Difference equations and Integrability

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Outline

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2. ABJM theory and its QSC
   - Pμ-system
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XXX Heisenberg spin chain and QQ-system

For a spin chain with a fundamental representation (on the space $\mathbb{C}^2$ with basis $|\uparrow\rangle$ and $|\downarrow\rangle$) at each site, the Hamiltonian is

$$H = \sum_{j=1}^{L} (1 - P_{j,j+1}) \,.$$

Its spectrum is described by Bethe equations

$$\left(\frac{u_{1,j} + i/2}{u_{1,j} - i/2} \right)^{L} = - \prod_{k=1}^{K} \frac{u_{1,j} - u_{1,k} + i}{u_{1,j} - u_{1,k} - i} \,, \qquad j = 1, \ldots, K \,,$$

where energies are given by

$$E = \sum_{j=1}^{K} \frac{1}{u_{1,j}^2 + 1/4} \,.$$

These Bethe equations are equivalent to canonical QQ-system relation $(f^{\pm}(u) \equiv f(u \pm i/2))$:

$$Q_1^+ Q_2^- - Q_1^- Q_2^+ = Q_\emptyset Q_{12}$$

where

$$Q_\emptyset = u^L, \quad Q_{12} = 1$$

and

$$Q_1(u) = \prod_{j=1}^{K} (u - u_{1,j}) \ ,$$

$$Q_2(u) = \mathrm{const} \times \prod_{j=1}^{L-K+1} (u - u_{2,j}).$$

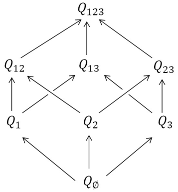
Q_1 describes excitations with respect to pseudovacuum $|0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$, while Q_2 - excitations with respect to pseudovacuum $|0'\rangle = |\downarrow\downarrow\ldots\downarrow\rangle$.

XXX Heisenberg spin chain and QQ-system

Similar in the case of $sl(3)$ spin chain the spectrum can be described with canonical QQ relations ($Q_{\emptyset} = u^L, Q_{123} = 1$):

$$Q_A Q_{Aab} = Q^+_{Aa} Q^-_{Ab} - Q^-_{Aa} Q^+_{Ab}$$

depicted in Hasse diagram



QQ relations and path in Hasse diagram give us Bethe equations:

$$\frac{Q^+_{\emptyset}}{Q^-_{\emptyset}} = -\frac{Q^{++}_a}{Q^{--}_a}\frac{Q^-_{ab}}{Q^+_{ab}}\Bigg|_{u=u_{a,n}}$$

$$1 = -\frac{Q^{++}_{ab}}{Q^{--}_{ab}}\frac{Q^-_{a}}{Q^+_{a}}\Bigg|_{u=u_{ab,n}}$$

ABJM theory and its QSC

ABJM is $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory with gauge group $U(N) \times U(N)$ on $\mathbb{R}^{1,2}$ and Chern-Simons levels k and $-k$. The field content is given by gauge fields A_μ and $\hat{A}_\mu$, four complex scalars Y^A and four Weyl spinors ψ_A.

In planar limit $k, N \to \infty, \quad \lambda \equiv \frac{N}{k} = \text{fixed}$ **it is dual to superstring theory on** $AdS_4 \times \mathbb{CP}^3$

We will be interested in anomalous dimensions of operators

$$\text{tr}\left[D_+^S (Y^1 Y_4^\dagger)^L\right]$$

with Dynkin labels $[L + S, S; L, 0, L]$ under $\text{OSp}(4|6)$

Aharony, Bergman, Jafferis, Maldacena 2008

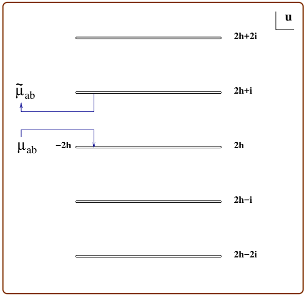
ABJM theory and its QSC: $\mathbf{P}\mu$-system

