Advances in Supersymmetric QFT and Deautonomization of Integrable Systems

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Advances in QFT

JINR, Dubna, October 2021

Plan

- $\mathcal{N} = 2$ supersymmetric gauge theory *beyond* loop calculus;
- Old story: prepotentials and integrable systems;
- Effective equations for partition function: deautonomization;
- Solutions: instanton interpretation *when possible*;
- Derivation: lift to 5d;
- Perspectives ...

with M.Bershtein, P.Gavrylenko, M.Semenyakin, A.Stoyan

Supersymmetric gauge theory

4d $\mathcal{N} = 2$ supersymmetric Yang-Mills theory:

$$\mathcal{L}_0 = \frac{1}{g_0^2}\mathrm{Tr}\ \left(\mathrm{F}_{\mu\nu}^2 + |D_\mu\Phi|^2 + [\Phi, \Phi^\dagger]^2 + \mathrm{fermions}\right) + \frac{\vartheta_0}{2\pi}\mathrm{Tr} F \wedge F \qquad (1)$$

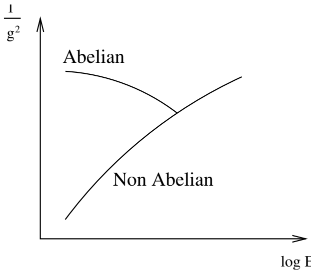
- Higgs condensate $\langle\Phi\rangle$ breaks gauge group to Abelian: Coulomb branch;
- *Moduli space* of the theory: $u \sim \langle\mathrm{Tr}\Phi^2\rangle$, generally

$$P(\lambda; u) = \langle\det(\lambda - \Phi)\rangle \qquad (2)$$

- Aim: effective $U(1)^{\mathrm{rank}G}$ Abelian theory in IR.

Supersymmetry and loop corrections

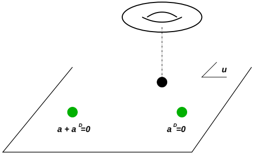
- Gauge coupling $\frac{1}{g^2} \sim \beta \log \frac{|u|}{\Lambda^2}$: *exact* 1-loop RG formula;
- $\beta = 2N - N_f \geq 0$ ... UV completion (?!);



- Complexification: $i\frac{4\pi^2}{g^2} + \frac{\vartheta}{2\pi} = \tau \sim \log \frac{u}{\Lambda^2}$ and $\mathcal{N} = 2$ *holomorphy*;
- Works at $u \gg \Lambda^2$, naively at $u = 0$ non-Abelian symmetry restores ...

SW theory: effective action

Obstruction: at $|u| < \Lambda^2$ e.g. one gets $\frac{1}{g^2} \sim \log \frac{|u|}{\Lambda^2} < 0$.
Quantum moduli space for $G = SU(2)$



- non-Abelian symmetry *never* restores: around $a^D = 0$ and $a + a^D = 0$, EM-dual Abelian theory;
- Effective couplings in IR $U(1)^{\mathrm{rank}G}$ theory $\mathcal{L}_{\mathrm{eff}} = \mathrm{Im}\ T_{ij}(a)\ F^i_{\mu\nu}F^j_{\mu\nu} + \ldots$
- $\mathcal{N} = 2$ special Kähler geometry : holomorphic prepotential $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$ (action $\mathrm{Im} \int d^4\theta \mathcal{F}(\Phi)$).

Prepotential expansions

At weak coupling:

$$\mathcal{F}(a) \underset{a\to\infty}{\to} \frac{1}{2}a^2 \log\frac{a}{\Lambda} + a^2 \sum_{k>0} f_k \left(\frac{\Lambda}{a}\right)^{4k}$$

- Logarithm from $\mathcal{N} = 2$ one loop;
- Expansion over instantons of charge k, in powers of $\Lambda^\beta = \Lambda^{2N} = \Lambda^4$: a way to compute $\{f_k\}$.

