From toy models to beta-functions in arbitrary renormalizable QFT

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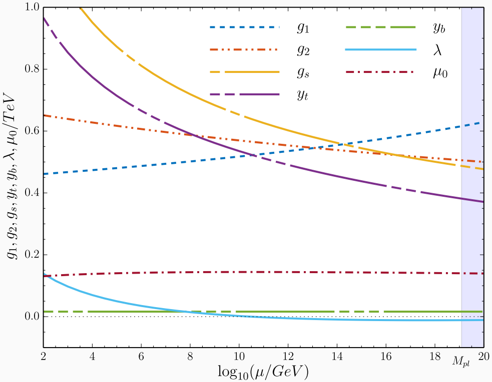
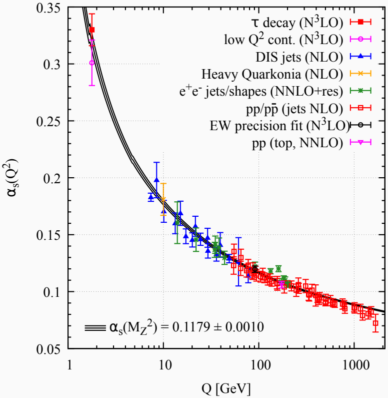
based on PRL 127 (2021) 4

Well known beta-functions use cases



Vacuum stability in the SM

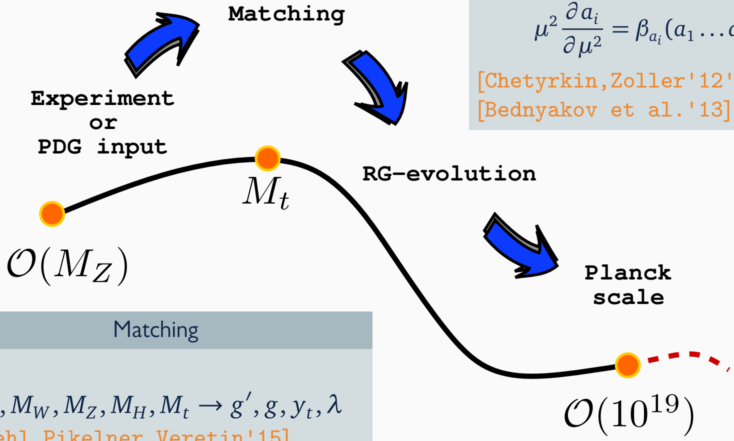
$\beta_\lambda = 0$



QCD running coupling

5-loop precision

More on the vacuum stability in the SM



Running

$$\mu^2 \frac{\partial a_i}{\partial \mu^2} = \beta_{a_i}(a_1 \dots a_n)$$

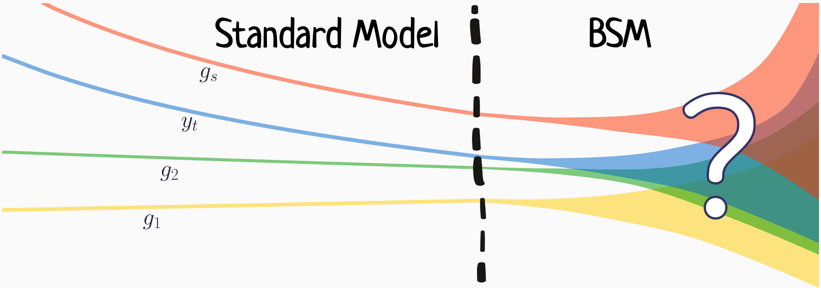
[Chetyrkin,Zoller'12'13]
[Bednyakov et al.'13]

Matching

$$G_F, M_W, M_Z, M_H, M_t \to g', g, y_t, \lambda$$

[Kniehl,Pikelner,Veretin'15]
[Martin,Robertson'19]

Going beyond SM beta functions



- At some scale if BSM scenario realizes we need precise RG evolution comparable with SM precision level

