin{aligned}
\mathfrak{L} \quad &= \quad \frac{1}{2(\mathrm{Im}\,\tau)^2}\Big[\mathcal{D}^2\tau\mathcal{D}^2\bar\tau - 2(R^{ab}-\frac{1}{3}\eta^{ab}R)\nabla_a\tau\nabla_b\bar\tau\Big] \\
&\quad + \frac{1}{12(\mathrm{Im}\,\tau)^4}\Big[\alpha\nabla^a\tau\nabla_a\tau\nabla^b\bar\tau\nabla_b\bar\tau + \beta\nabla^a\tau\nabla_a\bar\tau\nabla^b\tau\nabla_b\bar\tau\Big]
\end{aligned}
$$

where $\mathcal{D}^2\tau := \nabla^a\nabla_a\tau + \frac{\mathrm{i}}{\mathrm{Im}\,\tau}\nabla^a\tau\nabla_a\tau$, and α and β are numerical parameters.

Osborn (2003)

- $\int \mathrm{d}^4x\, e\, \mathfrak{L}$ is Weyl and $\mathsf{SL}(2,\mathbb{R})$ invariant.
- General structure of Weyl anomaly

$$
\delta_\sigma\Gamma(\tau,\bar\tau) \propto \int \mathrm{d}^4x\, e\,\sigma\,\mathfrak{L}
$$

What about a supersymmetric extension?

- Weyl invariance of $\int \mathrm{d}^4x\, e\, \mathfrak{L}$ follows from the fact that the fourth-order Fradkin-Tseytlin operator (1982)

$$\Delta_0 = (\nabla^a \nabla_a)^2 + 2\nabla^a \big(R_{ab} \nabla^b - \tfrac{1}{3} R \nabla_a\big)$$

  is conformal.
  (Δ_0 was re-discovered by Paneitz in 1983 and Riegert in 1984.)

- Soon after the 2003 work by Osborn several attempts were made to extend his construction to supersymmetric case. No success was achieved until 2020, perhaps, due to the following two reasons:

  1. $\mathcal{N} = 1$ supersymmetric extension of the Fradkin-Tseytlin operator operator was constructed only in 2013 by
     Butter, de Wit, SMK & Lodato (2013)
  2. Effective action corresponding to the vector multiplet model

$$S[V; \Phi, \bar{\Phi}] = -\frac{\mathrm{i}}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{E}\, \Phi W^\alpha(V) W_\alpha(V) + \mathrm{c.c.}\ ,$$

     where $W_\alpha(V) = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha V$, involves determinants of non-minimal differential operators which are much harder to evaluate than in the non-supersymmetric case.

Grimm-Wess-Zumino superspace geometry (1978)

Superspace covariant derivatives

$$\mathcal{D}_A := (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}) = E_A{}^M \partial_M + \Omega_A{}^{\beta\gamma} M_{\beta\gamma} + \bar\Omega_A{}^{\dot\beta\dot\gamma} \bar M_{\dot\beta\dot\gamma} .$$

Graded commutation relations

$$\begin{aligned} & \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} , \\ \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar R M_{\alpha\beta} , \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4R\bar M_{\dot\alpha\dot\beta} , \\ \left[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}\right] &= \mathrm{i}\varepsilon_{\alpha\beta}\Big(\bar R \bar{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot\beta})M_{\gamma\delta} + 2\bar W_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar M_{\dot\gamma\dot\delta}\Big) \\ & \quad +\mathrm{i}(\bar{\mathcal{D}}_{\dot\beta}\bar R)M_{\alpha\beta} . \end{aligned}$$

Torsion superfields R, $G_{\alpha\dot\alpha} = \bar G_{\alpha\dot\alpha}$ and $W_{\alpha\beta\gamma}$ obey the Bianchi identities:

$$\bar{\mathcal{D}}_{\dot\alpha} R = 0 , \qquad \bar{\mathcal{D}}_{\dot\alpha} W_{\alpha\beta\gamma} = 0 , \qquad \bar{\mathcal{D}}^{\dot\alpha} G_{\alpha\dot\alpha} = \mathcal{D}_\alpha R$$

R, $G_{\alpha\dot\alpha}$ and $W_{\alpha\beta\gamma}$ are supergravity analogues of the scalar curvature, traceless Ricci tensor and self-dual Weyl tensor, respectively.

