The Nicolai map for super Yang-Mills theory and application to the supermembrane

Dubna '21

- based on 2005.12324, 2104.00012, 2104.09654, 2109.00346 and works from the early 1980s (!)
- collaboration with
  Sudarshan Ananth, Hannes Malcha, Hermann Nicolai, Maximilian Rupprecht, Chetan Panday & Saurabh Pant
- Content

  1. Definition and construction of the Nicolai map
  2. The case of gauge theories
  3. Application to the supermembrane

① Definition & construction of the Nicolai map

- example: Wess-Zumino model in $\mathbb{R}^{1,3}$ $\quad (\phi, \psi, F)$

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + F^* F + \frac{i}{2} \bar{\psi} \bar{\sigma}\cdot\partial \psi - \frac{i}{2} \psi \sigma\cdot\partial \bar{\psi}$$
$$+ W'(\phi) F + W'(\phi)^* F^* - \frac{1}{2} \psi W''(\phi) \psi - \frac{1}{2} \bar{\psi} W''(\phi)^* \bar{\psi}$$

integrate out auxiliary (F, F^*) $\rightsquigarrow$ $F^* = -W'$

$$\mathcal{L} = |\partial\phi|^2 - |W'|^2 + \left(\frac{i}{2} \bar{\psi} \bar{\sigma}\cdot\partial \psi - \frac{1}{2} \psi W'' \psi + \text{h.c.} \right)$$

integrate out fermions $(\psi, \bar{\psi})$ $\rightsquigarrow$ $\det M = e^{\frac{i}{\hbar}\cdot(-i\hbar \operatorname{tr} \ln M)}$

$$S_g[\phi] = \int d^4x \left\{ |\partial\phi|^2 - |W'|^2 \right\} - i\hbar \operatorname{tr} \ln \begin{pmatrix} W'' & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & W''^* \end{pmatrix}$$
$$=: S_g^b[\phi] + \hbar\, S_g^f[\phi]$$

g = coupling constant(s) inside superpotential $W[\phi]$

$$\langle Y[\phi] \rangle_g = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S_g[\phi]} \, Y[\phi] \;, \quad \langle \mathbb{1} \rangle_g = 1$$

- Q: What trace remains of SUSY ? $S_g^b \leftrightarrow S_g^f$?

A [Nicolai 1980]:

$\exists$ (in general nonlocal & nonlinear) map $T_g : \phi \longmapsto \phi'[\phi; g]$

such that

$$\langle Y[\phi] \rangle_g = \langle Y[T_g^{-1}\phi] \rangle_0 \qquad \forall Y \qquad (1)$$

$\rightarrow$ relates interacting to free theory $(g=0)$

- equivalently:

$$\mathcal{D}\phi \, e^{\frac{i}{\hbar} S_g[\phi]} = \mathcal{D}(T_g\phi)\, e^{\frac{i}{\hbar} S_0[T_g\phi]}$$

$$= \mathcal{D}\phi \, e^{\frac{i}{\hbar} S_0[T_g\phi] + \mathrm{tr}\ln\left(\frac{\delta T_g\phi}{\delta\phi}\right)}$$

separate powers in $\hbar$:

$$S_0^b[T_g\phi] = S_g^b[\phi] \qquad \text{"free action condition"} \qquad (2a)$$

$$\underbrace{S_0^f[T_g\phi]}_{= \text{ constant}} - i\,\mathrm{tr}\underbrace{\ln\left(\frac{\delta T_g\phi}{\delta\phi}\right)}_{\text{Jacobian}} = \underbrace{S_g^f[\phi]}_{\text{MSS determinant}} \qquad \text{"determinant matching"} \qquad (2b)$$

- infinitesimal version [Lechtenfeld 1984] $\partial_g\,(1) \rightsquigarrow$

$$\partial_g \langle Y[\phi] \rangle_g \overset{(1)}{=} \langle (\partial_g + R_g[\phi])\, Y[\phi] \rangle_g \qquad (3)$$

with a "coupling flow operator"

$$R_g[\phi] = \int dx \, \left(\partial_g T_g^{-1} \circ T_g \right) \phi(x) \, \frac{\delta}{\delta\phi(x)} \qquad (4)$$

- reverse logic: somehow find $R_g \longrightarrow$ construct T_g

$$(T_g^{-1}\phi)(x) = e^{g(\partial_{g'} + R_{g'}[\phi])}\, \phi(x) \Big|_{g'=0} \xrightarrow{\text{invert}} T_g\phi$$

- more convenient is the observation

$$(\partial_g + R_g[\phi])\, T_g\phi(x) = 0 \qquad (5) \qquad \text{"fix point"}$$

this allows us to directly construct $T_g\phi$ from R_g ...