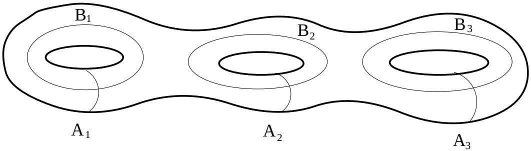
and at strong coupling (monopole point $a_D \to 0$)

$$\mathcal{F}_D(a_D) \underset{a_D\to 0}{\to} -\frac{1}{2}a_D^2 \log\frac{a_D}{\Lambda} - 8\Lambda a_D + a_D^2 \sum_{k>0} f_k^D \left(\frac{a_D}{\Lambda}\right)^{k}$$

Different powers: no instantons in monopole theory! No way to compute $\{f_k^D\}$ other, than to solve an equation ...

Step 1: SW theory

SW: determined by Σ of genus=rank G,



Smooth Riemann surface (of genus 3)
with fixed A- and B-cycles.

Lattice of charges $\Leftrightarrow$ $H_1(\Sigma)$ with symplectic $\langle , \rangle$, $\langle A_i, B_j \rangle = \delta_{ij}$.

Period matrix: $\mathrm{Im}\, T_{ij} \geq 0$, $T \underset{\text{degeneration}}{\rightarrow} \log a$

Prepotential and IS

Σ with *pair of differentials* or dS: $\delta dS \simeq \text{holomorphic}$, or by *an integrable system*:

$$a_i = \oint_{A_i} dS, \qquad a_i^D = \oint_{B_i} dS = \frac{\partial \mathcal{F}}{\partial a_i} \tag{3}$$

consistent by symmetricity of period matrix $\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = T_{ij}(a)$ (RBI).

Example (pure $\mathcal{N} = 2$ gauge theory)

$$\Lambda^N \left(w + \frac{1}{w} \right) = P_N(\lambda) = \langle \det(\lambda - \Phi) \rangle = \prod_{i=1}^{N} (\lambda - v_i) \tag{4}$$

with $\oint d\lambda = 0$, $\oint \frac{dw}{w} = 2\pi i \mathbb{Z}$ so that $dS = \lambda \frac{dw}{w}$.

Does it $\mathcal{F}$ satisfy any reasonable (integrable!?) equation?

Toda chains

- 'Oversimplified' $G = U(1)$: $\Lambda\left(w + \frac{1}{w}\right) = \lambda - v$ gives $a = \oint \lambda \frac{dw}{w} = v$, $\mathcal{F} = \frac{1}{2}a^2\tau + e^\tau$, with $\Lambda^2 = e^\tau$, satisfying

$$\frac{\partial^2 \mathcal{F}}{\partial \tau^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2} \tag{5}$$

  for the Toda tau-function $\mathcal{F} = \log \mathcal{T}$.
- Generally:
  - Non-autonomous version of 'Toda-like' equations;
  - In ε-deformed theory: instead of naive partition function $\mathcal{T} \overset{?}{\sim} Z(a,\tau) \sim e^{\mathcal{F}/\varepsilon^2}$ a Fourier transform ($\mathcal{F}_D = aa_D - \mathcal{F}$)

$$\mathcal{T} \underset{\varepsilon \to 0}{\sim} \exp\left(\frac{\mathcal{F}_D}{\varepsilon^2} + O(\varepsilon^2)\right) \cdot \Theta(\ldots)$$

- To derive: lift to 5d (with a compact dimension).

SU(2)/Painlevé

Outcome:

- Deautonomization: integrable (or isospectral) $\Rightarrow$ isomonodromic system;
- 'SW Toda' (sine-Gordon) $\Rightarrow$ Painlevé III

$$\frac{d^2 q}{d\tau^2} + e^{2\tau} \sinh q = 0 \tag{6}$$

- In conventional "isomonodromic" variables ($t \sim \Lambda^4$, $w \sim \sqrt{t} e^q$)

$$H(w, w'; t) = \frac{t w'^2}{4w^2} + \frac{w}{t} + \frac{1}{w} = \partial_t \log \mathcal{T}(t), \tag{7}$$

and

$$w(t)^{-1} = \partial_t t \partial_t \log \mathcal{T}(t) = -t^{1/2} \frac{\mathcal{T}_1(t)^2}{\mathcal{T}(t)^2} \tag{8}$$

Tau-functions

The isomonodromic tau functions

$$\mathcal{T}(t;a,\eta) \underset{t\to 0}{=} \sum_n e^{4\pi i n\eta} t^{(a+n)^2} \frac{\mathcal{B}(a+n,t)}{G(1+2(a+n))G(1-2(a+n))} \tag{9}$$

are expressed through partition functions of ε-deformed $SU(2)$ gauge theory.

