Matrix and tensor models: recent progress

Andrei Mironov

P.N.Lebedev Physics Institute; ITEP; IITP

Basic properties of matrix models

Gaussian Hermitian matrix model:

the integral over Hermitian $N \times N$ matrix H

$$Z_N(p_k) = \frac{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2 + \sum_k \frac{p_k \operatorname{Tr} H^k}{k}\right)}{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2\right)}$$

with the Haar measure dH, $Z_N(p_k)$ is understood as a formal power series in p_k.

Integrable property:

$Z_N(p_k)$ is a τ-function of the Toda chain hierarchy, satisfies an infinite set of integrable equations, the first one being

$$Z_N(p_k)\,\frac{\partial^2 Z_N(p_k)}{\partial p_1^2} - \left(\frac{\partial Z_N(p_k)}{\partial p_1}\right)^2 = Z_{N+1}(p_k)Z_{N-1}(p_k)$$

or

$$\frac{\partial^2 \phi_N}{\partial p_1^2} = \exp\Big(\phi_{N+1} - \phi_N\Big) - \exp\Big(\phi_N - \phi_{N-1}\Big)$$

The integrable times are $t_k = p_k/k$.

Ward identities:

vanishing integrals of total derivatives

$$\int dH \ \mathrm{Tr}\,\frac{\partial}{\partial H}\left[\mathrm{Tr}\,H^{n+1}\exp\left(-\frac{\mu^2}{2}\mathrm{Tr}\,H^2+\sum_k\frac{p_k\mathrm{Tr}\,H^k}{k}\right)\right]=0$$

at $n\geq -1$ gives rise to the infinite set of conditions:

$$L_n Z_N(p_k)=0,\qquad n\geq -1$$

$$L_n=\sum_k(k+n)p_k\frac{\partial}{\partial p_{k+n}}+\sum_{a=1}^{n-1}a(n-a)\frac{\partial^2}{\partial p_a\partial p_{n-a}}+$$
$$+2Nn\frac{\partial}{\partial p_n}+N^2\delta_{n,0}+Np_1\delta_{n+1,0}-\mu^2(n+2)\frac{\partial}{\partial p_{n+2}}$$

These are so called Virasoro constraints since

$$[L_n,L_m]=(n-m)L_{n+m}$$

Solutions of the Virasoro constraints

There are many solutions to the Toda chain equations, but the solution of the infinite set of Virasoro constraints is unique:

$$Z_N(p_k) = 1 + \frac{1}{\mu}\alpha_{[1]}p_1 + \frac{1}{\mu^2}\Big(\alpha_{[1,1]}p_1^2 + \alpha_{[2]}p_2\Big) + \ldots$$

where the coefficients α's are determined recursively.
Non-Gaussian potentials give rise to ambiguous integrals (dependence on the integration contour choice).
Convenient basis: the Schur polynomials, characters of representations of the $GL(N)$ group:

$$Z_N(p_k) = 1 + \frac{1}{\mu}c_{[1]}p_1 + \frac{1}{\mu^2}\Big(c_{[1,1]}\frac{(p_1^2 - p_2)}{2} + c_{[2]}\frac{(p_1^2 + p_2)}{2}\Big) + \ldots = \sum_R \frac{1}{\mu^{|R|}} c_R \chi_R\{p_k\}$$

$\chi_R\{p_k\}$ is the Schur function, R is the Young diagram.

Solutions of the Virasoro constraints

There are many solutions to the Toda chain equations, but the solution of the infinite set of Virasoro constraints is unique:

$$Z_N(p_k) = 1 + \frac{1}{\mu}\alpha_{[1]}p_1 + \frac{1}{\mu^2}\Big(\alpha_{[1,1]}p_1^2 + \alpha_{[2]}p_2\Big) + \ldots$$

where the coefficients α's are determined recursively.
Non-Gaussian potentials give rise to ambiguous integrals (dependence on the integration contour choice).
Convenient basis: the Schur polynomials, characters of representations of the $GL(N)$ group:

$$Z_N(p_k) = 1 + \frac{1}{\mu}c_{[1]}p_1 + \frac{1}{\mu^2}\left(c_{[1,1]}\frac{(p_1^2 - p_2)}{2} + c_{[2]}\frac{(p_1^2 + p_2)}{2}\right) + \ldots = \sum_R \frac{1}{\mu^{|R|}}c_R\chi_R\{p_k\}$$

$\chi_R\{p_k\}$ is the Schur function, R is the Young diagram.