- R_g is a derivation $\Longleftrightarrow$ T_g^{-1} acts distributively :

$$R_g Y[\phi] = \int \frac{\delta Y}{\delta\phi} \cdot R_g\phi \quad \Longleftrightarrow \quad T_g^{-1} Y[\phi] = Y[T_g^{-1}\phi]$$

- universal formula [Lechtenfeld, Rupprecht 2021]

$$T_g \phi = \mathcal{P}\, e^{-\int_0^g dh\, R_h[\phi]} \cdot \phi \qquad (6)$$

path-ordered exponential

$$= \sum_{s=0}^{\infty} (-1)^s \int_0^g dh_s \cdots \int_0^{h_3} dh_2 \int_0^{h_2} dh_1\; R_{h_s}[\phi] \cdots R_{h_2}[\phi]\, R_{h_1}[\phi] \cdot \phi$$

- expansion in powers of g

$$R_g[\phi] = \sum_{k=1}^{\infty} g^{k-1}\, r_k[\phi] = r_1[\phi] + g\, r_2[\phi] + g^2 r_3[\phi] + \ldots$$

$$T_g[\phi] = \sum_{\vec{n}} g^n\, c_{\vec{n}}\; r_{n_s}[\phi] \cdots r_{n_2}[\phi]\, r_{n_1}[\phi] \cdot \phi \qquad (7)$$

with $\vec{n} = (n_1, n_2, \ldots, n_s)$, $n_i \in \mathbb{N}$, $\sum_{i=1}^{s} n_i = n$ multi-index

and $c_{\vec{n}} = (-1)^s \left[n_1 \cdot (n_1+n_2) \cdot (n_1+n_2+n_3) \cdots (n_1+n_2+\ldots+n_s) \right]^{-1}$

- explicit start of perturbation series:

$$T_g \phi = \phi - g\, r_1 \phi - \tfrac{1}{2} g^2 (r_2 - r_1^2)\phi - \tfrac{1}{6} g^3 (2 r_3 - r_1 r_2 - 2 r_2 r_1 + r_1^3)\phi + \ldots$$

- When does R_g exist and how to find it ?

  express $\quad \partial_g S_{SUSY}[\phi,\psi] = \delta_\alpha \Delta_\alpha[\phi,\psi] \qquad (8)$

  for SUSY transformation δ_α , α = Majorana spinor index

  ex. WZ model: $\quad \Delta_\alpha = \frac{1}{2}\int d^4x\, \psi_\alpha\, \partial_g W'(\phi) \qquad$ linear in ψ

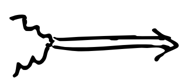
- construction of R_g : $\qquad$ uses SUSY Ward identity

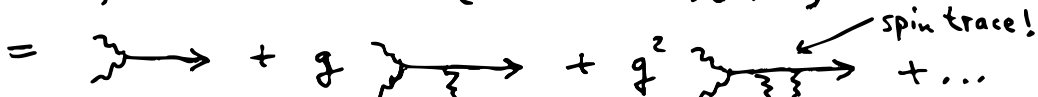
$$\partial_g \int\mathcal{D}\phi\int\mathcal{D}\psi\; e^{\frac{i}{\hbar}S_{SUSY}}\, Y[\phi] = \int\mathcal{D}\phi\int\mathcal{D}\psi\; e^{\frac{i}{\hbar}S_{SUSY}} \left(\partial_g + i\Delta_\alpha \delta_\alpha\right) Y[\phi]$$

  integrate out fermions contracts bilinears $\underbrace{\psi\ \psi}$ (fermion propagator)