Summary of the general model results

- 2-loop results by Mahacek and Vaughn [Machacek,Vaughn'83,83,84]

$$\beta^g(g, Y^a_{ij}, \lambda_{abcd}), \quad \beta^y_{a,ij}(g, Y^a_{ij}, \lambda_{abcd}), \quad \beta^\lambda(g, Y^a_{ij}, \lambda_{abcd})$$

- 3-loop gauge coupling, simple group [Pickering,Gracey,Jones'01]

$$\beta^g(g, Y^a_{ij}, \lambda_{abcd})$$

- 3-loop gauge, non-simple group [Mihaila'13][Poole,Thomsen'19]

$$\beta^g_I(g_I, Y^a_{ij}, \lambda_{abcd})$$

- 6-loop pure scalar theory [Bednyakov,AP'21]

$$\beta^\lambda_{abcd}(\lambda_{abcd})$$

Standard RG calculations toolchain

- Implement model, usually by hand
- Generate diagrams with `DIANA`
- Make algebraic manipulations with `FORM`
- Perform single scale integrals reduction
  - Massless propagators `MINCER` and `FORCER`
  - Fully massive tadpoles `MATAD` and `FMFT`
  - Laporta algorithm implementations
- Renormalization of all calculated Green functions
- RG functions derivation from renormalization constants

Dreams about general model codes

- From Lagrangian to Feynman rules for specific model
  - LanHEP
  - FeynRules
- From model to amplitudes and observables
  - FeynArts
  - FeynCalc, FormCalc
  - CompHEP,CalcHEP
  - GoSam MadGraph5_aMC@NLO
- General model RG equations from the simple model description
  - SARAH 4 [Staub'13]
  - PyR@TE 3 [Sartore,Schienbein'20]
  - ARGES [Litim,Steudtner'20]
  - RGBeta [Thomsen'21]

General model ansatz idea

- TS in results of Machacek and Vaughn can be generated w/o any loop integrations and renormalization

$$\beta_g^{2-\text{loop}} = -g^3 \left(A_1 \cdot C(G) + B_1 \cdot T(R) + C_1 \cdot T(S)\right) - g^3 \Big(g^2 A_2 \cdot C(G)^2 + g^2 B_2 \cdot C(G)T(R) + g^2 C_2 \cdot C(G)T(S) + g^2 D_2 \cdot \frac{TrC(S)^2}{N_A} + E_2 \cdot \frac{TrC(R)\bar{Y}_a Y_a}{N_A} + g^2 F_2 \cdot \frac{TrC(R)^2}{N_A} \Big)$$

- To construct TS no need for loop integrals and renormalization
- Coefficients $\{A, B, \dots\}$ - rational numbers and MZV @ high orders

Known applications:

- TS ansatz for general Gauge-Yukawa model (4-3-2) [Poole,Thomsen'19]
- TS ansatz and CS fixing from existing models for scalar [Steudtner'20]
- And general Yukawa theory [Steudtner'21]

RGBeta session

QCD 1-loop example

```
(* Define gauge group *)
   AddGaugeGroup[g, SUc, SU[Nc]];
(* Fermions in fundamental representation *)
   AddFermion[qL, GaugeRep -> {SUc[fund]}];
   AddFermion[qR, GaugeRep -> {Bar@SUc[fund]}];
(* 1-loop beta-function *)
   BetaTerm[g, 1]
```

Some features:

- Advanced tool to construct model specific tensor structures T_i
- Collection of model independent coefficients C_i

	Gauge	Yukawa	Lambda
TS	4	3	2
CS	3→4	2→3	2

- All included TS clasified and available from [Poole,Thomsen'19]

Fixing coefficiets from WCC

Due to the Weyl Consistency Conditions(WCC) and postulated a-function we have relations between different terms of β_i [Osborn'91]

WCC provides us with relations connecting different β_i [Antipin et al.'13]