Super-Weyl transformations

Howe & Tucker (1978)

$$
\begin{aligned}
\delta_\sigma \mathcal{D}_\alpha &= (\bar{\sigma} - \frac{1}{2}\sigma)\mathcal{D}_\alpha + (\mathcal{D}^\beta \sigma)\, M_{\alpha\beta} \ , \\
\delta_\sigma \bar{\mathcal{D}}_{\dot\alpha} &= (\sigma - \frac{1}{2}\bar{\sigma})\bar{\mathcal{D}}_{\dot\alpha} + (\bar{\mathcal{D}}^{\dot\beta}\bar{\sigma})\bar{M}_{\dot\alpha\dot\beta} \ , \\
\delta_\sigma \mathcal{D}_{\alpha\dot\alpha} &= \frac{1}{2}(\sigma + \bar{\sigma})\mathcal{D}_{\alpha\dot\alpha} + \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\sigma})\mathcal{D}_\alpha + \frac{\mathrm{i}}{2}(\mathcal{D}_\alpha\sigma)\bar{\mathcal{D}}_{\dot\alpha} \\
&\quad + (\mathcal{D}^\beta{}_{\dot\alpha}\sigma)M_{\alpha\beta} + (\mathcal{D}_\alpha{}^{\dot\beta}\bar{\sigma})\bar{M}_{\dot\alpha\dot\beta} \ ,
\end{aligned}
$$

where the super-Weyl parameter σ is covariantly chiral, $\bar{\mathcal{D}}_{\dot\alpha}\sigma = 0$.
The torsion tensors transform as follows:

$$
\begin{aligned}
\delta_\sigma R &= 2\sigma R + \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar{\sigma} \ , \\
\delta_\sigma G_{\alpha\dot\alpha} &= \frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha\dot\alpha} + \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}(\sigma - \bar{\sigma}) \ , \\
\delta_\sigma W_{\alpha\beta\gamma} &= \frac{3}{2}\sigma W_{\alpha\beta\gamma} \ .
\end{aligned}
$$

Supersymmetric extension of Fradkin-Tseytlin operator

Butter, de Wit, SMK & Lodato (2013)

- Operator

$$\Delta_c \bar{\Phi} := -\frac{1}{64}(\bar{\mathcal{D}}^2 - 4R)\Big\{\mathcal{D}^2\bar{\mathcal{D}}^2\bar{\Phi} + 8\mathcal{D}^\alpha\big(G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}\big)\Big\} , \quad \bar{\mathcal{D}}^{\dot{\alpha}}\Delta_c\bar{\Phi} = 0$$

is superconformal: its super-Weyl transformation law is

$$\delta_\sigma \Delta_c \bar{\Phi} = 3\sigma \Delta_c \bar{\Phi} .$$

- Given two super-Weyl neutral chiral scalars Φ and Ψ, the functional

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\Psi\Delta_c\bar{\Phi} = \frac{1}{16}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\Big\{\mathcal{D}^2\Psi\bar{\mathcal{D}}^2\bar{\Phi} - 8\mathcal{D}^\alpha\Psi\, G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}\Big\}$$

is super-Weyl invariant.

- Important identity

$$\delta_\sigma\Big\{\mathcal{D}^2\bar{\mathcal{D}}^2\bar{\Phi} + 8\mathcal{D}^\alpha\big(G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}\big)\Big\} = (\sigma + \bar{\sigma})\Big\{\mathcal{D}^2\bar{\mathcal{D}}^2\bar{\Phi} + 8\mathcal{D}^\alpha\big(G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}\big)\Big\}$$
$$+2\bar{\mathcal{D}}_{\dot{\alpha}}\Big\{(\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi})\mathcal{D}^2\sigma + 4\mathrm{i}(\mathcal{D}^{\alpha\dot{\alpha}}\bar{\Phi})\mathcal{D}_\alpha\sigma\Big\} .$$

Duality-invariant supersymmetric electrodynamics

- $\mathsf{SL}(2,\mathbb{R})$ duality-invariant coupling of the dilaton-axion multiplet to general models for self-dual supersymmetric electrodynamics

$$S_\Lambda[V;\Phi,\bar\Phi] = -\frac{\mathrm{i}}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\Phi W^2 + \mathrm{c.c.}$$
$$-\frac{1}{16}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,(\Phi-\bar\Phi)^2 W^2\,\bar W^2\Lambda\,(\omega,\bar\omega)\ ,$$

where Φ is the dilaton-axion multiplet,

$$\omega = \frac{\mathrm{i}}{2}(\bar\Phi-\Phi)u\ , \qquad u := \frac{1}{8}(\mathcal{D}^2-4\bar R)W^2\ ,$$

and $\Lambda(\omega,\bar\omega)$ is the same as in the supersymmetric case.