Schur functions

The definition of the Schur functions:

$$\chi_R = \det_{i,j} h_{R_i - i + j} \qquad \exp\left(\sum_k \frac{p_k z^k}{k}\right) = \sum_n h_n z^n$$

The Schur function as a character:

$\chi_R\{p_k\}$ is a symmetric function of x_i, $p_k = \sum x_i^k$, and x_i are the eigenvalues of the group element g in representations R. In other words,

$$\mathrm{Tr}_{\,R}\, g = \chi_R\{p_k = \mathrm{Tr}_{\,f}\, g^k\}$$

Examples:

$$\chi_{[1]}\{p_k\} = p_1 = \sum_i x_i, \qquad \chi_{[1,1]}\{p_k\} = \mathrm{Tr}_{\,[1,1]}\, g = \frac{\left(\mathrm{Tr}_{\,f}\, g\right)^2 - \mathrm{Tr}_{\,f}\, g^2}{2} = \frac{p_1^2 - p_2}{2} = \sum_{i<j} x_i x_j$$

Schur functions

The definition of the Schur functions:

$$\chi_R = \det_{i,j} h_{R_i - i + j} \qquad \exp\left(\sum_k \frac{p_k z^k}{k}\right) = \sum_n h_n z^n$$

The Schur function as a character:

$\chi_R\{p_k\}$ is a symmetric function of x_i, $p_k = \sum x_i^k$, and x_i are the eigenvalues of the group element g in representations R. In other words,

$$\mathrm{Tr}_{\,R}\, g = \chi_R\{p_k = \mathrm{Tr}_{\,f}\, g^k\}$$

Examples:

$$\chi_{[1]}\{p_k\} = p_1 = \sum_i x_i, \qquad \chi_{[1,1]}\{p_k\} = \mathrm{Tr}_{\,[1,1]}\, g = \frac{\left(\mathrm{Tr}_{\,f}\, g\right)^2 - \mathrm{Tr}_{\,f}\, g^2}{2} = \frac{p_1^2 - p_2}{2} = \sum_{i<j} x_i x_j$$

Character property of matrix models

Cauchy formula:

$$\exp\left(\sum_k \frac{p_k \operatorname{Tr} H^k}{k}\right) = \sum_R \chi_R\{\operatorname{Tr} H^k\}\chi_R\{p_k\}$$

This means that

$$Z_N(p_k) = \frac{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2 + \sum_k \frac{p_k \operatorname{Tr} H^k}{k}\right)}{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2\right)} = \sum_R \Big\langle \chi_R\{\operatorname{Tr} H^k\}\Big\rangle \chi_R\{p_k\}$$

The main property:

$$\boxed{\frac{1}{\mu^{|R|}} c_R = \Big\langle \chi_R\{\operatorname{Tr} H^k\}\Big\rangle = \frac{\chi_R\{N\}\cdot\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}}}$$

Character property of matrix models

Cauchy formula:

$$\exp\left(\sum_k \frac{p_k \operatorname{Tr} H^k}{k}\right) = \sum_R \chi_R\{\operatorname{Tr} H^k\}\chi_R\{p_k\}$$

This means that

$$Z_N(p_k) = \frac{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2 + \sum_k \frac{p_k \operatorname{Tr} H^k}{k}\right)}{\int dH \exp\left(-\frac{\mu^2}{2}\operatorname{Tr} H^2\right)} = \sum_R \Big\langle \chi_R\{\operatorname{Tr} H^k\}\Big\rangle \chi_R\{p_k\}$$

The main property:

$$\boxed{\frac{1}{\mu^{|R|}} c_R = \Big\langle \chi_R\{\operatorname{Tr} H^k\}\Big\rangle = \frac{\chi_R\{N\}\cdot\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}}}$$

Universality of the character property

Chern-Simons theory (unitary matrix model).

The Wilson average of a simple closed contour is equal to

$$\langle\chi_R(e^{m_i})\rangle := \int \chi_R(e^{m_i}) \prod_{i<j}^N \sinh^2\left(\frac{m_i - m_j}{2}\right) \prod_{i=1}^N \exp\left(-\frac{m_i^2}{2g^2}\right) dm_i \;=\; q^{N|R|}\cdot q^{2C_2(R)}\cdot \chi_R\{p^*\}$$

$q = e^{g^2/2}$, $C_2(R)$ is the eigenvalue of the 2nd Casimir operator, $p_k^* = \frac{q^{Nk}-q^{-Nk}}{q^k-q^{-k}}$ is q-deformed $p_k = N$.