For example 3-2-1 relations in the SM with $a_i = \{a_1, a_2, a_3, a_t, a_\lambda\}$:

$$\beta_1 = 2a_1^2\left\{\cdots + \left(\frac{3}{4} + \frac{n_G}{2}\right)a_2 + \frac{22n_G}{9}a_3 + \cdots + a_t\left[-\frac{17}{12} - \ldots\right] + a_\lambda\left(\frac{3}{4}a_1 + \frac{3}{4}a_2 - \frac{3}{2}a_\lambda\right)\right\}$$

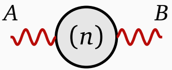
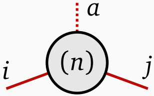
$$\beta_2 = 2a_2^2\left\{\cdots + \left(\frac{1}{4} + \frac{n_G}{6}\right)a_1 + \cdots + 2n_G a_3 \cdots + a_t\left[-\frac{3}{4} - \ldots\right] + a_\lambda\left(\frac{1}{4}a_1 + \frac{3}{4}a_2 - \frac{3}{2}a_\lambda\right)\right\}$$

$$\beta_3 = 2a_3^2\left\{\cdots + \frac{11n_G}{36}a_1 + \frac{3n_G}{4}a_2 + \cdots + a_t\left[-1 - \ldots\right]\right\}$$

$$\beta_t = 2a_t\left\{\frac{9}{4}a_t - 4a_3 - \frac{17}{24}a_1 - \frac{9}{8}a_2 + 3a_\lambda^2 - 6a_t a_\lambda - \ldots\right\}$$

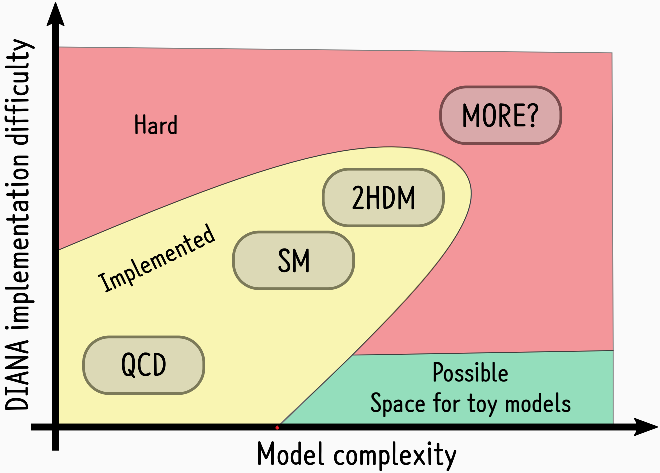
$$\beta_\lambda = \frac{9}{16}a_2^2 - \frac{9}{2}a_\lambda a_2 + \frac{3}{16}a_1^2 - \frac{3}{2}a_\lambda a_1 + \frac{3}{8}a_1 a_2 + 12a_\lambda^2 + 6a_\lambda a_t - 3a_t^2$$

TODO part after all known constraints

$\beta^{(4)}_{AB} = \sum_{n=1}^{202} \Big(\mathfrak{g}^{(4)}_n \cdot$  $\Big), \quad \beta^{(3)}_{aij} = \sum_{n=1}^{308} \Big(\mathfrak{y}^{(3)}_n \cdot$  $\Big)$

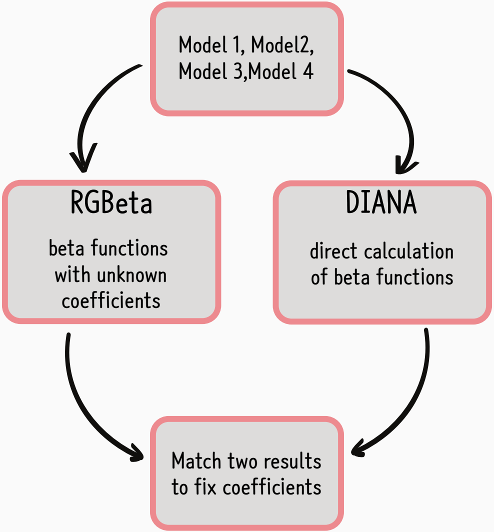
- WCC at the 4-3-2 order are quite powerfull and provide us with 128+133 constraints [Poole,Thomsen'19]
- More constrints from three-loop Yukawa beta-function with matrix couplings [Bednyakov,AP,Velizhanin'14]
- 2HDM three-loop beta-functions [Herren,Mihaila,Steinhauser'17]
- 4-loop gauge [Bednyakov,AP'15,Zoller'15,Davies et al.'19]
- What is left is to fix 11 coefficients in four-loop gauge beta function and 14 in three-loop Yukawa