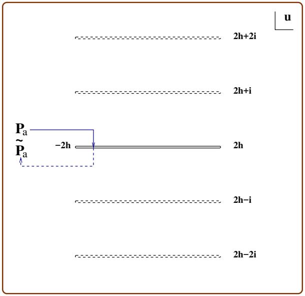
Vector form (CP^3 isometry group $SO(6) \simeq SU(4)$):

$$\mathbf{P}_A(u)\Big|_{A=1,\dots,6}, \quad \mu_{AB}(u) = -\mu_{BA}(u)\Big|_{A,B=1,\dots,6}$$

$$\widetilde{\mathbf{P}}_A - \mathbf{P}_A = \mu_{AB}\,\eta^{BC}\,\mathbf{P}_C, \qquad \widetilde{\mu}_{AB} - \mu_{AB} = \mathbf{P}_A\widetilde{\mathbf{P}}_B - \mathbf{P}_B\widetilde{\mathbf{P}}_A.$$

$$\mathbf{P}_5\mathbf{P}_6 - \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_1\mathbf{P}_4 = 1, \quad \mu_{AB}\,\eta^{BC}\,\mu_{CD} = 0, \quad \widetilde{\mu}_{AB}(u) = \mu_{AB}(u+i)$$





Gromov, Kazakov, Leurent, Volin, 2013
Cavaglia, Fioravanti, Gromov, Tateo, 2014

Spinor form (CP^3 isometry group $SO(6) \simeq SU(4)$): the matrix $\mu_{AB}(u)$ is decomposed in terms of $4+4$ functions ν_a, ν^a as

$$\mu_{AB} = \nu^a \, (\sigma_{AB})_a^{\ b} \, \nu_b \,, \quad \nu^a \, \nu_a = 0.$$
$$\widetilde{\nu}_a(u) = e^{i\mathcal{P}} \, \nu_a(u+i), \quad \widetilde{\nu}^a(u) = e^{-i\mathcal{P}} \, \nu^a(u+i)$$

Riemann-Hilbert problem to solve:

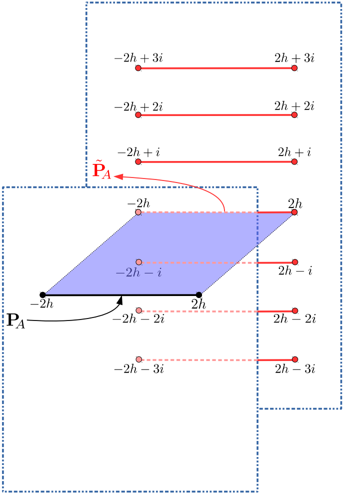
$$\boxed{\begin{array}{c} \widetilde{\mathbf{P}}_{ab} - \mathbf{P}_{ab} = \nu_a \tilde{\nu}_b - \nu_b \tilde{\nu}_a, \quad \widetilde{\mathbf{P}}^{ab} - \mathbf{P}^{ab} = -\nu^a \tilde{\nu}^b + \nu^b \tilde{\nu}^a, \\ \tilde{\nu}_a = -\mathbf{P}_{ab} \, \nu^b, \quad \tilde{\nu}^a = -\mathbf{P}^{ab} \, \nu_b. \end{array}}$$

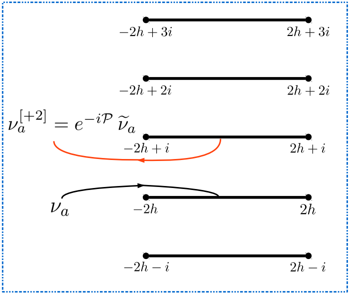
$$\mathbf{P}_{ab} = \mathbf{P}_A \sigma^A_{ab} = \begin{pmatrix} 0 & -\mathbf{P}_1 & -\mathbf{P}_2 & -\mathbf{P}_5 \\ \mathbf{P}_1 & 0 & -\mathbf{P}_6 & -\mathbf{P}_3 \\ \mathbf{P}_2 & \mathbf{P}_6 & 0 & -\mathbf{P}_4 \\ \mathbf{P}_5 & \mathbf{P}_3 & \mathbf{P}_4 & 0 \end{pmatrix}, \quad \mathbf{P}^{ab} \text{ is inverse matrix}$$

Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017

ABJM theory and its QSC: $\mathbf{P}\mu$-system

$$f^{[n]}(u) = f(u + ni/2)$$





Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017

Boundary conditions in $sl(2)$ sector (large u):

$$\mathbf{P}_a \simeq (A_1 u^{-L},\, A_2 u^{-L-1},\, A_3 u^{+L+1},\, A_4 u^{+L},\, A_0 u^0),$$
$$-A_1 A_4 = \frac{(-\Delta + L - S)(-\Delta + L + S - 1)(\Delta + L - S + 1)(\Delta + L + S)}{L^2(2L+1)},$$
$$-A_2 A_3 = \frac{(-\Delta + L - S + 1)(-\Delta + L + S)(\Delta + L - S + 2)(\Delta + L + S + 1)}{(L+1)^2(2L+1)},$$

$$\nu_a \sim \left(u^{\Delta - L}, u^{\Delta + 1}, u^{\Delta}, u^{\Delta + L + 1}\right).$$

$L \in \mathbb{N}^+$ (twist), $S \in \mathbb{N}^+$ (spin) and Δ is the conformal dimension. The anomalous dimension γ is given by $\gamma = \triangle - L - S$.

Cavaglia, Fioravanti, Gromov, Tateo, 2014

We will look for solution at weak coupling in the form $(\mathbf{P}_0 = \mathbf{P}_5 = \mathbf{P}_6)$

$$\mathbf{P}_1 = (xh)^{-L}\mathbf{p}_1 = (xh)^{-L}\left(1 + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{1,k}^{(l)}\frac{h^{2l+k}}{x^k}\right), \quad \nu_i(u) = \sum_{l=1}^{\infty} h^{2l-L}\nu_i^{(l)}(u),$$

$$\mathbf{P}_2 = (xh)^{-L}\mathbf{p}_2 = (xh)^{-L}\left(\frac{h}{x} + \sum_{k=2}^{\infty}\sum_{l=0}^{\infty} c_{2,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_0 = (xh)^{-L}\mathbf{p}_0 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_0^{(l)} h^{2l} u^L + \sum_{j=0}^{L-1}\sum_{l=0}^{\infty} m_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{0,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_3 = (xh)^{-L}\mathbf{p}_3 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_3^{(l)} h^{2l} u^{2L+1} + \sum_{j=0}^{2L}\sum_{l=0}^{\infty} k_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{3,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

where

$$x \equiv x(u) = \frac{u + \sqrt{u^2 - 4h^2}}{2h}$$

$c_{i,k}^{(l)}$ are some functions of spin S only, otherwise they are just constants. The analytically continued though the cut functions are defined as

$$\tilde{\mathbf{P}}_i = \left(\frac{x}{h}\right)^L \tilde{\mathbf{p}}_i\,, \quad \tilde{\mathbf{p}}_i = \mathbf{p}_i\Big|_{x \to 1/x}.$$

Initial conditions for iterative solution:

$$\mathbf{p}_{1,0} = 1, \qquad \mathbf{p}_{2,0} = 0,$$
$$\widetilde{\mathbf{p}}_{1,0}(u) \sim 1 + O(u), \qquad \widetilde{\mathbf{p}}_{2,0}(u) \sim u + O(u^2).$$

