Evaluating five-loop propagators

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in collaboration with Alessandro Georgoudis, Vasco Gonçalves, Erik Panzer, Raul Pereira and Alexander Smirnov [JHEP,2021].

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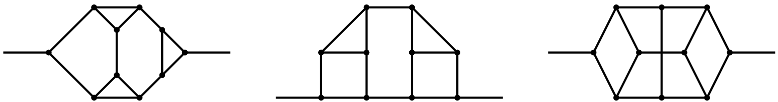

- 1 loop

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- Conclusion

1 loop

$$\int \frac{\mathrm{d}^d k}{(k^2)^{\lambda_1}[(q-k)^2]^{\lambda_2}}$$
$$= \pi^{d/2} \frac{\Gamma(2-\epsilon-\lambda_1)\Gamma(2-\epsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\epsilon)} \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)}{(q^2)^{\lambda_1+\lambda_2+\epsilon-2}}$$
$$\equiv \pi^{d/2} \frac{G(\lambda_1,\lambda_2)}{(q^2)^{\lambda_1+\lambda_2+\epsilon-2}}$$

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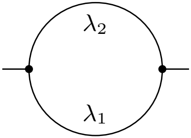
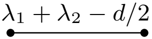


$= \pi^{d/2} G(\lambda_1, \lambda_2) \times$



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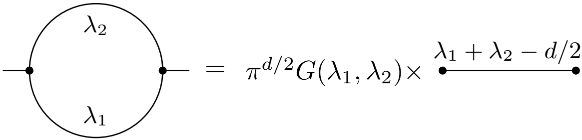
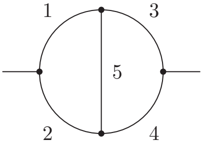


Recursively one-loop integrals: they can be evaluated at general d by successfully applying this one-loop formula.

2 loops



$$F_\Gamma = \int\int \frac{1}{\prod D_i^{a_i}}\, \mathsf{d}^d k_1 \mathsf{d}^d k_2$$

where $D_1 = k_1^2,\ D_1 = (p - k_1)^2,\ D_3 = k_2^2,$
$D_4 = (p - k_2)^2,\ D_5 = (k_1 - k_2)^2.$

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be a family of h-loop Feynman integrals and D_i are denominators of propagators which are quadratic or linear wrt loop momenta k_j and external momenta p_r.

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IBP relations: insert $\frac{\partial}{\partial k_a} \cdot k_b$ or $\frac{\partial}{\partial k_a} \cdot p_r$
into the integrand of the general integral and set the resulting expression to zero.

$$\int \dots \int \left(\frac{\partial}{\partial k_a} \cdot k_b \frac{1}{\prod D_i^{a_i}} \right) \mathrm{d}^d k_1 \dots \mathrm{d}^d k_h = 0 \,,$$

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Express the results of the differentiation in terms of D_i to obtain a system of difference equations of the integrals considered as functions of integer variables a_i, with operators of shifting indices a_i and multiplication by indices similar to creation and annihilation operators.

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Express the results of the differentiation in terms of D_i to obtain a system of difference equations of the integrals considered as functions of integer variables a_i, with operators of shifting indices a_i and multiplication by indices similar to creation and annihilation operators.
Solve these equations.

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The number of master integrals is finite

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The whole problem of evaluation→

- constructing a reduction procedure
- evaluating master integrals

Public codes to solve IBP relations:

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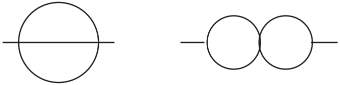
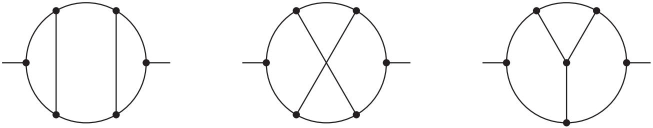
- KIRA [P. Maierhöfer, J. Usovitsch, P. Uwer]

2 loops

Master integrals in 2 loops:



3 loops



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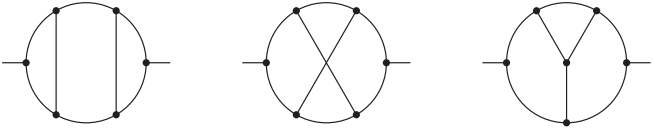

Manual solution [K.G. Chetyrkin & F.V. Tkachov'81]

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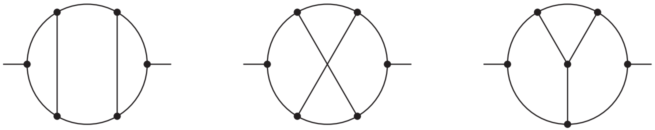

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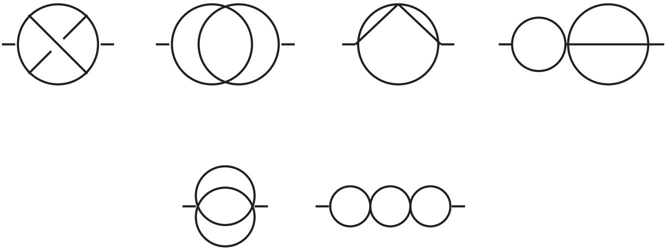