- $t \sim \Lambda^4$, (a,η) are two yet *independent* integration constants, t^{a^2} – classical part ;
- Barnes G-functions $G(a+1) = \Gamma(a)G(a) \underset{a\to\infty}{\sim} \exp\left(\frac{1}{2}a^2 \log a\right)$;
- $\mathcal{B}(a,t) = \sum_{\lambda,\mu} t^{|\lambda|+|\mu|} \left(\frac{a+\ldots}{a+\ldots}\right)$: Nekrasov instanton partition function, $|\lambda| + |\mu| = k$;
- Lagrangian submanifold ($\eta \to a_D = \partial\mathcal{F}/\partial a$) appears at singularities of solution ...

Painlevé/SYM

- Analytic properties of the Painlevé solutions contain important information about non-perturbative SYM: Already $t \sim \Lambda^4$ gives $4 = 2N$ pure $SU(2)$ beta-function ...
- Expansion in $t = \Lambda^4$ at $t \to 0$ and in $t^{-1/4} = \Lambda^{-1}$ at $t \to \infty$;
- Non-autonomous Toda equation

$$\partial_t t \partial_t \log \mathcal{T}(t) = -t^{1/2} \frac{\mathcal{T}_1(t)^2}{\mathcal{T}(t)^2} \tag{10}$$

  an analog of $\frac{\partial^2 \mathcal{F}}{\partial \tau^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2}$.
- 'Gravitational flows': the Nakajima-Yoshioka *blow-up equations* from simple analysis.

Blow-up from Painlevé

At singularity of solution

$$\mathcal{T}_1(t;a,\eta_\star)=0$$
$$e^{4\pi i\eta_\star}=\frac{\Gamma\left(1+2a\right)^2}{\Gamma\left(1-2a\right)^2}t^{-2a}\exp\left(\frac{\partial f(a,t)}{\partial a}\right), \tag{11}$$

explicitly

$$\sum_{n\in\frac{1}{2}+\mathbb{Z}}\frac{t^{n^2}\frac{\Gamma(1+2a)^{2n}}{\Gamma(1-2a)^{2n}}}{G(1+2(a+n))G(1-2(a+n))}\exp\left(n\frac{\partial f(a,t)}{\partial a}\right)\mathcal{B}(a+n,t)=0 \tag{12}$$

which has the form

$$\sum(\ldots)\,Z(\ldots;\epsilon_1-\epsilon_2)Z(\ldots;\epsilon_1+\epsilon_2)=0 \tag{13}$$

at both $\epsilon_1+\epsilon_2=0$ and $\epsilon_1\to 0$. Extends to strong coupling ...

Step 2: derivation from 5d

Lift SW theory to 5d: $\mathbb{R}^4 \times \mathbb{S}^1$

$$\lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} = u \tag{14}$$

at $z \to 0$ for $u = 2\cosh a$, $\lambda = 2\cosh\zeta$

$$\mu = 4\sinh\frac{\zeta - a}{2}\sinh\frac{\zeta + a}{2} \tag{15}$$

The period "matrix" (= complexified coupling $\tau \sim \frac{\vartheta}{2\pi} + i\frac{4\pi^2}{g^2_{\rm YM}}$)

$$\tau \sim \int_{-a}^{a} d\log\sinh\frac{\zeta - a}{2} \sim \log\sinh a \tag{16}$$

collects contributions from 5d KK modes.

5d Nekrasov functions from quantum mechanics on instanton moduli space!