Baxter equations to solve (state quantum numbers are specified by LO Baxter polynomial $Q(u) \sim \nu_1^{[1]}(u)$) :

$$\frac{\nu_1^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_1^{[-1]}}{\mathbf{P}_1^{[-1]}} - \sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]} = -\sigma\left(\frac{\mathbf{P}_2^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_2^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]}.$$

$$\frac{\nu_2^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_2^{[-1]}}{\mathbf{P}_1^{[-1]}} + \sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]} = \sigma\left(\frac{\mathbf{P}_3^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_3^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]}.$$

where

$$\sigma \equiv e^{i\mathcal{P}} = Q^{[1]}(0)/Q^{[-1]}(0)\,, \quad Q \text{ is LO Baxter polynomial}$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

Coefficients are fixed from equations:

$$\left.\begin{aligned} \nu_a(u)+\tilde{\nu}_a(u) &= \nu_a(u)+\sigma\,\nu_a^{[2]}(u) \\ \frac{\nu_a(u)-\tilde{\nu}_a(u)}{\sqrt{u^2-4h^2}} &= \frac{\nu_a(u)-\sigma\,\nu_a^{[2]}(u)}{\sqrt{u^2-4h^2}} \end{aligned}\right\} \quad \text{free of cuts on real axis}$$

$$\begin{aligned} \left(\nu_1+\sigma\,\nu_1^{[2]}\right)\left(\mathbf{p}_0-(hx)^L\right) &= \mathbf{p}_2\left(\nu_2+\sigma\,\nu_2^{[2]}\right)-\mathbf{p}_1\left(\nu_3+\sigma\,\nu_3^{[2]}\right), \\ \left(\nu_2+\sigma\,\nu_2^{[2]}\right)\left(\mathbf{p}_0+(hx)^L\right) &= \mathbf{p}_3\left(\nu_1+\sigma\,\nu_1^{[2]}\right)+\mathbf{p}_1\left(\nu_4+\sigma\,\nu_4^{[2]}\right). \end{aligned}$$

$$\sigma\,\nu_1^{[2]}=\mathbf{P}_0\,\nu_1-\mathbf{P}_2\,\nu_2+\mathbf{P}_1\,\nu_3, \quad \tilde{\mathbf{P}}_2-\mathbf{P}_2=\sigma\left(\nu_3\nu_1^{[2]}-\nu_1\nu_3^{[2]}\right),$$

$$\sigma\,\nu_2^{[2]}=-\mathbf{P}_0\,\nu_2+\mathbf{P}_3\,\nu_1+\mathbf{P}_1\,\nu_4, \quad \tilde{\mathbf{P}}_1-\mathbf{P}_1=\sigma\left(\nu_2\nu_1^{[2]}-\nu_1\nu_2^{[2]}\right).$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

The most complex part is the solution at each perturbative order k of two Baxter equations, which for $L = 1$ take the form:

$$\mathcal{B}_1^S[q_1] \equiv (u+i/2)q_1^{(k)[2]}(u) - i(2S+1)q_1^{(k)}(u) - (u-i/2)q_1^{(k)[-2]}(u) = V_1^{(k)}\,,$$
$$\mathcal{B}_2^S[q_2] \equiv (u+i/2)q_2^{(k)[2]}(u) + i(2S+1)q_2^{(k)}(u) - (u-i/2)q_2^{(k)[-2]}(u) = V_2^{(k)}\,.$$

$$q_1^{(k)}(u) = \nu_1^{(k)[1]}(u)\,, \quad q_2^{(k)}(u) = \nu_2^{(k)[1]}(u)$$

For fixed integer spins the solution is expressed in terms of polynomials, rational functions and generalized Hurwitz functions

$$\eta_{a_1,a_2,\ldots,a_k}(u) = \sum_{n_k > n_{k-1} > \cdots > n_1 \geq 0} \prod_{i=1}^{k} \frac{(\mathrm{sgn}(a_i))^{n_i - n_{i-1} - 1}}{(u + i n_i)^{|a_i|}}\,,$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