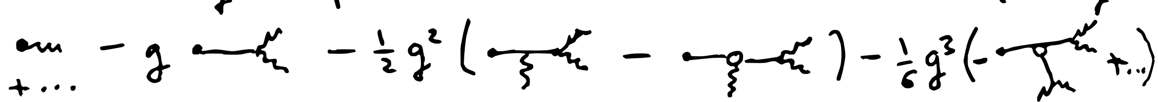
$$R_g[\phi] = i\int dx\, \underbrace{\Delta_\alpha[\phi]\ \ \delta_\alpha}\phi(x)\, \frac{\delta}{\delta\phi(x)} \qquad (9)$$

- diagrammatics $\qquad$ [Flume, Lechtenfeld 1983]

  $R_g[\phi] \;\hat{=}$  $\qquad$ [for $W = \frac{1}{3}g\phi^3$]

  linear tree for R_g exponentiates to branched tree for T_g

  $T_g\phi \;\hat{=}$ 

② The case of gauge theories

- complication: gauge redundancy
  $\longrightarrow$ gauge fixing $\mathcal{G}(A)=0$, ghost fields $c, \bar{c}$
  reduces gauge to BRST symmetry

- $S_{SUSY}[A, \lambda, D, c, \bar{c}] = \int dx \, \mathrm{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \mathcal{G}(A)^2 + \right.$
  $\left. + \text{fermions} + \text{ghosts} + \text{auxiliaries} \right\}$
  with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu]$ $su(N)$-valued
  and $A_\mu = A^A_\mu T^A$, $\lambda_\alpha = \lambda^A_\alpha T^A$ etc. , $[T^A, T^B] = f^{ABC} T^C$

  consider $\mathbb{R}^{1, D-1}$ and Majorana gaugini $\lambda^A \in \mathbb{C}^r$ $\leftarrow$ dim of Clifford rep'n
  $\mu = 0, 1, \dots, D-1$ $A = 1, \dots, N^2-1$ $\alpha = 1, \dots, r$

- construction of R_g uses broken-SUSY & BRST Ward identities
  [$D=4$: Flume, Lechtenfeld 1983 , $D \neq 4$ ALMNPP 2020]

- result: $\partial_g \langle Y[A] \rangle_g = \langle (\partial_g + R_g[A] + Z_g[A])\, Y[A] \rangle_g$

with $R_g = i \underbrace{\Delta_\alpha}\, \delta_\alpha - \underbrace{\Delta_\alpha (\delta_\alpha} \underbrace{\Delta_{gh})\, s}$ ← Slavnov variation

where $\Delta_\alpha = -\frac{1}{2r} \int dr\, (\gamma^{\mu\nu}\lambda)_\alpha\, A_\mu A_\nu$ and $\Delta_{gh} = \int dr\, \bar{C}\, \mathcal{G}(A)$

and $Z_g = (s \underbrace{\Delta_\alpha)(\delta_\alpha} \Delta_{gh}) - \left(\frac{D-1}{r} - \frac{1}{2}\right) \int dr\, \underbrace{\bar{\lambda} \slashed{A} \lambda} + i \int dr\, \bar{C} \underbrace{\frac{\partial \mathcal{G}}{\partial A_\mu} A_\mu C}$

⇝ a multiplicative piece destroys derivative property

- computation reveals that in Landau gauge $\mathcal{G} = \partial \cdot A$, $\xi \to 0$

$$Z_g = 0 \iff \frac{r}{2} = D-2 \iff D = 3, 4, 6, 10 \;!$$

⇝ critical SYM dimensions recovered!

- for $D \leq 4$ a different strategy works in any gauge,
[Lechtenfeld 1984, Lechtenfeld, Rupprecht 2021, Nicolai, Malcha 2021]
coupling flow conserves gauge surface: $R_g \mathcal{G} \sim \mathcal{G}$