IS on cluster varieties

- 'Relativization' of Toda, 'trigonometric' dependence;
- Integrable systems in Lie groups: cluster varieties;
- *Quiver* $\mathcal{Q}$ with $|\mathcal{Q}|$ vertices, oriented edges $\Rightarrow$ logarithmically constant bracket

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \ldots, |\mathcal{Q}| \tag{17}$$

(no sum!) with skew-symmetric

$$\epsilon_{ij} = \#\text{arrows } (i \to j) = -\epsilon_{ji} \tag{18}$$

- Discrete flows from cluster mutations:

$$\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik}, \text{ if } i = j \text{ or } k = j, \quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}| + \epsilon_{jk}|\epsilon_{ij}|}{2} \quad \text{otherwise,}$$

the x-variables (Poisson map)

$$\mu_j : \quad x_j \to \frac{1}{x_j}, \qquad x_i \to x_i \left(1 + x_j^{\mathrm{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}, i \neq j \tag{19}$$

Cluster integrable system

- Defined by a convex Newton Polygon Δ: a curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \tag{20}$$

  endowed with $d\lambda/\lambda \wedge d\mu/\mu$, modulo $SL(2,\mathbb{Z})$.
- Realized on a cluster variety with Poisson structure

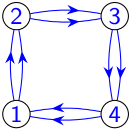
$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2\mathrm{Area}(\Delta)}. \tag{21}$$

  determined by $\mathcal{Q}$, with $\epsilon_{ij} = \#\mathrm{arrows}(i \to j)$.
- Integrability: Pick's formula

$$\dim\mathcal{X} = 2\mathrm{Area}(\Delta) - 1 = (B-3) + 2g \tag{22}$$

Deautonomization: $SU(2)$ Toda quiver

- Poisson quiver $\mathcal{Q}$:



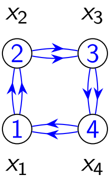
defines the bracket

$$\{x_i, x_{i+1}\} = 2x_i x_{i+1}, \qquad i = 1, \dots, 4 \tag{23}$$

- $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra;
- Integrable system (relativistic Toda) on Poisson submanifold in affine group at $q = 1$ (!);
- straightforward quantization $\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$ (q and p – *two* ε-parameters).

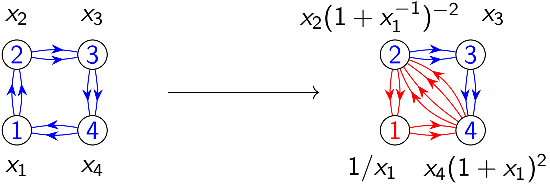
Discrete flow

Abelian subgroup of the quiver MCG:



Discrete flow

Abelian subgroup of the quiver MCG:



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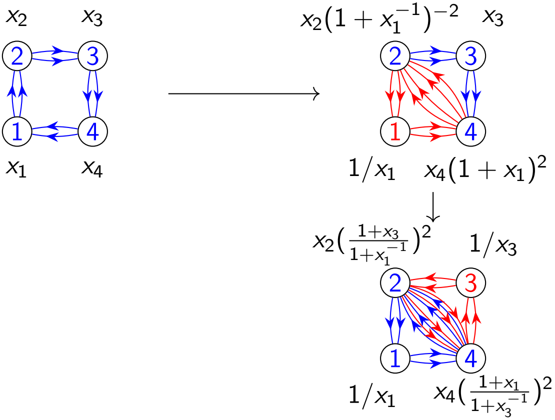