Solution of Baxter equations could be written as:

$$q_1^{(k)} = \mathcal{F}_1^S\left[V_1^{(k)}\right] + Q_S\Phi_1^{per,(k)} + \mathcal{Z}_S\Phi_1^{anti,(k)}$$

$$q_2^{(k)} = \mathcal{F}_2^S\left[V_2^{(k)}\right] + Q_S\Phi_2^{anti,(k)} + \mathcal{Z}_S\Phi_2^{per,(k)}$$

where ($\nabla_\pm\Psi_\pm g = g, \nabla_\pm g = g \mp g^{[2]}$):

$$\mathcal{F}_1^S[f] = -Q_S\Psi_+\left(\frac{1}{u+i/2}\Psi_-\left(Q_S(-1)^S f\right)^{[2]}\right) - Q_S\Psi_+\left(P_S(-1)^S f\right) + P_S\Psi_-\left(Q_S(-1)^S f\right)$$

$$\mathcal{F}_2^S[f] = -Q_S\Psi_-\left(\frac{1}{u+i/2}\Psi_+\left(Q_S(-1)^S f\right)^{[2]}\right) + Q_S\Psi_-\left(P_S(-1)^S f\right) - P_S\Psi_+\left(Q_S(-1)^S f\right)$$

and

$$P_S(u) = i\sum_{k=0}^{\left[\frac{S-1}{2}\right]}\frac{1}{S-k}Q_{S-1-2k}(u)$$

$$Q_S\left(u\right) = \frac{(-1)^S\Gamma\left(\frac{1}{2}+iu\right)}{S!\Gamma\left(\frac{1}{2}+iu-S\right)}\,{}_2F_1\left(-S, \frac{1}{2}+iu; \frac{1}{2}+iu-S; -1\right),$$

$$\mathcal{Z}_S(u) = i\sigma\sum_{k=0}^{\left\lfloor\frac{S-1}{2}\right\rfloor}\frac{1}{S-k}Q_{S-1-2k}\left(u\right) + \sigma\eta_{-1}(u+i/2)Q_S\left(u\right)$$

Lee, AO, 2018

To obtain the solution for arbitrary integer spins we will need to introduce new class of functions: sums of Baxter polynomials

$$\langle Q(u)\,|w_1(\bullet), w_2(\bullet), \ldots, w_n(\bullet)\rangle = \sum_{S\geq j_1>j_2\cdots>j_n>0} Q_{S-j_1}(u)\prod_k w_k(j_k)\,,$$
$$\langle Q(u)\,|\rangle = Q_S(u)\,.$$

For example

$$\left\langle Q(u)\,|\frac{(-1)^\bullet}{(\bullet)^3}, \frac{1}{(S+1-\bullet)^2}\right\rangle = \sum_{S\geq j_1>j_2>0} Q_{S-j_1}(u)\frac{(-1)^{j_1}}{j_1^3}\frac{1}{(S+1-j_2)^2}$$

The weights are not arbitrary, but take the form

$$\frac{1}{\bullet^n} = n_+(\bullet)\,, \qquad \frac{(-)^\bullet}{\bullet^n} = n_-(\bullet)\,,$$
$$\frac{1}{(S+1-\bullet)^n} = \overline{n}_+(\bullet)\,, \qquad \frac{(-)^\bullet}{(S+1-\bullet)^n} = \overline{n}_-(\bullet)\,,$$
$$\frac{1}{(2S+1-\bullet)^n} = \hat{n}_+(\bullet)\,, \qquad \frac{(-)^\bullet}{(2S+1-\bullet)^n} = \hat{n}_-(\bullet)\,.$$