Toy models idea



- Calculations in SM and 2HDM are too complicated to consider its generalisations with more fields and interactions
- Since we have access to the most general ansatz construct unphysical, but easy to implement in `DIANA` models

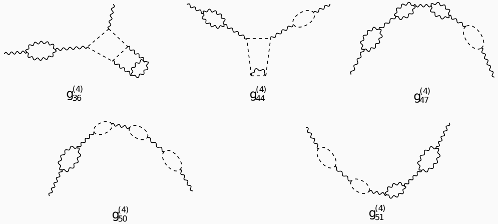
Toy models workflow



M1

$$\mathscr{L}_{\mathrm{M1}} = -\frac{1}{2{g_i}^2}\mathrm{Tr}(F^i_{\mu\nu}F^i_{\mu\nu}) + (D_\mu\phi)^\dagger_{\alpha\rho}(D_\mu\phi)_{\alpha\rho}$$
$$-\frac{\lambda_1}{2}(\phi^\dagger_{\alpha\rho}\phi_{\alpha\rho})(\phi^\dagger_{\beta\sigma}\phi_{\beta\sigma}) - \frac{\lambda_2}{2}(\phi^\dagger_{\alpha\rho}\phi_{\alpha\sigma})(\phi^\dagger_{\beta\sigma}\phi_{\beta\rho})$$

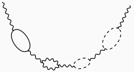
- $SU(n_1)\times SU(n_2)$ version of QCD with “scalar quarks”



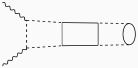
M2

$$\mathscr{L}_{\mathrm{M2}} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + i\bar{Q}\gamma^\mu D_\mu Q + i\bar{u}_R \gamma^\mu \partial_\mu u_R + \frac{1}{2}(\partial_\mu s)^2 + |D_\mu h|^2$$
$$- y_s \bar{Q} Q s - y_u \left[(\bar{Q} h) u_R + \mathrm{h.c.} \right] - \frac{\lambda_s s^4}{24} - \frac{\lambda_{sh}}{2} s^2 (h^\dagger h) - \frac{\lambda_h}{2} (h^\dagger h)^2$$

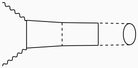
- $SU(n)$ gauge theory with singlet fermion and singlet scalar



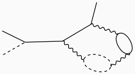
$g^{(4)}_{54}$



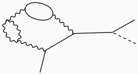
$g^{(4)}_{190}$



$g^{(4)}_{193}$



$y^{(3)}_{36}$

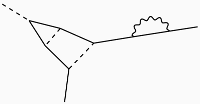


$y^{(3)}_{37}$

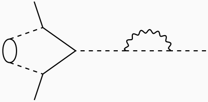
M3

$$\mathscr{L}_{\mathrm{M3}} = -\frac{1}{2{g_i}^2}\mathrm{Tr}(F^i_{\mu\nu}F^i_{\mu\nu}) + \mathrm{Tr}[(D_\mu\phi_i)(D_\mu\phi_i)] + i\bar{\Psi}\gamma_\mu(D_\mu\Psi)$$
$$- y_i\left[\bar{\Psi}\phi_i\Psi + \mathrm{h.c.}\right] - \frac{\lambda_{ij}}{8}\mathrm{Tr}(\phi_i\phi_i)\mathrm{Tr}(\phi_j\phi_j) - \frac{\lambda_i}{24}\mathrm{Tr}(\phi_i\phi_i\phi_i\phi_i)$$