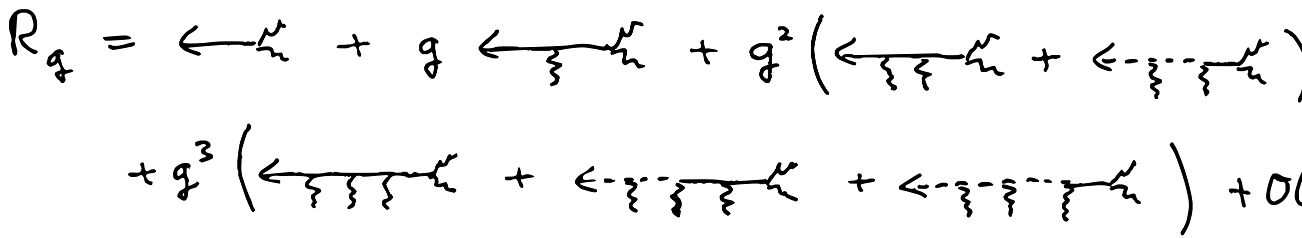
- $$\overleftarrow{R}_g[A] \underset{\substack{\uparrow \\ \text{Landau gauge}}}{=} \frac{1}{2r} \iiint dr\, \frac{\overleftarrow{\delta}}{\delta A_\mu} \underbrace{\left\{ \delta_\mu^\nu \mathbb{1} - D_\mu \underbrace{C \bar{C}} \partial^\nu \right\}}_{\text{nonabelian projector}} \underbrace{\left\{ \gamma_\nu \underbrace{\bar{\lambda}\; \lambda}\, \gamma^{\rho\lambda} A_\rho A_\lambda \right\}_{\alpha\alpha}}_{\text{linear tree}}$$

(10)

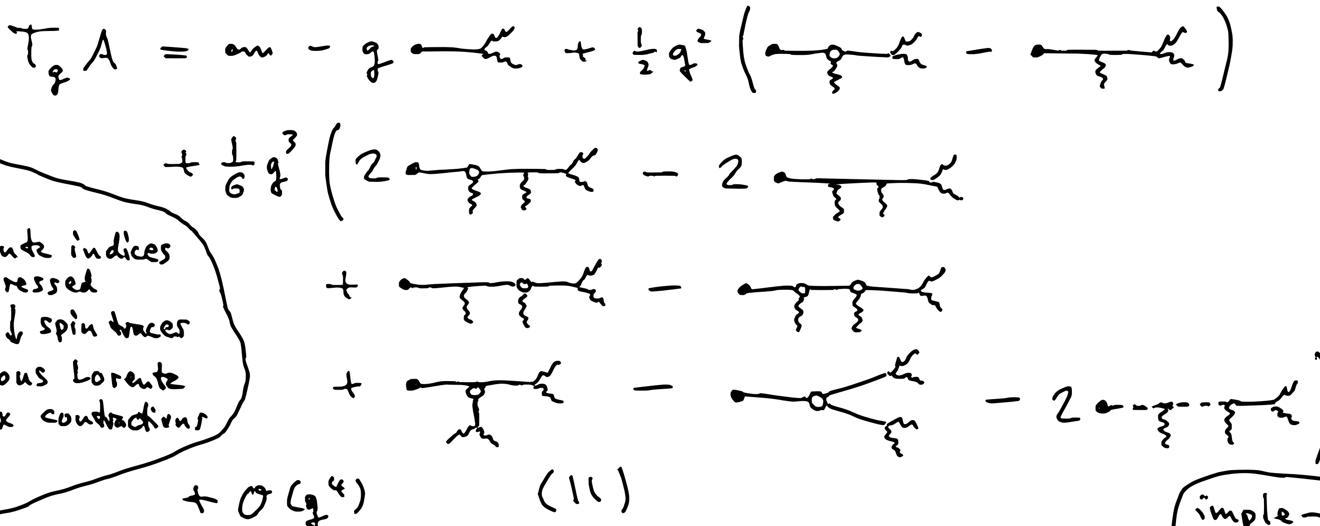
• diagrammatics (Landau gauge)

— $\hat{=} (i\partial\!\!\!/)^{-1}$

- - - $\hat{=} \Box^{-1}$



iterate and plug into universal formula ...