Lee, AO, 2019

Moreover, the introduced sums have a number of nice properties under shifts ($a = \pm 1$):

$$Q_S^{[2a]} = Q_S + 2\sum_{k=1}^{S} a^k Q_{S-k} = Q_S + 2\langle Q|0_a\rangle$$

$$\langle Q|w,W\rangle^{[2a]} = \langle Q|w,W\rangle + 2\langle Q|0_a, 0_a \cdot w, W\rangle \ .$$

and partial fractions:

$$\frac{Q_S}{u + a\frac{i}{2}} = \frac{(-a)^S}{u + a\frac{i}{2}} + 2ia\left\langle Q|0_a, \overline{1}_-\right\rangle + 2ia\left\langle Q|\overline{1}_{-a}\right\rangle$$

$$\frac{\langle Q|w,W\rangle}{u + a\frac{i}{2}} = \frac{(-a)^S}{u + a\frac{i}{2}}\langle 0_{-a}\cdot w, W\rangle + 2ia\left\langle Q|0_a, \overline{1}_-, 0_{-a}\cdot w, W\right\rangle + 2ia\left\langle Q|\overline{1}_{-a}, 0_{-a}\cdot w, W\right\rangle \ .$$

In addition this class of functions is closed under differentiation.

Lee, AO, 2019

ABJM theory and its QSC: solution for $sl(2)$ sector

To find particular solutions for arbitrary spins we introduce the idea of dictionary - find recursively images $\mathcal{F}_i^S[f]$ for canonical inhomogeneities f. Consider for example $\mathcal{F}_1^S$ image for $\langle Q(u)|w, W\rangle$:

$$\mathcal{B}_1^S\left[\langle Q|w, W\rangle\right] = \sum_{j=1}^{S} \mathcal{B}_1^S\left[Q_{S-j}\right] w(j)\,|W\rangle_j$$

$$= -2i \sum_{j=1}^{S} Q_{S-j}\, j w(j)\,|W\rangle_j = -2i\,\langle Q|(-1)_+ \cdot w, W\rangle$$

Now, replacing $w \to 1_+ \cdot w$ and taking $\mathcal{F}_1^S$ from both side and using $\mathcal{F}_1^S[\mathcal{B}_1^S[f]] = f$ we get

$$\mathcal{F}_1^S\left[\langle Q|w, W\rangle\right] = \frac{i}{2}\,\langle Q|1_+ \cdot w, W\rangle$$

This way we may construct all required images for arbitrary order of perturbation theory and stay all the time within the introduced class of functions.

Lee, AO, 2019

At four loop order we obtained

$$\gamma(S) = \gamma^{(0)}(S)h^2 + \gamma^{(1)}(S)h^4 + \ldots$$

where

$$\gamma^{(0)}(S) = 4\left(\bar{H}_1 + \bar{H}_{-1} - 2\bar{H}_i\right)$$

$$\begin{aligned}\gamma^{(1)}(S) = 16\Big\{&3\bar{H}_{-2,-1} - 2\bar{H}_{-2,i} - \bar{H}_{-2,1} - \bar{H}_{-1,-2} + 2\bar{H}_{-1,2i} - \bar{H}_{-1,2} - 6\bar{H}_{i,-2} \\ &+ 12\bar{H}_{i,2i} - 6\bar{H}_{i,2} - 6\bar{H}_{2i,-1} + 4\bar{H}_{2i,i} + 2\bar{H}_{2i,1} - \bar{H}_{1,-2} + 2\bar{H}_{1,2i} - \bar{H}_{1,2} + 3\bar{H}_{2,-1} \\ &- 2\bar{H}_{2,i} - \bar{H}_{2,1} + 2\bar{H}_{-1,i,-1} - 2\bar{H}_{-1,i,1} + 8\bar{H}_{i,-1,-1} - 12\bar{H}_{i,-1,i} + 4\bar{H}_{i,-1,1} - 16\bar{H}_{i,i,-1} \\ &+ 16\bar{H}_{i,i,i} + 4\bar{H}_{i,1,-1} - 4\bar{H}_{i,1,i} + 2\bar{H}_{1,i,-1} - 2\bar{H}_{1,i,1}\Big\} + 8\left(H_{-1} - H_1\right)\zeta_2\end{aligned}$$

and we introduced new sums

$$H_{a,b,\ldots}(S) = \sum_{k=1}^{S} \frac{\Re[(a/|a|)^k]}{k^{|a|}} H_{b,\ldots}(k) \qquad H_{a,\ldots} = H_{a,\ldots}(S) \qquad \bar{H}_{a,\ldots} = H_{a,\ldots}(2S)$$