Kontsevich model:

$$\langle Q_R\{\mathrm{Tr}\,X^k\}\rangle = \frac{\int dX Q_R\{\mathrm{Tr}\,X^k\}\exp\left(-\mathrm{Tr}\,X^2\Lambda\right)}{\int dX \exp\left(-\mathrm{Tr}\,X^2\Lambda\right)} = \begin{cases} \dfrac{Q_{R/2}\{\mathrm{Tr}\,\Lambda^{-k}\}Q_{R/2}\{\delta_{k,1}\}}{Q_R\{\delta_{k,1}\}} & \text{if } R|2 \\ 0 & \text{otherwise} \end{cases}$$

Universality of the character property

Chern-Simons theory (unitary matrix model).

The Wilson average of a simple closed contour is equal to

$$\langle\chi_R(e^{m_i})\rangle := \int \chi_R(e^{m_i}) \prod_{i<j}^{N} \sinh^2\left(\frac{m_i - m_j}{2}\right) \prod_{i=1}^{N} \exp\left(-\frac{m_i^2}{2g^2}\right) dm_i \;=\; q^{N|R|}\cdot q^{2C_2(R)}\cdot\chi_R\{p^*\}$$

$q = e^{g^2/2}$, $C_2(R)$ is the eigenvalue of the 2nd Casimir operator, $p_k^* = \frac{q^{Nk}-q^{-Nk}}{q^k-q^{-k}}$ is q-deformed $p_k = N$.

Kontsevich model:

$$\langle Q_R\{\mathrm{Tr}\,X^k\}\rangle \;=\; \frac{\int dX Q_R\{\mathrm{Tr}\,X^k\}\exp\left(-\mathrm{Tr}\,X^2\Lambda\right)}{\int dX \exp\left(-\mathrm{Tr}\,X^2\Lambda\right)} \;=\; \begin{cases} \dfrac{Q_{R/2}\{\mathrm{Tr}\,\Lambda^{-k}\}Q_{R/2}\{\delta_{k,1}\}}{Q_R\{\delta_{k,1}\}} & \text{if } R|2 \\ 0 & \text{otherwise} \end{cases}$$

Tensor models

The rectangular complex matrix model:

$$\left\langle \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\}\right\rangle = \int_{N_1\times N_2} \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} e^{-\mathrm{Tr}\,M\bar{M}} d^2M = \frac{\chi_R\{N_1\}\chi_R\{N_2\}}{\chi_R\{\delta_{k,1}\}}$$

Universality of the tensor model.

Consider the integral over tensor $M_{a_1\ldots a_r}$. Then,

$$\left\langle \chi_{R_1,\ldots,R_r}\right\rangle = \int \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} e^{-\mathrm{Tr}\,M\bar{M}} d^2M = C_{R_1,\ldots,R_r}\cdot \frac{\chi_{R_1}\{N_1\}\cdot\ldots\cdot\chi_{R_r}\{N_r\}}{\chi_{R_1}\{\delta_{k,1}\}\cdot\ldots\cdot\chi_{R_r}\{\delta_{k,1}\}}$$

Here $\chi_{R_1,\ldots,R_r}$ is the generalized character, and $C_{R_1,\ldots,R_r}$ is the Clebsh-Gordan coefficient for the representations of the symmetric group.

Tensor models

The rectangular complex matrix model:

$$\left\langle \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\}\right\rangle = \int_{N_1\times N_2} \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} e^{-\mathrm{Tr}\,M\bar{M}} d^2M = \frac{\chi_R\{N_1\}\chi_R\{N_2\}}{\chi_R\{\delta_{k,1}\}}$$

Universality of the tensor model.

Consider the integral over tensor $M_{a_1\ldots a_r}$. Then,

$$\boxed{\left\langle \chi_{R_1,\ldots,R_r}\right\rangle = \int \chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} e^{-\mathrm{Tr}\,M\bar{M}} d^2M = C_{R_1,\ldots,R_r}\cdot \frac{\chi_{R_1}\{N_1\}\cdot\ldots\cdot\chi_{R_r}\{N_r\}}{\chi_{R_1}\{\delta_{k,1}\}\cdot\ldots\cdot\chi_{R_r}\{\delta_{k,1}\}}}$$

Here $\chi_{R_1,\ldots,R_r}$ is the generalized character, and $C_{R_1,\ldots,R_r}$ is the Clebsh-Gordan coefficient for the representations of the symmetric group.