MINCER: an implementation of the manual solution in FORM
[S.G. Gorishny, S.A. Larin, L.R. Surguladze &
F.V. Tkachov'89]
[S.A. Larin, F.V. Tkachov & J. Vermaseren'91]

Glue-and-Cut

The dependence on the external momentum of a propagator massless Feynman integral corresponding to a graph Γ follows from power counting:

$$F_{\Gamma}(p;d) = \left(\pi^{d/2}\right)^{h} C_{\Gamma}(\varepsilon)(p^2)^{\omega - h\varepsilon} \ ,$$

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Gluing by lines. Let us suppose that UV- and IR-convergent graphs, Γ_1 and Γ_2, have $\omega_1 = \omega_2 = -1$ and that the graphs obtained by connecting the two external vertices by additional lines are the same.

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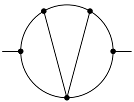
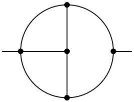
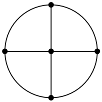
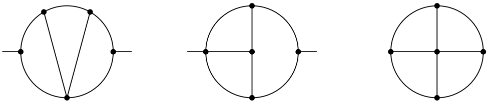
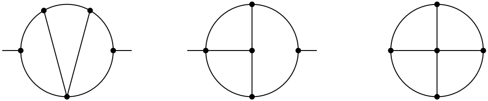
Then $C_{\Gamma_1}(0) = C_{\Gamma_2}(0)$.



At $\varepsilon = 0$, the first two diagrams are equal to $20\zeta(5)$.



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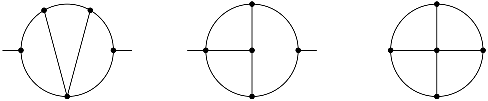
They can be obtained by cutting a line in the third vacuum diagram.



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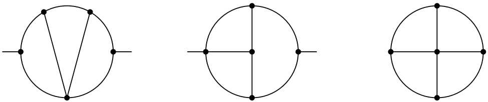
[P.A. Baikov & K.G. Chetyrkin'10]



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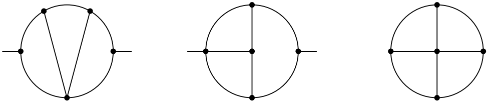

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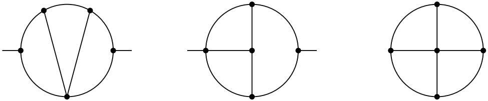

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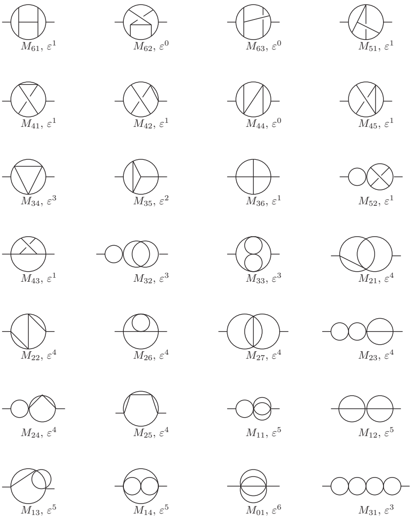
- Solve resulting linear equations for the coefficients of the master integrals in their ε-expansions.

4 loops


M_{61}, ε^1
M_{62}, ε^0
M_{63}, ε^0
M_{51}, ε^1
M_{41}, ε^1
M_{42}, ε^1
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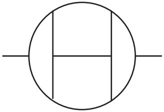

M_{61}

$$= -\frac{10\zeta_5}{\varepsilon} + 50\zeta_5 - 10\,\zeta_3^2 - 25\zeta_6$$

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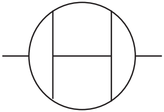

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[R. Lee, A. Smirnov and V. Smirnov'11]: evaluation up to transcendental weight 12.

5 loops

[P.A. Baikov & K.G. Chetyrkin'10]:
The five-loop problem can also be solved. The identities stemming from the GaC symmetry will express all five-loop MI's in terms of significantly smaller set of p-integrals. One could certainly expect that:

- *in general the five-loop master p-integrals will contain irrational terms of weight not higher than 9;*
- *the 'small set' of five-loop integrals will include ones primitive as well as those expressible in terms of the generalized F-function.*

All the five-loop propagators integrals can be described by 46 families of integrals associated with graphs with triple vertices.

$$F_{a_1,a_2,\ldots,a_{20}} = \int \ldots \int \frac{1}{\prod_{i=1}^{20} D_i^{a_i}} \, \mathrm{d}^d l_1 \ldots \mathrm{d}^d l_5$$

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For each family, first 14 indices can be positive and the last 6 indices are always non-positive.