- Superconformal model $S[V;\Phi,\bar\Phi]$: $\qquad \Lambda(\omega,\bar\omega)=0$
- Logarithmic divergence of the effective action

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi,\bar\Phi]} = \int[\mathrm{D}V]\,\mathrm{e}^{\mathrm{i}S[V;\Phi,\bar\Phi]}$$

should be invariant under $\mathsf{SL}(2,\mathbb{R})$ and super-Weyl transformations.

Superconformal higher-derivative sigma model

- Let $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ be the Kähler potential of a Kähler manifold $\mathcal{M}$. Associated with $\mathcal{M}$ is a higher-derivative sigma model described in terms of covariantly chiral scalar superfields Φ^I, $\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^I = 0$.

$$S = \frac{1}{16}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, E\Big[g_{I\bar{J}}(\Phi,\bar{\Phi})\Big\{\nabla^2\Phi^I\bar{\nabla}^2\bar{\Phi}^{\bar{J}} - 8G_{\alpha\dot{\alpha}}\mathcal{D}^\alpha\Phi^I\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}^{\bar{J}}\Big\}$$
$$+\Big\{\alpha R_{I\bar{J}K\bar{L}}(\Phi,\bar{\Phi}) + \beta g_{I\bar{J}}(\Phi,\bar{\Phi})g_{K\bar{L}}(\Phi,\bar{\Phi})\Big\}\mathcal{D}^\alpha\Phi^I\mathcal{D}_\alpha\Phi^K\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Phi}^{\bar{J}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}^{\bar{L}}\Big]$$

  where α and β are constants, $g_{I\bar{J}} = \partial_I\partial_{\bar{J}}K$ is the Kähler metric, $R_{I\bar{J}K\bar{L}}$ the Riemann curvature of the Kähler manifold, and

$$\nabla^2\Phi^I = \mathcal{D}^2\Phi^I + \Gamma^I_{KL}\mathcal{D}^\alpha\Phi^K\mathcal{D}_\alpha\Phi^L\ .$$

- The action proves to be super-Weyl invariant provided the chiral scalars Φ^I are neutral under the super-Weyl transformations.

General structure of super-Weyl anomalies

- Purely supergravity-dependent part part

$$\delta_\sigma \Gamma = 2(a-c)\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{E}\, \sigma W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \mathrm{c.c.}$$
$$+2a\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, E\,(\sigma+\bar\sigma)(G^a G_a + 2R\bar R)\ .$$

Bonora, Pasti &Tonin (1985)
Buchbinder & SMK (1986)

- Contribution of the local couplings

$$\delta_\sigma\Gamma = \frac{1}{16}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, E\,(\sigma+\bar\sigma)$$
$$\times\Big[g_{I\bar J}(\Phi,\bar\Phi)\Big\{\nabla^2\Phi^I\bar\nabla^2\bar\Phi^{\bar J} - 8G_{\alpha\dot\alpha}\mathcal{D}^\alpha\Phi^I\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar J}\Big\}$$
$$+\alpha R_{I\bar J K\bar L}(\Phi,\bar\Phi)\mathcal{D}^\alpha\Phi^I\mathcal{D}_\alpha\Phi^K\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar J}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar L}\Big]\ .$$

SMK (2020)

- Wess-Zumino consistency condition: $[\delta_{\sigma_2}, \delta_{\sigma_1}]\Gamma = 0$

Final results for the vector multiplet model

$$S[V;\Phi,\bar{\Phi}] = -\frac{\mathrm{i}}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\Phi W^\alpha(V)W_\alpha(V) + \mathrm{c.c.}$$

- Logarithmically divergent part of the effective action

$$\Gamma_{\mathrm{div}} = -\frac{1}{512\pi^2\omega}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\left\{\frac{\nabla^2\Phi\bar{\nabla}^2\bar{\Phi} - 8\mathcal{D}^\alpha\Phi\,G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}}{(\Phi-\bar{\Phi})^2} - \frac{\mathcal{D}^\alpha\Phi\mathcal{D}_\alpha\Phi\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Phi}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}}{(\Phi-\bar{\Phi})^4}\right\}.$$

- Super-Weyl anomaly

$$\delta_\sigma\Gamma = \frac{1}{512\pi^2}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,(\sigma+\bar{\sigma}) \times\left\{\frac{\nabla^2\Phi\bar{\nabla}^2\bar{\Phi} - 8\mathcal{D}^\alpha\Phi\,G_{\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}}{(\Phi-\bar{\Phi})^2} - \frac{\mathcal{D}^\alpha\Phi\mathcal{D}_\alpha\Phi\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Phi}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}}{(\Phi-\bar{\Phi})^4}\right\}.$$