Lee, AO, 2017

ABJM theory and its QSC: solution for $sl(2)$ sector

At six loops for fixed spin values we get

$$
\begin{aligned}
\gamma^{(2)}(5) &= \frac{16928\zeta(3)}{25} - \frac{749207584}{1771875} + \frac{92912\pi^2}{1575} + \frac{322\pi^4}{75} - \frac{33856}{225}\pi^2\log(2) \\
\gamma^{(2)}(10) &= \frac{10143008\zeta(3)}{11025} - \frac{303562045526159958 4}{143248910889459375} + \frac{4641541857896\pi^2}{173241313245} \\
&+ \frac{1126\pi^4}{225} - \frac{20286016\pi^2\log(2)}{99225} \\
\gamma^{(2)}(15) &= \frac{265411493888\zeta(3)}{225450225} - \frac{36242750794665141402652795479 04}{6659016057333576400844067187 5} \\
&+ \frac{3593709256322943648256\pi^2}{98455081120952180625} + \frac{182144\pi^4}{32175} - \frac{530822987776\pi^2\log(2)}{2029052025} \\
\gamma^{(2)}(20) &= \frac{30827191890924032\zeta(3)}{23520996524025} + \frac{194885704751142318496410297520349122808564758 4}{748214428437133284539377524885424237701562 5} \\
&+ \frac{8973760124028289167906009359 68\pi^2}{106034967193798706401189726875} + \frac{62075752\pi^4}{10392525} - \frac{61654383781848064\pi^2\log(2)}{211688968716225}
\end{aligned}
$$

and the results for arbitrary integer spins can be found on arXiv

Lee, AO, 2019

$\mathcal{N} = 4$ SYM theory and its QSC

$\mathcal{N} = 4$ SYM is $\mathcal{N} = 4$ superconformal Yang-Mills theory with gauge group $SU(N)$ on $\mathbb{R}^{1,3}$. The field content is given by gauge field A_μ, six scalars ϕ_{ab} in antisymmetric representation of $SU(4)_R$ together with four chiral ψ_α^a fermions in fundamental and four anti-chiral $\bar{\psi}_{\dot{\alpha}a}$ fermions in anti-fundamental representation of $SU(4)_R$.

In planar limit $N \to \infty, \quad \lambda \equiv g^2 N = \text{fixed}$ **it is dual to superstring theory on** $AdS_5 \times S_5$

We will be interested in anomalous dimensions of operators

$$\mathrm{tr}\left[D_+^S \phi_{34}^L\right]$$

with Dynkin labels $[L+S, S, S; 0, L, 0]$ under $\mathrm{PSU}(2,2|4)$

Brink, Scherk, Schwarz, 1977

$\mathcal{N} = 4$ SYM theory and its QSC: $\mathbf{P}\mu$-system

P-functions in this case carry indexes corresponding to isometry group of S_5 ($SU(4)_R$ R-symmetry of $\mathcal{N} = 4$ SYM) and Riemann-Hilbert problem for $sl(2)$-sector takes the form $(a, b = 1, \ldots, 4)$:

$$\mu_{ab} - \tilde{\mu}_{ab} = \tilde{\mathbf{P}}_a \mathbf{P}_b - \tilde{\mathbf{P}}_b \mathbf{P}_a \,,$$
$$\tilde{\mathbf{P}}_a = (\mu\chi)_a{}^b \mathbf{P}_b \,,$$
$$\tilde{\mu}_{ab} = \mu_{ab}^{[2]} \,,$$

