Solution of differential equations for multiloop integrals with **Libra**[1] package

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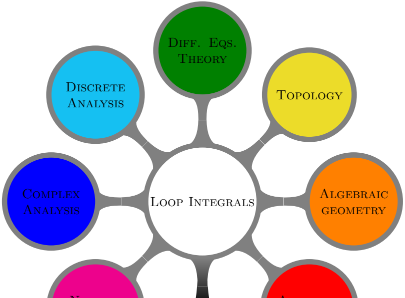
[1] RL, CPC **267** (2021) 108058.

Congratulations!

My congratulations to all heros of the event!
I am very grateful to Vladimir Smirnov for pleasant and fruitful collaboration during many years.

Motivation

- If we want to detect deviations (new physics) from SM, we need to know the predictions of the latter with high precision. In particular, we have to be able to calculate two-loop radiative corrections to various processes [A. Arbuzov's talk yesterday].
- Only quite recently the methods of multiloop calculations have reached the point where they can be really helpful with this goal.
- Besides these practical purposes, multiloop calculations provide a perfect polygon for trying the methods from various mathematical fields: differential equations, complex analysis, number theory



Modern approach to diagrams calculation

1. Consider a family of integrals

$$j(n_1, \dots, n_N) = \int \frac{d^d l_1 \dots d^d l_L}{D_1^{n_1} \dots D_N^{n_N}} .$$

Integrals are functions of kinematic variables x_i and $d = 4 - 2\epsilon$.

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3. Find differential equations [Kotikov, 1991, Remiddi, 1997] (and/or dimensional recurrences [Tarasov, 1996]) for master integrals

Differential equations

$$\frac{\partial}{\partial x_i}\boldsymbol{j} = M(\boldsymbol{x},\epsilon)\boldsymbol{j}$$

Dimensional recurrences

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M and R are $n \times n$ matrices rational in $\boldsymbol{x}$ and ϵ.

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4. Find **general solution**.
5. Use other methods for **boundary conditions**.

IBP reduction

Note on phase-space integrals

- Phase-space integrals can be transformed in a standard way into loop integrals with cut propagator:

$$\frac{d^{d-1}p}{(2\pi)^{d-1}2\varepsilon_p} = \frac{d^d p}{(2\pi)^d}2\pi\delta_+\left(p^2-m^2\right) = \frac{d^{d-1}p}{(2\pi)^d}(-i)\oint_{\varepsilon_p} dp_0\left(p^2-m^2\right)^{-1}$$

- As IBP identities are insensitive to the integration contour, provided that it does not lead to surface terms, we can treat the cut propagators in the same way as uncut ones.
- There are two things to take care of: first, shift symmetries which mix cut with uncut denominators should be omitted, second, positive (integer) powers of cut denominators are equal to zero.

ϵ-form

Differential equation

$$\partial \boldsymbol{J}(x) / \partial x = M(x, \epsilon) \boldsymbol{J}(x)$$

Function change

$$\boldsymbol{J}(x) = T(x, \epsilon) \tilde{\boldsymbol{J}}(x)$$

In particular, we can choose master integrals in infinitely many ways.

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How to find canonical basis?

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Algorithmic approach [RL, 2015]: general idea

Perform many "elementary" transformations gradually improving properties of the system.

General structure of reduction algorithm

Algorithm proceeds in three major stages, each involving a sequence of "elementary" transformations.

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1. *"Fuchsification"*: Eliminating higher-order poles

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3. *Factorization*: Factoring out ϵ

Input: Matrix from the previous step.
Output: Matrix in ϵ-form, $M(x, \epsilon) = \epsilon S(x) = \epsilon \sum_k \frac{S_k}{x - a_k}$.

Balance

Balance transformation

$$T(x) = \overline{P} + \frac{x - x_2}{x - x_1} P\,,$$

where P is some projector and $\overline{P} = I - P$. When $x_1 = \infty$ or $x_2 = \infty$ omit denominator/numerator.

Balance transformation changes properties (pole order and eigenvalues of matrix residue) of the differential system at $x = x_1$ and $x = x_2$ only.

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Programs

Further details of the algorithm deserve a special talk, but the good news is that we now have **programs**! At least, three public ones: `epsilon`, `Fuchsia`, `Libra`.

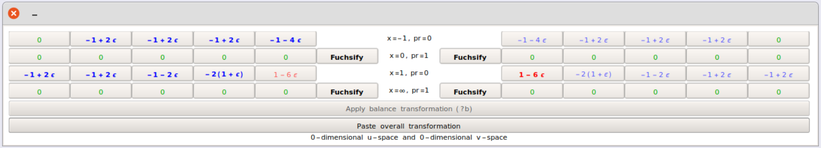
Libra interface for reduction

Automatic tool (useful for simple cases)

In[1]: t=Rookie[M,x,ϵ];

Interactive tool (useful for most cases)

In[1]: t=VisTransformation[M,x,ϵ];



Manual tool (useful for really hard cases)

In[1]: u=GetSubspaces[M,{x,0},ϵ][[1]];v=GetSubspaces[M...