Correlators in tensor models

r=2

The invariant operators are given by a permutation

$$\mathcal{K}_\sigma = \prod_{p=1}^{n} M_{a_p b_p} \bar{M}^{a_p b_{\sigma(p)}}$$

where σ is a permutations from S_n. It can be symmetrically written as depending on two permutations,

$$\mathcal{K}_{\sigma_1,\sigma_2} = \prod_{p=1}^{n} M_{a_p b_p} \bar{M}^{a_{\sigma_1(p)} b_{\sigma_2(p)}} = \mathcal{K}_{id,\sigma_1^{-1}\circ\sigma_2}$$

r=3

The invariant $2n$-point operators are parameterized by three permutations from S_n:

$$\mathcal{K}_{\sigma_1\sigma_2\sigma_3} = \prod_{p=1}^{n} M_{a_p b_p c_p} \bar{M}^{a_{\sigma_1(p)} b_{\sigma_2(p)} c_{\sigma_3(p)}}$$

Correlators in tensor models

r=2

The invariant operators are given by a permutation

$$\mathcal{K}_\sigma = \prod_{p=1}^{n} M_{a_p b_p} \bar{M}^{a_p b_{\sigma(p)}}$$

where σ is a permutations from S_n. It can be symmetrically written as depending on two permutations,

$$\mathcal{K}_{\sigma_1,\sigma_2} = \prod_{p=1}^{n} M_{a_p b_p} \bar{M}^{a_{\sigma_1(p)} b_{\sigma_2(p)}} = \mathcal{K}_{id,\sigma_1^{-1}\circ\sigma_2}$$

r=3

The invariant $2n$-point operators are parameterized by three permutations from S_n:

$$\mathcal{K}_{\sigma_1\sigma_2\sigma_3} = \prod_{p=1}^{n} M_{a_p b_p c_p} \bar{M}^{a_{\sigma_1(p)} b_{\sigma_2(p)} c_{\sigma_3(p)}}$$

Generalized characters

$r = 2$ characters

$$\chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} = \frac{1}{n!}\sum_R \psi_R(\sigma)\mathcal{K}_\sigma$$

where σ is a permutation from S_n, and $\psi_R(\sigma)$ is the character of the symmetric group S_n in the representation R. It can be also written in terms of operators depending on two permutations:

$$\chi_{R_1,R_2} \equiv \frac{1}{n!}\sum_{\sigma_1,\sigma_2\in S_n} \psi_{R_1}(\sigma_1)\psi_{R_2}(\sigma_2)\mathcal{K}_{\sigma_1,\sigma_2} = \delta_{R_1,R_2}\frac{\chi_{R_2}\{\mathrm{Tr}\,(M\bar{M})^k\}}{\chi_{R_1}\{\delta_{k,1}\}}$$

The Clebsh-Gordan coefficients

$$C_{R_1R_2} = \frac{1}{n!}\sum_{\gamma\in S_n} \psi_{R_1}(\gamma)\psi_{R_2}(\gamma) = \delta_{R_1R_2}\chi_R\{\delta_{k,1}\}$$

Generalized characters

$r = 2$ characters

$$\chi_R\{\mathrm{Tr}\,(M\bar{M})^k\} = \frac{1}{n!}\sum_R \psi_R(\sigma)\mathcal{K}_\sigma$$

where σ is a permutation from S_n, and $\psi_R(\sigma)$ is the character of the symmetric group S_n in the representation R. It can be also written in terms of operators depending on two permutations:

$$\chi_{R_1,R_2} \equiv \frac{1}{n!}\sum_{\sigma_1,\sigma_2\in S_n} \psi_{R_1}(\sigma_1)\psi_{R_2}(\sigma_2)\mathcal{K}_{\sigma_1,\sigma_2} = \delta_{R_1,R_2}\frac{\chi_{R_2}\{\mathrm{Tr}\,(M\bar{M})^k\}}{\chi_{R_1}\{\delta_{k,1}\}}$$

The Clebsh-Gordan coefficients

$$C_{R_1R_2} = \frac{1}{n!}\sum_{\gamma\in S_n} \psi_{R_1}(\gamma)\psi_{R_2}(\gamma) = \delta_{R_1R_2}\chi_R\{\delta_{k,1}\}$$

$r = 3$ characters

$$\chi_{R_1R_2R_3} := \frac{1}{n!}\sum_{\{\sigma_i\}\in S_n} \psi_{R_1}(\sigma_1)\psi_{R_2}(\sigma_2)\psi_{R_3}(\sigma_3)\cdot \mathcal{K}_{\sigma_1\sigma_2\sigma_3}$$