- $SU(n_1)\times SU(n_2)\times SU(n_3)$ with adjoint scalars



$y^{(3)}_{191}$

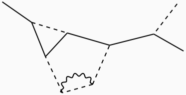


$y^{(3)}_{216}$

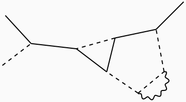
M4

$$\mathscr{L}_{\mathrm{M4}} = -\frac{1}{2g^2}(F_{\mu\nu}F_{\mu\nu}) + |D_\mu h|^2 + \frac{1}{2}(\partial_\mu s)^2 + i\bar{\Psi}_i\gamma_\mu(D_\mu\Psi_i) + i\bar{\psi}\gamma_\mu(D_\mu\psi)$$
$$-\left[(y_1)_{ij}s\bar{\Psi}_iP_R\Psi_j + (y_2)_i h\bar{\Psi}_iP_R\psi + (y_3)_i h^*\bar{\psi}P_R\Psi_i + y_4 s\bar{\psi}P_R\psi + \mathrm{h.c.}\right]$$
$$-\frac{\lambda_s s^4}{24} - \frac{\lambda_{sh}}{2}s^2(h^\dagger h) - \frac{\lambda_h}{2}(h^\dagger h)^2$$

- $U(1)$ model with charged and neutral Higgs



$y^{(3)}_{172}$



$y^{(3)}_{173}$

Summary of unknown coefficients fixing

Type of the beta function	Gauge		Yukawa	
Number of equations and unknowns	$r = \mathrm{n} + c$	$u_{\mathfrak{g}}$	$r = \mathrm{n} + c$	$u_{\mathfrak{y}}$
Initial number of unknown coefficients	-	202	-	308
Weyl Consistency Conditions	128+0	74	133+0	175
Four-loop SM gauge beta functions	63+84	11	-	
Three-loop matrix Yukawa beta functions in the SM	-		128+17	47
Three-loop matrix Yukawa beta functions in THDM	-		33+213	14
Four-loop QCD beta function for general group	2+11	9	-	
$SU(n_1) \times SU(n_2)$ gauge theory (M1)	5+25	4	-	
$SU(n)$ gauge theory (M2)	2+55	2	4+76	10
$SU(n_1) \times SU(n_2) \times SU(n_3)$ gauge theory (M3)	-		9+89	1
$U(1)$ gauge theory (M4)	-		1+199	0
Constraints from symmetric T_{IJ}	2+8	0	-	
Final number of unknowns		0		0

- Several cross checks stemming from different models
- Validation of the conjecture about symmetric T_{IJ}, part of WCC

Application to 2HDM and NHDM at 4-loops

- Compact result for 2HDM model and NHDM generalisation
- All fermion indices are contracted in Yukawa matrices traces
- Up to 3 traces at four-loop order have form

$$\mathrm{Tr}\left[Y_a \ldots Y_a \ldots Y_b \ldots\right] \cdot \mathrm{Tr}\left[Y_c \ldots Y_c \ldots Y_b \ldots\right] \cdot \ldots$$

- Four-loop 2HDM result naturally extends three-loop 2HDM resuls for Yukawa to 4-3-2 running [Herren,Mihaila,Steinhauser'17]

Conclusion

- Extensive use of already known results in SM, 2HDM and from WCC reduced the problem to fixing of the small number of numerical coefficients in general ansatz for beta-functions

- Combination of the standard workflow for the direct calculation in toy models with `RGBeta` for efficient manipulation of the general model ansatz allows to fix all remaining coefficients at 4-loops for gauge and at 3-loops for Yukawa

- All our results confirmed two days ago by Davies, Herren and Thomsen in arXiv:2110.05496

Thank you for attention!