Factoring ϵ

In[2]: t=FactorOut[M,x,ϵ,μ];

General solution

Differential system in ϵ-form

$$\partial_x \boldsymbol{J} = \epsilon S(x) \boldsymbol{J}\,, \qquad S(x) = \sum_k \frac{S_k}{x - a_k}.$$

General solution (evolution operator)

$$U(x, x_0) = \mathrm{Pexp}\left[\epsilon \int_{x_0}^{x} dx_1\, S(x_1)\right]$$

```
In[3]:  U=PexpExpansion[{S,6},x];
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Goncharov polylogarithms

So, $U(x, x_0)$ is expressed via Goncharov polylogs [Goncharov, 1998]

$$G(a_n, \ldots, a_1|x) = \iiiint\limits_{x>x_n>\ldots>x_1>0} \frac{dx_1}{x_1 - a_1} \cdots \frac{dx_n}{x_n - a_n}$$

Perfect class of functions: numerical evaluation, analytic continuation, series representation, functional identities, relations to classical polylogs.

Boundary conditions

Suppose we have found a transformation $T(x) = T(x, \epsilon)$ to ϵ-form, $\boldsymbol{j} = T\boldsymbol{J}$. Then we can write

$$\begin{aligned}\boldsymbol{J}(x) &= U(x, x_0)\boldsymbol{J}(x_0),\\ \boldsymbol{j}(x) &= T(x)U(x, x_0)[T(x_0)]^{-1}\boldsymbol{j}(x_0)\end{aligned}$$

But the point x_0 should be somewhat special to simplify the evaluation of $\boldsymbol{j}(x_0)$ as compared to $\boldsymbol{j}(x)$. With no known exceptions, "special" boils down to "singular", i.e., we can expect simplifications for x_0 being a singular point of the differential system. Let it be $x_0 = 0$ for simplicity.

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Problem

$U(x, x_0)$ diverges when x_0 tends to zero. Therefore, we have to consider not the values, but the asymptotics of $\boldsymbol{j}(x_0)$ at $x = 0$.

Boundary conditions

Regularized evolution operator

$$U(x,\underline{0}) = \lim_{x_0\to 0} U(x,x_0)x_0^{\epsilon S_0},$$

where $S_0 = \mathrm{Res}_{x=0}\, S(x)$.

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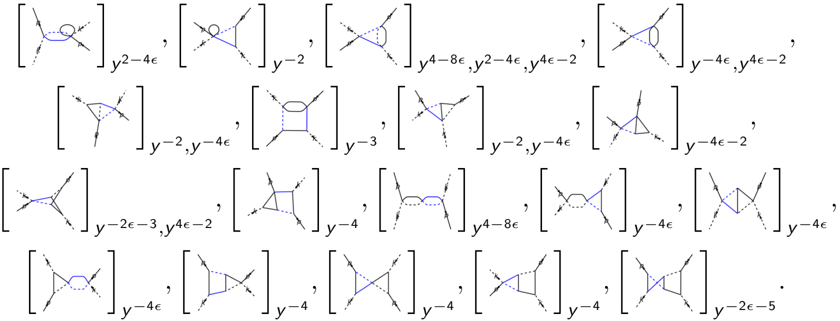
Good news

Libra can determine which asymptotic coefficients, $\boldsymbol{c}$, are sufficient to calculate and find the "adapter" matrix L in $\boldsymbol{C} = L\boldsymbol{c}$.
NB: for regular point, of course, $\boldsymbol{c} = \boldsymbol{j}(0)$ and $L = T^{-1}(0)$.