The Clebsh-Gordan coefficients

$$C_{R_1R_2R_3} = \frac{1}{n!}\sum_{\gamma\in S_n} \psi_{R_1}(\gamma)\psi_{R_2}(\gamma)\psi_{R_3}(\gamma)$$

$r = 3$ characters

$$\boxed{\chi_{R_1R_2R_3} := \frac{1}{n!} \sum_{\{\sigma_i\} \in S_n} \psi_{R_1}(\sigma_1)\psi_{R_2}(\sigma_2)\psi_{R_3}(\sigma_3) \cdot \mathcal{K}_{\sigma_1\sigma_2\sigma_3}}$$

The Clebsh-Gordan coefficients

$$C_{R_1R_2R_3} = \frac{1}{n!} \sum_{\gamma \in S_n} \psi_{R_1}(\gamma)\psi_{R_2}(\gamma)\psi_{R_3}(\gamma)$$

Single equation

Single equation instead of infinite set of Ward identities in the Hermitian model.

Infinite set of Virasoro constraints

$$L_n Z_N(p_k) = 0, \qquad n \geq -1$$

$$L_n = \sum_k (k+n) p_k \frac{\partial}{\partial p_{k+n}} + \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} + \\ + 2Nn \frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + N p_1 \delta_{n+1,0} - \underline{\mu^2 (n+2) \frac{\partial}{\partial p_{n+2}}}$$

Single equation

$$\sum_{n\geq 1} p_n L_{n-2} Z_N\{p_k\} = 0$$

Single equation

Single equation instead of infinite set of Ward identities in the Hermitian model.

Infinite set of Virasoro constraints

$$L_n Z_N(p_k) = 0, \qquad n \geq -1$$

$$L_n = \sum_k (k+n) p_k \frac{\partial}{\partial p_{k+n}} + \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} +$$
$$+2Nn \frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + N p_1 \delta_{n+1,0} - \underline{\mu^2 (n+2) \frac{\partial}{\partial p_{n+2}}}$$

Single equation

$$\sum_{n\geq 1} p_n L_{n-2} Z_N\{p_k\} = 0$$

W-representation

The single equation can be presented in the form

$$\left(\hat{d} - 2\hat{O}^{(2)}\right) Z_N\{p_k\} = 0$$

where $\hat{d} = \sum_k kp_k \frac{\partial}{\partial p_k}$ is the grading operator, and $\hat{O}^{(2)}$ is an operator of grading 2.
Since $\hat{d}$ is the grading operator, any operator of grading k commutes as

$$[\hat{d}, \hat{O}^{(k)}] = k\hat{O}^{(k)}$$

This implies that

$$\hat{d}e^{\hat{O}^{(k)}} = e^{\hat{O}^{(k)}}(\hat{d} + k\hat{O}^{(k)})$$

and the equation

$$(\hat{d} - k\hat{O}^{(k)}) \cdot Z = 0$$

is solved by

$$Z = e^{\hat{O}^{(k)}} \cdot 1$$

This is called W-representation.

A more general example

Consider the case when there are two infinite sets of Ward identities (matrix model with quartic potential depending on external matrix):

$$
\begin{aligned}
{}_3\hat{L}_n Z\{p\} &= 0, \qquad n \geq -1 \\
{}_3\hat{W}_n Z\{p\} &= 0, \qquad n \geq -2
\end{aligned}
$$

The single equation is

$$
\sum_{n=1} p_{3n-1} \cdot \ {}_3\hat{W}_{n-3} Z\{p\} - \sum_{n=1} p_{3n-2} \cdot \ {}_3\hat{L}_{n-2} Z\{p\} = 0
$$

This equation reduces to

$$
\left(\hat{d} - 4\hat{O}_4 - 8\hat{O}_8\right) Z\{p\} = 0
$$

and the W-representation is

$$
Z\{p\} = P\exp\left(\int^x \left(4x'^4\hat{O}_4 + 8x'^8\hat{O}_8\right)\frac{dx'}{x'}\right) \cdot 1
$$

The rule of thumb: in order to get the single equation, one has to construct a combination bilinear in p_k and Ward identities such that it contains $\hat{d}$.

Two new basic properties of matrix models:

The average of a proper character is proportional to a character

The infinite set of Ward identities can be written as a single equation, which leads to the W-representation

Thank you for your attention!