Interesting open problem

- It is interesting to extend the quantum analysis to the case of local $\mathcal{N}=2$ supersymmetry. The corresponding vector multiplet action:

$$S[\mathbb{V};X,\bar{X}]=-\frac{\mathrm{i}}{8}\int \mathrm{d}^4x\mathrm{d}^4\theta\,\mathcal{E}\,X\Big(W(\mathbb{V})\Big)^2+\mathrm{c.c.}$$

  X is a background chiral scalar superfield containing the dilaton and axion as the lowest component
  W is the field strength of a vector multiplet,

$$\bar{\mathcal{D}}^{\dot\alpha}_i W=0~,\qquad \Big(\mathcal{D}^{ij}+4S^{ij}\Big)W=\Big(\bar{\mathcal{D}}^{ij}+4\bar{S}^{ij}\Big)\bar{W}$$

- $\mathcal{N}=2$ superconformal higher-derivative sigma model
  Gomis, Hsin, Komargodski, Schwimmer, Seiberg & Theisen (2016)

$$S=\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,E\,K(X^I,\bar{X}^{\bar{J}})~,\qquad \bar{\mathcal{D}}^{\dot\alpha}_i X^I=0~,\qquad \delta_\sigma X^I=0~.$$

  The action is (i) Kähler invariant; and (ii) super-Weyl invariant, with the chiral multiplets X^I being super-Weyl inert.

$\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction

- $\mathcal{N} = 2$ superconformal higher-derivative sigma model in Minkowski superspace

$$S = \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\, K(X^I, \bar{X}^{\bar{J}}) \ , \qquad \bar{D}^{\dot{\alpha}}_i X^I = 0 \ .$$

- $\mathcal{N} = 2$ chiral superfield X^I is equivalent to three $\mathcal{N} = 1$ chiral superfields Φ^I, λ^I_α and Z^I defined by

$$\Phi^I := X^I\big|_{\theta_{\underline{2}}=0} \ , \quad \sqrt{2}\Omega^I_\alpha := D^{\underline{2}}_\alpha X^I\big|_{\theta_{\underline{2}}=0} \ , \quad Z^I := -\frac{1}{4}(D^{\underline{2}})^2 X^I\big|_{\theta_2=0} \ .$$

- Reducing the action to $\mathcal{N} = 1$ superspace gives

$$S = \frac{1}{16}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\left\{g_{I\bar{J}}\nabla^2\Phi^I\bar{\nabla}^2\bar{\Phi}^{\bar{J}} + R_{I\bar{J}K\bar{L}}D^\alpha\Phi^I D_\alpha\Phi^K \bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{I}}\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\bar{L}}\right\}$$
$$+ \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, g_{I\bar{J}}\Big\{\mathbb{Z}^I\bar{\mathbb{Z}}^{\bar{J}} - \mathrm{i}\Omega^{I\alpha}\nabla_{\alpha\dot{\alpha}}\bar{\Omega}^{\bar{J}\dot{\alpha}}\Big\} \ ,$$

where

$$\mathbb{Z}^I := Z^I - \frac{1}{4}\Gamma^I_{JK}\Omega^{J\alpha}\Omega^K_\alpha \ , \qquad \nabla_{\alpha\dot{\alpha}}\bar{\Omega}^{\bar{I}\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}\bar{\Omega}^{\bar{J}\dot{\alpha}} + \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\partial_{\alpha\dot{\alpha}}\bar{\Phi}^{\bar{J}}\bar{\Omega}^{\bar{K}\dot{\alpha}} \ .$$

Interesting open problem

- In the absence of scalars, $\mathsf{U}(n)$ is the largest duality group of n Abelian vector fields.
- $\mathsf{Sp}(2n, \mathbb{R})$ is the maximal duality group of n Abelian vector fields in the presence of scalar τ's parametrising the homogeneous space $\mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$.

Gaillard & Zumino (1981)

- It would be interesting to compute the low-energy effective for a superconformal $\mathsf{Sp}(2n, \mathbb{R})$ duality-invariant model of $n > 2$ vector multiplets V_i coupled to a background $n \times n$ matrix chiral superfield $\Phi = (\Phi^{ij})$,

$$\Phi^{\mathrm{T}} = \Phi \ , \qquad \mathrm{i}(\bar{\Phi} - \Phi) > 0 \ ,$$

which parametrises the Hermitian symmetric space $\mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$. The classical action is

$$S[V; \Phi, \bar{\Phi}] = -\frac{\mathrm{i}}{4} \int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathcal{E} \, W^{\alpha}(V)^{\mathrm{T}} \Phi W_{\alpha}(V) + \mathrm{c.c.}$$

Thank you!