```
In[4]:  {L,cs}=GetLcs[S,T,{x,0}];
```

Example: boundary conditions for $\sigma_{e^-\gamma\to e^-\gamma}$ @NLO

The following threshold ($s \to 1$, $x \to 0$) asymptotic coefficients are to be calculated:

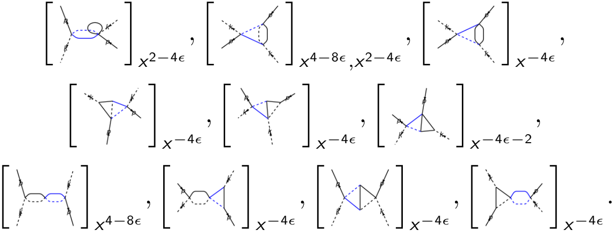


Here $\left[\text{integral}\right]_{x^\alpha}$ denotes the coefficient in front of x^α in the small-x asymptotics of integral ($x \approx \frac{1}{2}\sqrt{s-1}$).

Example: Boundary conditions for $\sigma_{e^-\gamma\to e^-\gamma}$ @NLO

[RL,Lyubyakin,Stocky(2020); RL,Schwartz,Zhang (2021)]

- Selection rule by the fractional power $n\epsilon$ leaves us with 11 possibly nonzero constants.The fractional power -4ϵ corresponds to the hard momentum flowing over the black lines, while -8ϵ — to soft momentum.



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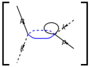
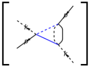
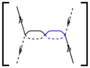
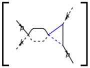
- Selection rule by integer power reduces the number to 5.

 $_{x^{2-4\epsilon}}$,  $_{x^{4-8\epsilon},x^{2-4\epsilon}}$,

 $_{x^{4-8\epsilon}}$,  $_{x^{-4\epsilon}}$

General solution, Frobenius method

In many applications there is a natural small parameter. E.g., for $e^+e^- \to X$ the electron mass is small, but can not be put to zero. Instead one should expand the regularized evolution operator

$$U(x, \underline{0}) = \lim_{x_0 \to 0} U(x, x_0) x_0^{S_0}$$

in generalized power series. **Libra** has tools for it. It closely follows the approach described in Ref. [RL, Smirnov, and Smirnov, 2018].

$U(x, \underline{0})$ as generalized power series

Recursion data for (matrix) coefficients of U:

```
In[1]:  sdata=SeriesSolutionData[S,x,x];
```

Using data for expanding to fixed order:

```
In[2]:  Uexp=ConstructSeriesSolution[sdata,x,6];
```

Algebraic extensions

- Sometimes, in order to find the transformation to ϵ-form, one has to extend the class of transformations by passing from x to y, such that $x = x(y)$ is some rational function. **Libra** has tool for it:

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- Moreover, in many cases there is no common rationalizing variable. Thus, **Libra** implements a more powerful way to treat such algebraic extensions, with
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  In[1]:  AddNotation[ds,y → x(1-y*y) - 4 y*y];
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  One may add as many notations as needed, and **Libra** will take care of them (minimizing their appearance, correctly treating their differentiation).

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  One may add as many notations as needed, and **Libra** will take care of them (minimizing their appearance, correctly treating their differentiation).
- There is a bunch of functions related to treating the algebraic extensions: `QuolyMod, DiffMod, SeriesCoefficientMod, EValuesMod,` etc. These functions can be used also to treat irreducible denominators, like $x^2 + x + 1$ in a way which do not introduce radicals.

Irreducible cases

- As we know, even with algebraic extensions it is not always possible to reduce the system to ϵ-form. Sometimes the integrals just can not be expressed via polylogs, [see D. Broadhurst's talk yesterday].
- In Ref. [RL and Pomeransky, 2017] the criterion of irreducibility has been derived. The reducibility has been shown to correspond to triviality of some holomorphic vector bundle on the Riemann sphere. Thanks to Birkhoff-Grothendieck theorem, it is possible to constructively decide this. **Libra** has the corresponding tool: the command
  ```
  In[3]:  {T1,T2,T3}= BirkhoffGrothendieck[T,x];
  ```
  decomposes Laurent-polynomial matrix T into the product $T_1 T_2 T_3$, where T_1, T_1^{-1} are polynomial in x, T_3, T_3^{-1} are polynomial in x^{-1}, and $T_2 = \mathrm{diag}(x^{n_1}, \ldots x^{n_k})$. The bundle is trivial iff $T_2 = 1$.

Summary

- Modern multiloop calculation techniques can really help in NNLO calculation useful for the experiments.
- **Libra** can really help in applying the differential equations method. It has tools
  - for the reduction of the differential system to ϵ-form,
  - for the construction of general solution in terms of Goncharov's polylogs,
  - for determining the minimal set of asymptotic coefficients to be evaluated to fix the boundary conditions,
  - for constructing Frobenius expansion,
  - for treating the algebraic extensions,
  - for detecting the irreducible cases.
  - It can work with univariate and multivariate systems.

Outlook

- `Libra` improvements:
  - Improve automatic tool `Rookie[M,x,`ϵ`]`.
  - Better treatment of algebraic extensions, especially, for multivariate case.
- Differential equations method:
  - construct a systematic approach to irreducible cases. In particular, Birkhoff-Grothendieck factorizations seems to be carry a lot of information yet to be properly understood and used.

Thank you!

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