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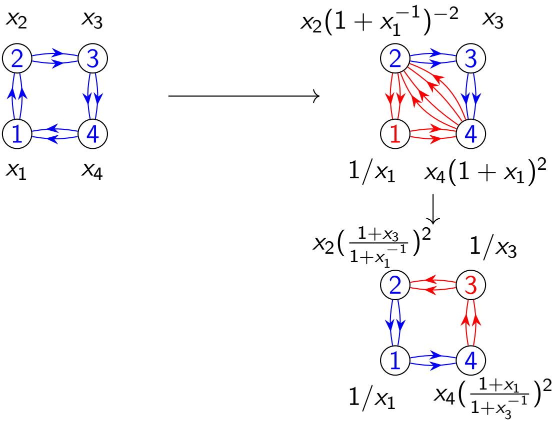
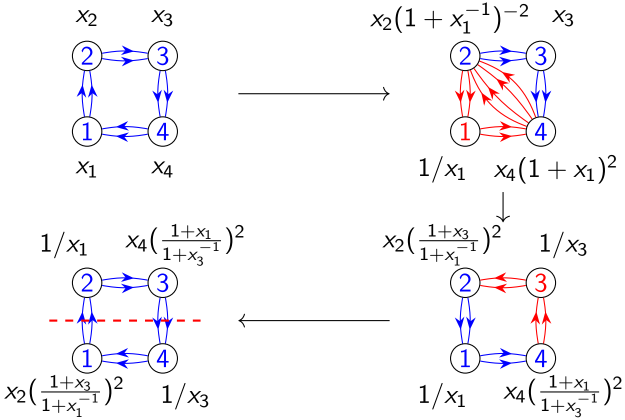


Discrete flow

Abelian subgroup of the quiver MCG:



Deautonomization: flow

For $q = 1$ the flow

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 \left(\frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left(\frac{1 + x_1}{1 + x_3^{-1}} \right)^2, x_3^{-1} \right)$$

or

$$T : (x_1, x_2, {\color{red} z}, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, {\color{red} qz}, q \right) \underset{q=1}{=} \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, z, q \right)$$

preserves the Hamiltonian $\mathcal{H} = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Deautonomization: Painlevé

Let $x_1 x_2 x_3 x_4 = q \neq 1$

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)$$

Consider z as "time" $T : x(z) \mapsto x(qz)$, then $x_1 = x(z)$, $x_2 = x^{-1}(q^{-1}z)$, satisfy

$$x(qz)x(q^{-1}z) = \left(\frac{x(z) + z}{x(z) + 1} \right)^2$$

or q-Painlevé III_3 equation.

Deautonomization: tau-functions

For the tau-functions $x(z) = z^{1/2}\frac{\mathcal{T}_1(z)^2}{\mathcal{T}_0(z)^2}$ one gets bilinear (non-autonomous!) *Hirota equations*

$$\mathcal{T}_0(qz)\mathcal{T}_0(q^{-1}z) = \mathcal{T}_0(z)^2 + z^{1/2}\mathcal{T}_1(z)^2$$
$$\mathcal{T}_1(qz)\mathcal{T}_1(q^{-1}z) = \mathcal{T}_1(z)^2 + z^{1/2}\mathcal{T}_0(z)^2$$

Generally for the $SU(N)_k$-Toda:

$$\mathcal{T}_j\left(qz\right)\mathcal{T}_j\left(q^{-1}z\right) = \mathcal{T}_j(z)^2 + z^{1/N}\mathcal{T}_{j+1}\left(q^{k/N}z\right)\mathcal{T}_{j-1}\left(q^{-k/N}z\right)$$
$$j \in \mathbb{Z}/N\mathbb{Z}$$

Origin: mutation of tau-variables (dual to x-variables) ...

Deautonomization: solutions

Autonomous case: solution of Hirota relations in theta functions (Fay identities)

Deautonomization $q \neq 1$:

$$\mathcal{T}_j^{N,k}(\vec{u}, \vec{s}; q|z) = \sum_{\vec{\Lambda} \in Q_{N-1} + \omega_j} s^{\Lambda} \mathcal{Z}_{N,k}(\vec{u} q^{\vec{\Lambda}}; q^{-1}, q|z) \tag{24}$$

with $\mathcal{Z}_{N,k} = \mathcal{Z}_{\mathrm{cl}}^{N,k} \cdot \mathcal{Z}_{\mathrm{1loop}}^{N} \cdot \mathcal{Z}_{\mathrm{inst}}^{N,k}$ being 5d Nekrasov functions.