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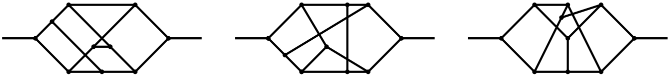

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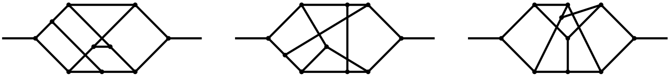

For example, for family 46, $\{D_1, D_2, \ldots, D_{20}\} = \{l_1^2, (l_1 - p)^2, l_2^2, (l_2 + p)^2, l_3^2, l_4^2, (l_3 + l_4)^2, l_5^2, (l_1 + l_3 + l_4 + l_5 - p)^2, (l_1 + l_3 + l_4 - p)^2, (l_2 - l_3 - l_5 + p)^2, (l_1 - l_2 + l_3 + l_4 + l_5 - p)^2, (l_1 + l_4)^2, (l_2 - l_3 + p)^2, l_1 \cdot l_3, l_2 \cdot l_3, l_2 \cdot l_4, l_2 \cdot l_5, l_3 \cdot l_5, l_4 \cdot l_5\}$

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If the code doesn't produce a good basis (i.e. with IBP reductions with good denominators) then it is reasonable to look for a hidden relation between current master integrals.

We have found the following additional relation

$$
\begin{aligned}
&F_{1,1,1,0,1,1,1,1,1,0,1,1,0,1,0,0,0,0,0,0} \\
&= \frac{1}{3d-11}\Big[4(2d-7)F_{0,1,1,1,1,1,1,1,1,0,1,1,1,0,0,0,0,0,0,0} \\
&\qquad -5(d-5)F_{1,1,0,1,1,0,1,1,1,0,1,1,1,1,0,0,0,0,0,0} \\
&\qquad +(d-5)F_{0,1,1,0,1,1,1,1,1,0,1,1,1,1,0,0,0,0,0,0}\Big] + \dots
\end{aligned}
$$

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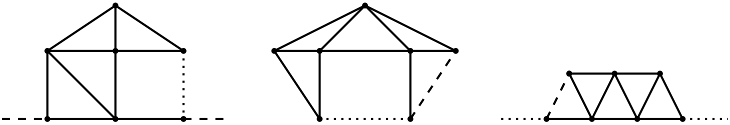

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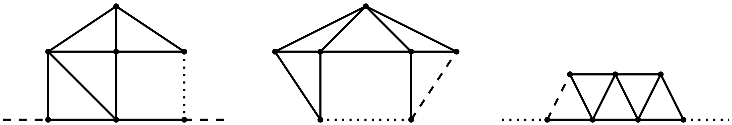

- Reduce all the integrals involved to master integrals using FIRE.

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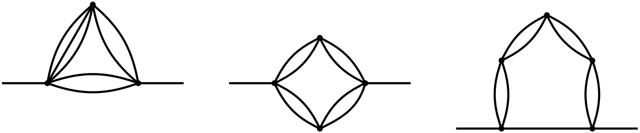
We have obtained results for all the terms of the ε-expansion of all the 281 master integrals up to weight 9.

To fix the solution, we took into account information about the ε-expansion of 21 recursively one-loop master integrals, e.g.



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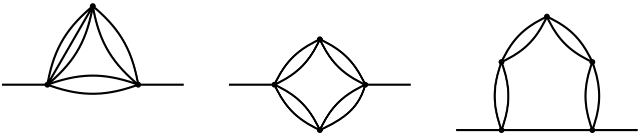
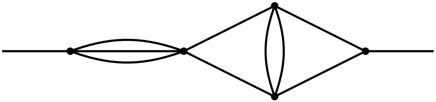
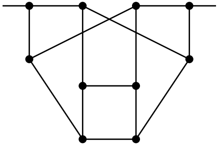


In addition, we needed the ε-expansion of only one factorizable master integral.



An example of our results



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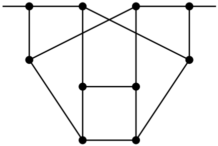

$$
\begin{aligned}
= \;& \frac{6\zeta(5)}{\varepsilon^2} + \left(\frac{174\zeta(3)^2}{5} - 29\zeta(5) - \frac{42\zeta(7)}{5} + \frac{\pi^6}{63}\right)\frac{1}{\varepsilon} \\
- \;& 210\zeta(5) - \frac{29\pi^6}{378} - \frac{1261\zeta(3)^2}{5} - \frac{1919\zeta(7)}{10} + \frac{29\pi^4\zeta(3)}{25} \\
- \;& 204\zeta(3)\zeta(5) + \frac{2887\pi^8}{78750} - \frac{3888}{25}\zeta(3,5) \\
+ \;& \frac{614\zeta(9)}{3} + 48\zeta(3)^3 + O(\varepsilon)\,,
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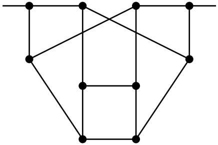

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$$

$$
\zeta(m_1, \ldots, m_k) = \sum_{i_1=1}^{\infty} \sum_{1}^{i_1-1} \cdots \sum_{1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}} \, .
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- We have evaluated all the five-loop propagator master integrals in an ε-expansion up to weight 9.

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- Standard applications: evaluating β-functions or anomalous dimensions 5 loops can be feasible.
- The bottleneck is IBP reduction.