(11)

Lorentz indices suppressed

↓ spin traces

various Lorentz index contractions

explicitly evaluated by ALMNPP 2020 to $O(g^3)$

and by Malcha, Nicolai 2021 to $O(g^4)$

implementable computer algorithm

③ Application to the supermembrane

[Lechtenfeld, Nicolai 2021]

- The D=11 supermembrane [Bergshoeff, Sezgin, Townsend 1987] can be obtained as an $N\to\infty$ limit of the BFSS matrix model

  [Claudson, Halpern 1985; Baake, Reinicke, Rittenberg 1985; Flume 1985; deWit, Hoppe, Nicolai 1988; Banks, Fischler, Shenker, Susskind 1996]

- main problem remains: how to quantize?
- proposal: quantize it as APD gauge theory!

$$\begin{tikzcd}
 & \text{D=11 supermembrane }(T) \arrow[r, Leftrightarrow] \arrow[dl, "\text{double dim'l red.}"'] \arrow[dr, shift left, "\text{lightcone regularization}"] & \text{APD gauge theory }(g) \\
\text{D=10 IIA string }(\alpha', g_s) \arrow[rr, "N\ D\emptyset\text{ branes}"] \arrow[d, "\alpha'\to 0"] & & \text{BFSS SU(N) matrix model }(g) \arrow[ul, shift left, "N\to\infty"] \\
\text{D=10 SYM }(g) \arrow[urr, "\text{KK reduction}"] & &
\end{tikzcd}$$

read off: $T \sim g$

- concretely: use Nicolai map for D=10 SYM for a g-expansion of the BFSS matrix model

$\Longrightarrow$ small-T expansion for the supermembrane!

- the map for the APD gauge theory

$X_a(t,\sigma^1,\sigma^2)$, $\Theta_\alpha(t,\sigma^1,\sigma^2)$, temporal gauge, $a=1,\dots,9$, $\alpha=1,\dots,32$

APD bracket $\{A(t,\vec\sigma), B(t,\vec\sigma)\} \sim \partial_{\sigma^1}A\,\partial_{\sigma^2}B - \partial_{\sigma^2}A\,\partial_{\sigma^1}B$

propagator $\partial_t^{-1} = \frac{1}{2}\,\mathrm{sgn}(t-t')$, suppressing common σ arguments:

$$
\begin{aligned}
T_g X_a(t) = {} & X_a(t) - \tfrac{1}{2}g^2 \int ds\,du\; \varepsilon(t-s)\,\varepsilon(s-u)\,\{X_b(s),\{X_b(u),X_a(u)\}\} \\
& + \tfrac{1}{8}g^4 \int ds\,du\,dv\,dw\; \varepsilon(t-s)\,\varepsilon(s-u)\,\varepsilon(u-v)\,\varepsilon(v-w)\,\Big[\\
& \qquad 6\,\{X_b(s),\{X_c(u),\{X_{[a}(v),\{X_b(w),X_{c]}(w)\}\}\}\} \\
& \qquad + 2\,\{X_b(s),\{X_{[b}(u),\{X_{|c|}(v),\{X_{a]}(w),X_c(w)\}\}\}\} \\
& \qquad + 2\,\{X_a(s)-X_a(t),\{X_b(u),\{X_c(v),\{X_b(w),X_c(w)\}\}\}\}\Big] \\
& + \tfrac{1}{8}g^4 \int ds\,du\,dv\,dw\; \varepsilon(t-s)\,\varepsilon(s-u)\,\varepsilon(s-v)\,\varepsilon(v-w)\;\times \\
& \qquad \times\;\{\{X_a(u),X_b(u)\},\{X_c(v),\{X_b(w),X_c(w)\}\}\} \;+\; \mathcal{O}(g^6)
\end{aligned}
\qquad (12)
$$

Outlook

- a new angle of attack on the supermembrane
- perturbative small-tension expansion $\Rightarrow$ quantization
- establish quantum target-space Lorentz invariance
- compute physically relevant correlation functions

  e.g. graviton emission vertex operators

$$V_h[X,\theta] = h_{ab}\Big[D_t X^a D_t X^b - \{X^a, X^c\}\{X^b, X^c\} - i\bar\theta\gamma^a\{X_b,\theta\}$$
$$-\tfrac{1}{2}D_t X^a\bar\theta\gamma^{bc}k_c - \tfrac{1}{2}\{X^a,X^c\}\bar\theta\gamma^{bcd}\theta k_c + \tfrac{1}{2}\bar\theta\gamma^{ac}\theta\,\bar\theta\gamma^{bd}\theta\,k_c k_d\Big]e^{-i\vec{k}\cdot\vec{X}+ik^- t}$$

- convergence of perturbation series from universal formula
- special rôle of Landau gauge for $D>4$?
- traces of "integrability" for $N=4$ $D=4$ SYM ?