Deautonomization: 5d SYM

Here:

$$\mathcal{Z}_{\rm cl}^{N,k} = \exp\left(\log z \frac{\sum (\log u_i)^2}{-2\log q_1 \log q_2} + k \frac{\sum (\log u_i)^3}{-6\log q_1 \log q_2}\right)$$

$$\mathcal{Z}_{\rm 1loop}^{N} = \prod_{1\le i\neq j\le N} (u_i/u_j; q_1, q_2)_\infty, \quad \mathcal{Z}_{\rm inst}^{N,k} = \sum_{\vec{\lambda}} \frac{z^{|\vec{\lambda}|}\prod_{i=1}^N \mathsf{T}_{\lambda^{(i)}}(u;q_1,q_2)^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)}$$

with

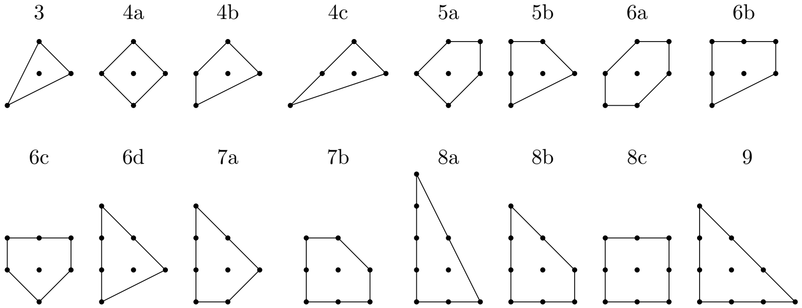
$$\mathsf{N}_{\lambda,\mu}(u,q_1,q_2) = \prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})$$

$$\mathsf{T}_\lambda(u;q_1,q_2) = u^{|\lambda|}q_1^{\frac{1}{2}(\|\lambda^t\|-|\lambda^t|)}q_2^{\frac{1}{2}(\|\lambda\|-|\lambda|)} = \prod_{(i,j)\in\lambda} uq_1^{i-1}q_2^{j-1},$$

and $\vec{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(N)})$, $|\vec{\lambda}| = \sum|\lambda^{(i)}|$, $|\lambda| = \sum \lambda_j$, $\|\lambda\| = \sum \lambda_j^2$.

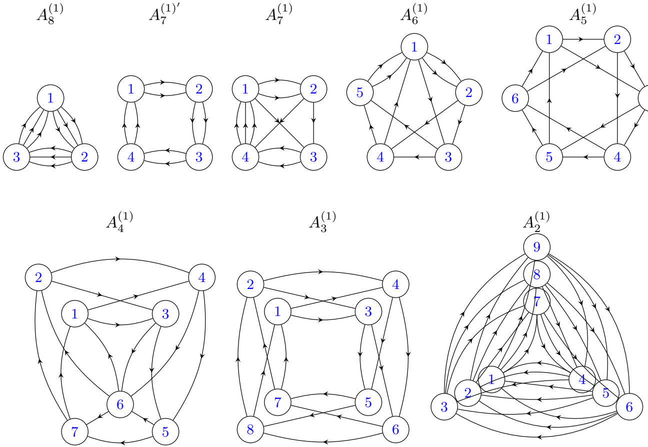
Application: Painlevé Newton Polygons

with a single internal point and $3 \le B \le 9$ boundary points:



Here Σ: $f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is torus with $g = 1$.

Application: Painlevé quivers



$\beta = 2N - N_f = 4 - N_f$, $0 \leq k \leq 2$

Conclusions and outlook

- Dual partition functions of (ε-deformed) $\mathcal{N} = 2$ supersymmetric gauge theories satisfy non-autonomous equations of Painlevé type;
- Follow from deautonomization of the SW integrable systems;
- Natural picture for lifted to 5d theories: exploits language of cluster varieties, q-difference equations;
- Extends to regime of strong coupling, where Nekrasov functions are not known (no monopoles in dual Abelian theories);
- Suggests some UV completion of 5d theories ... speculations about 6d.