where μ_{ab} is antisymmetric matrix, $(\mu\chi)_a{}^b \equiv \mu_{ac}\chi^{cb}$ and

$$\chi^{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

Gromov, Kazakov, Leurent, Volin, 2013

Boundary conditions in $sl(2)$ sector (large u):

$$\mathbf{P}_1 \simeq A_1 u^{-\frac{L+2}{2}}\,, \mathbf{P}_2 \simeq A_2 u^{-\frac{L}{2}}\,, \mathbf{P}_3 \simeq A_3 u^{\frac{L-2}{2}}\,, \mathbf{P}_4 \simeq A_4 u^{\frac{L}{2}}\,,$$
$$\mu_1 \sim u^{\Delta-L},\ \mu_2 \sim u^{\Delta-1},\ \mu_3 \sim u^{\Delta},\ \mu_4 \sim u^{\Delta+L},\ \mu_5 \sim u^{\Delta+L}$$

with

$$A_1 A_4 = \frac{[(L-S+2)^2-\Delta^2][(L+S)^2-\Delta^2]}{16iL(L+1)}\,,$$
$$A_2 A_3 = \frac{[(L+S-2)^2-\Delta^2][(L-S)^2-\Delta^2]}{16iL(L-1)}\,.$$

$L \in \mathbb{N}^+$ (twist), $S \in \mathbb{N}^+$ (spin) and Δ is the conformal dimension. The anomalous dimension γ is given by $\gamma = \triangle - L - S$.

Gromov, Kazakov, Leurent, Volin, 2013

The solution in this case goes similar to ABJM case considered before. In particular we introduce similar ansatz for **P**-functions and reduce the perturbative solution of Riemann-Hilbert problem to iterative solution of two inhomogeneous second-order Baxter and one inhomogeneous first-order difference equations. The coefficients in the ansatz for **P**-functions are then determined from the similar constraint equations.

The two Baxter equations are both of the form

$$B_S[q] = (u+\frac{i}{2})^2 q^{[2]}(u) + (u-\frac{i}{2})^2 q^{[-2]}(u) - (2u^2 - \frac{1}{2} - S(S+1))q(u) = V(u)$$

while the first-order difference equation is given by

$$\nabla r(u) = r(u) - r(u+i) = V(u)\,.$$

$\mathcal{N}=4$ SYM theory and its QSC: solution for $sl(2)$ sector

For this model and twist 2 operators we may also introduce class of functions - sums of Baxter polynomials and find their rules under shifts, multiplication by simple fractions, differentiation and so on. The corresponding analysis is however more involved as leading order Baxter polynomials in this case are more complex functions

$$Q_S(u) = {}_3F_2\left(-S, S+1, \frac{1}{2} - i\,u; 1, 1; 1\right)$$

Within this class of functions we may recursively determine all the required images both for second-order Baxter and first-order difference equations needed for finding corresponding particular solutions for in principle arbitrary prescribed order in perturbation theory. As an example we re-derived known four-loop anomalous dimensions of twist 2 operators. The results can be found on arXiv. Interesting fact is that the weights in sums of Baxter polynomials are the same as for ABJM model, where we have seen the appearance of new harmonic sums decorated by fourth root of unity. Recently, the latter also appeared in the reconstruction of NNNLO eigenvalue of BFKL kernel performed by Velizhanin.

AO, 2021

Conclusion and future directions

- We have introduced new class of special functions relevant to the solution of long-range spin-chains and studied their properties
- We have also performed similar analysis for twist 2 operators within ABJM model.
- Obtained results gives us hope that similar techniques will also work for higher twists and other models.
- An extension of computational techniques to twisted ABJM and $\mathcal{N} = 4$ theories
- Develop techniques for strong coupling and large spin expansion

Thank you for your attention!